Multipole Point Representation of Seismic Sources for Joint Model-Source FWI

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RICE

- 2010 - University of Texas at El Paso (UTEP),
  B.S. in Physics and Applied Mathematics

- 2015 - Rice University,
  M.A. in Computational and Applied Mathematics (CAAM)
  Thesis: "*Discontinuous Galerkin and Finite Difference Methods for the Acoustic Equations with Smooth Coefficients*"

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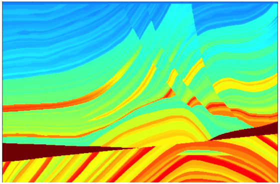
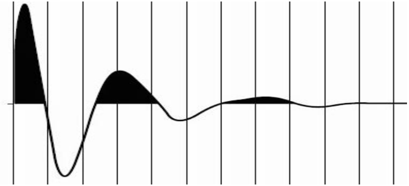
  Thesis: "*Joint Model and Minimal-Source Full Waveform Inversion for Seismic Imaging Under General (Anisotropic) Sources*"

Outline

1. Multipole Point-Source (MPS) Approximation and Representation
2. MPS Numerical Considerations and Results
3. Future Work

Motivation





accurate model estimation $\Longleftrightarrow$ accurate **estimation** & **representation** of seismic sources

Estimation: joint model-source parameter estimation

Representation: multipole point-source (MPS)

- preserve point-source representation
- account for source radiation pattern

Importance of Modeling Source Anisotropy: Case Study [Minkoff & Symes, 1997]

- plane-wave marine data from Gulf of Mexico
- estimated:
  - P-wave background velocity
  - 3 elastic parameter reflectivities
  - anisotropic source term (airgun array)
- viscoelastic layered model, primary reflections:

$$d(t,p) = f(t,p) * \tilde{r}(t,p)$$

- t = time
- p = slowness
- d = data
- f = source term
- $\tilde{r}$ = reflectivity

- anisotropic source model:

$$f(t,p) = \sum_{i=0}^{N} f_i(t) L_i(p)$$

- $L_i = i^{th}$ Legendre polynomial

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Importance of Modeling Source Anisotropy: Case Study [Minkoff & Symes, 1997]

Better data fit when inverting for anisotropic source:

- fixed isotropic source (airgun model): 55% misfit
- estimated isotropic source: 53% misfit
- estimated anisotropic source: 29% misfit

∗ Better match with P-wave impedance well-logs and estimated P-wave reflectivity.

∗ v_P/v_S reflectivity relation helped detect gas-sand target, ONLY with estimated anisotropic source

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Multipole Point-Source (MPS) Approximation

Consider acoustic wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - c^2(\mathbf{x})\nabla^2\right) p(\mathbf{x},t) = f(\mathbf{x},t).$$

General source term f can be approximated via truncated MPS series centered at some $\mathbf{x}^*$, [Santosa & Symes, 2000]:

$$f(\mathbf{x},t) \approx \sum_{|\mathbf{s}|=0}^{N} (-1)^{|\mathbf{s}|} f_{\mathbf{s}}(t) \mathbf{D}^{\mathbf{s}} \delta(\mathbf{x}-\mathbf{x}^*),$$

where $\mathbf{s} = [s_1, s_2, s_3]$ is a multi-array index, and

$$\mathbf{D}^{\mathbf{s}} = \left(\frac{\partial}{\partial x_1}\right)^{s_1} \left(\frac{\partial}{\partial x_2}\right)^{s_2} \left(\frac{\partial}{\partial x_3}\right)^{s_3}.$$

Multipole Point-Source (MPS) Approximation

Number of terms in MPS series depends on

- desired accuracy
- size of source region relative to wavelengths of waves generated
- complexity of source radiation pattern

In most seismic applications, point-source representation is justified

- size of source region $\sim 10m$
- wavelengths $\sim 100m$

My interest: use of MPS series to represent source terms, modeling anisotropy/directivity of seismic sources.

Multipole Point-Source (MPS) Representation

- Acoustic equations (velocity-pressure form):

$$\frac{\partial}{\partial t} v_i - \frac{1}{\rho} \frac{\partial}{\partial x_i} p = f_i^{[v]}$$

$$\frac{\partial}{\partial t} p - \kappa \frac{\partial}{\partial x_i} v_i = f^{[p]}$$

- $p(\mathbf{x}, t) =$ pressure field
- $v_i(\mathbf{x}, t) =$ particle velocity field
- $\rho(\mathbf{x}) =$ density
- $\kappa(\mathbf{x}) =$ bulk modulus

- Linear elasticity equations (velocity-stress form):

$$\rho \frac{\partial}{\partial t} v_i - \frac{\partial}{\partial x_j} \sigma_{ij} = f_i^{[v]}$$

$$\frac{\partial}{\partial t} \sigma_{ij} - C_{ijmn} \frac{\partial}{\partial x_n} v_m = f_{ij}^{[\sigma]}$$

- $\sigma_{ij}(\mathbf{x}, t) =$ stress field
- $C_{ijmn}(\mathbf{x}) =$ Hooke's tensor

- source terms: $f^{[p]}, f_i^{[v]}, f_{ij}^{[\sigma]}$

Multipole Point-Source (MPS) Representation

Represent source terms as "linear combination" of MPS basis:

$$f^{[p]}(\mathbf{x},t) \approx \sum_n f_n(t) b^n(\mathbf{x}-\mathbf{x}^*)$$

$$f_i^{[v]}(\mathbf{x},t) \approx \sum_n f_n(t) b_i^n(\mathbf{x}-\mathbf{x}^*)$$

$$f_{ij}^{[\sigma]}(\mathbf{x},t) \approx \sum_n f_n(t) b_{ij}^n(\mathbf{x}-\mathbf{x}^*)$$

- $f_n(t)$ - MPS coefficients
- $b^n(\mathbf{x})$ - MPS basis, related to $\mathbf{D}^{\mathbf{s}}\delta(\mathbf{x})$

$*$ Impose constraints on source radiation pattern via choice of MPS basis $b^n(\mathbf{x})$

$*$ Forward map will be linear w.r.t. MPS coefficients $f_n(t)$

MPS Numerical Considerations

Problem:

- numerically solve PDEs with singular source terms

$$\mathbf{D^s}\delta(\mathbf{x}-\mathbf{x}^*)$$

Interested in:

- approximation of singular sources terms in FD methods, uniform grids
- preserving spatial convergence rate of FD methods

MPS Numerical Considerations

Solution: approximate singularity with continuous function of compact support, [Waldén, 1998], [Tornberg & Engquist, 2004], [Petersson & Sjögreen, 2010]

$$D^s\delta(x) \approx \delta_\varepsilon^s(x) \in C_0([-\varepsilon,\varepsilon])$$

where δ_ε^s satisfies certain properties (discrete moments) dependent on

- q - accuracy order of approximation
- s - derivative order
- ε - width of support

$*$ Condition on width of support:

$$2\varepsilon \geq (q+s)h$$

MPS Numerical Considerations

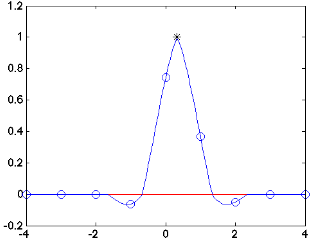


Figure 1: $\delta_\varepsilon^s(x-\alpha)$ with $s=0$, $q=4$, $\alpha=h/3$, $\varepsilon=4h$.

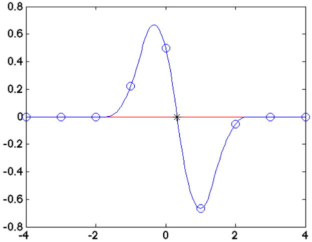
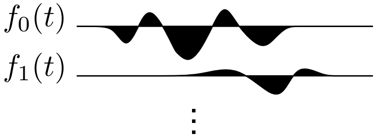


Figure 2: $\delta_\varepsilon^s(x-\alpha)$ with $s=1$, $q=3$, $\alpha=h/3$, $\varepsilon=4h$.

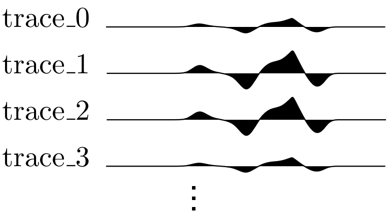
IWave Implementation of MPS Representation

From MPS representation

$$f(x,t) = \sum_n f_n(t) b^n(x - x^*)$$



To time traces at FD grid points



Numerical Experiments

Goal: preserve spatial convergence rates of FD methods

Setup:

- solved acoustic equations in velocity-pressure form
- staggered grid FD solver (2-2 and 2-4)
- homogenous medium; $\rho = 1\ kg/m^3$, $c = 4\ km/s$
- source term , with $s = 0, 1$,

$$f^{[p]}(\boldsymbol{x}, t) = f(t) \left(\frac{\partial}{\partial x_2} \right)^s \delta(\boldsymbol{x} - \boldsymbol{x}^*)$$

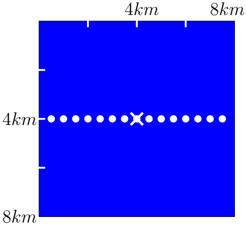


Figure 3: Homogeneous model.

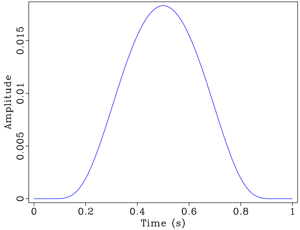


Figure 4: MPS coefficient $f(t)$.

Numerical Experiments

Convergence rate calculated approximately by

$$R(\boldsymbol{x}_r) = \log_2\left(\frac{\|p_h(\boldsymbol{x}_r,\cdot) - p_{h/2}(\boldsymbol{x}_r,\cdot)\|}{\|p_{h/2}(\boldsymbol{x}_r,\cdot) - p_{h/4}(\boldsymbol{x}_r,\cdot)\|}\right),$$

where norm $\|\cdot\|$ is chosen to be either $\|\cdot\|_2$ or $\|\cdot\|_\infty$ defined by

$$\|p(\boldsymbol{x}_r,\cdot)\|_2 := \sqrt{\Delta t \sum_k |p(\boldsymbol{x}, t_k)|^2},$$

$$\|p(\boldsymbol{x}_r,\cdot)\|_\infty := \max_k |p(\boldsymbol{x}_r, t_k)|.$$

h-refinement [m]:

- 2-2: 10, 20, 40
- 2-4: 20, 40, 80

$*$ $\Delta t = 0.5\, ms$, sufficiently small that spatial error dominates, i.e., approximate semi-discrete solution

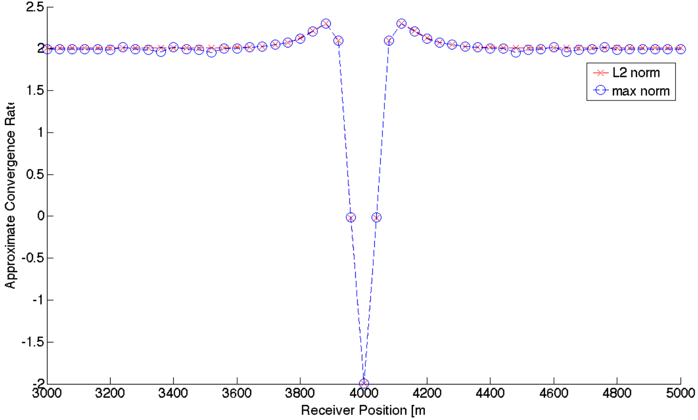


Figure 5: Rates for 2-2 staggered grid scheme with $\delta(\boldsymbol{x}-\boldsymbol{x}^*)$ source term and approximation $\delta^0_{\varepsilon_1}\delta^0_{\varepsilon_2}$ with $q=2$.

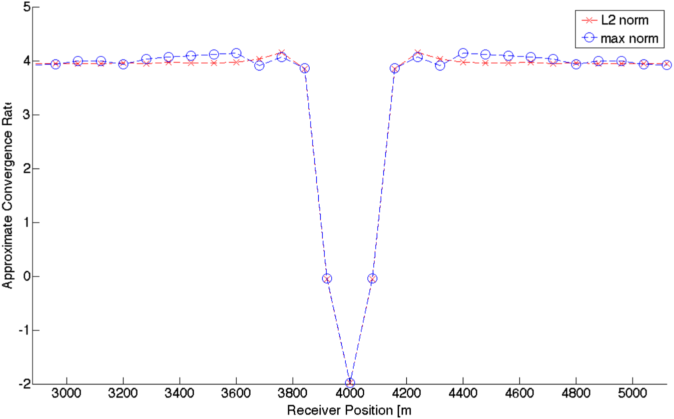


Figure 6: Rates for 2-4 staggered grid scheme with $\delta(\boldsymbol{x} - \boldsymbol{x}^*)$ source term and approximation $\delta^0_{\varepsilon_1}\delta^0_{\varepsilon_2}$ with $q = 4$.

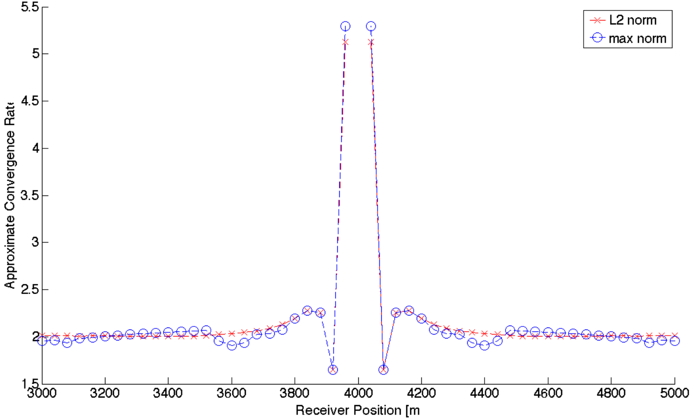


Figure 7: Rates for 2-2 staggered grid scheme with $\frac{\partial}{\partial x_2}\delta(\boldsymbol{x}-\boldsymbol{x}^*)$ source term and approximation $\delta^0_{\varepsilon_1}\delta^1_{\varepsilon_2}$ with $q = 2$.

Numerical Results

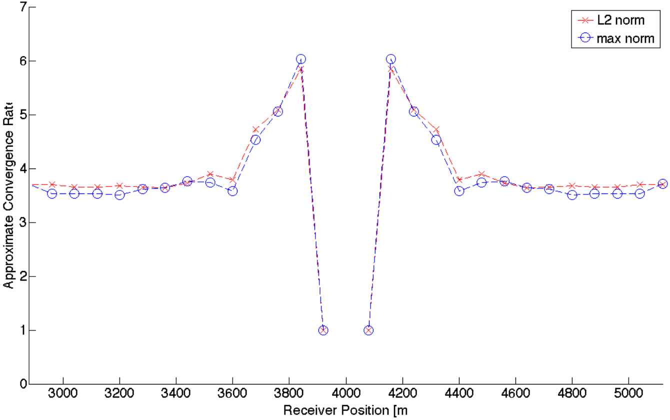


Figure 8: Rates for 2-4 staggered grid scheme with $\frac{\partial}{\partial x_2}\delta(\boldsymbol{x}-\boldsymbol{x}^*)$ source term and approximation $\delta^0_{\varepsilon_1}\delta^1_{\varepsilon_2}$ with $q=4$.

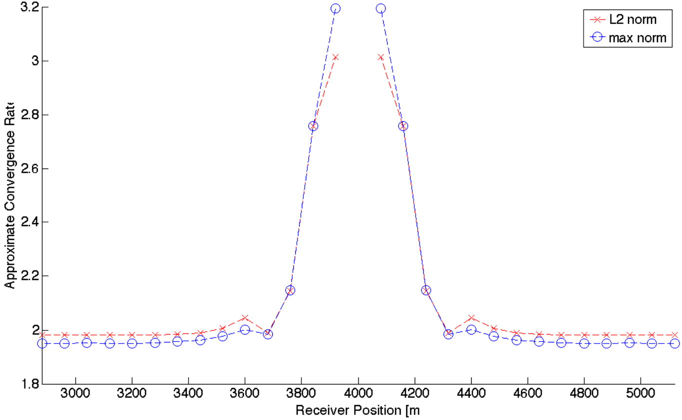


Figure 9: Rates for 2-4 scheme with $\frac{\partial}{\partial x_2}\delta(\boldsymbol{x}-\boldsymbol{x}^*)$ source term and approximation $\delta^0_{\varepsilon_1}\delta^1_{\varepsilon_2}$ with $q=2$.

Future Directions

Idea: modify domain norm of forward map w.r.t. source such that

$$\|\text{input}\| \sim \|\text{output}\|$$

- better bounded forward map
- better condition on inverse problem for source

Consider $f^{[p]}$ for acoustics in velocity-pressure form:

$$f^{[p]}(\mathbf{x},t) = f(t)\left(\frac{\partial}{\partial x_2}\right)^s \delta(\mathbf{x}-\mathbf{x}^*).$$

Resulting data (pressure field) will look like some derivative of MPS coefficient $f(t)$,

$$p(\mathbf{x},t) \sim f^{s+r}.$$

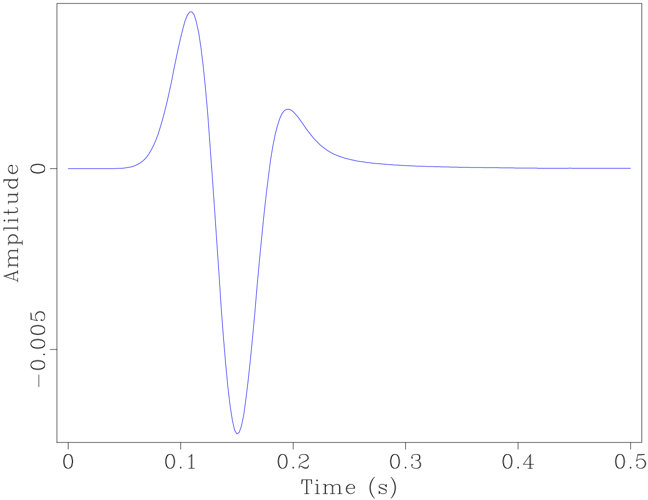


Figure 10: Data trace for source with $s = 0$.

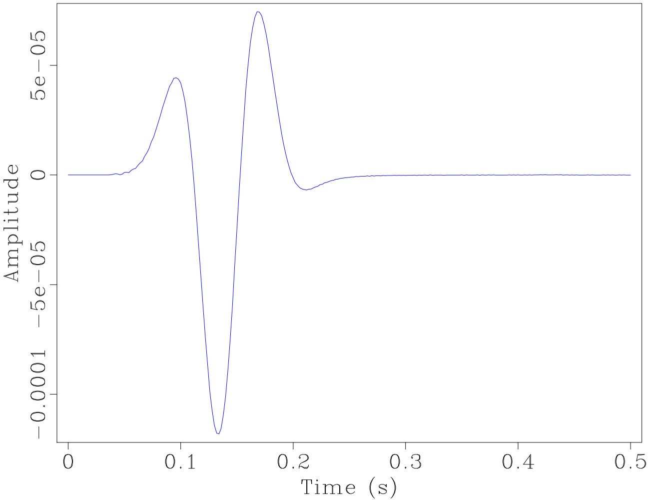


Figure 11: Data trace for source with $s = 1$.

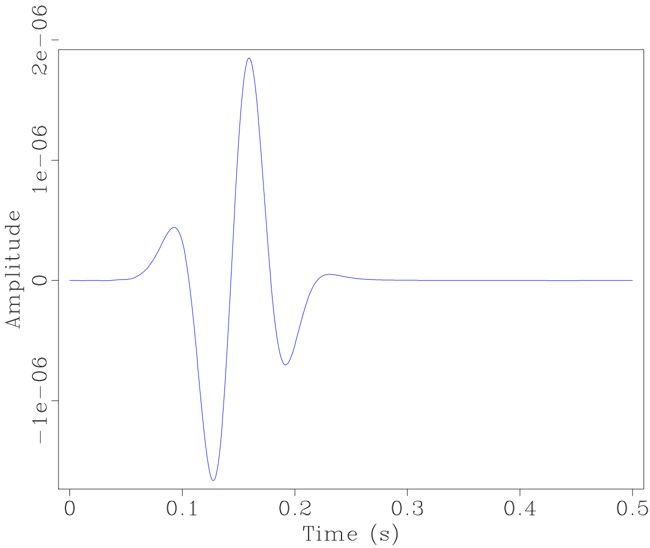


Figure 12: Data trace for source with $s = 2$.

Future Directions

Denote forward map $F : \mathfrak{M} \times \mathfrak{F} \to \mathfrak{D}$,

$$F(\mathbf{m}, \mathbf{f}) \equiv F(\mathbf{m})\mathbf{f} = \mathbf{d}$$

- $\mathfrak{M}$ - model space; **m** - model parameters
- $\mathfrak{F}$ - source space; **f** - MPS coefficients
- $\mathfrak{D}$ - data space; **d** - data vector

Introduce weight W such that

$$\|W\mathbf{f}\|_{\mathfrak{F}} \sim \|F(\mathbf{m})\mathbf{f}\|_{\mathfrak{D}}.$$

E.g.,

$$W = \left(\frac{d}{dt}\right)^{s+r}$$

Future Directions

MPS inversions:

- better condition of source estimation from weighted inner product (preconditioning in CG)
- non-uniqueness of MPS representation of seismic sources (null space of forward map)

MPS + model inversions:

- acoustics $\rightarrow$ elasticity

Thank You