Linearized Extended Waveform Inversion

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Outline

- Introduction
- Linearized extended waveform inversion and reduced objective function
- Image stability of normal operator
- Smoothness and unimodality of reduced objective function (ROF)
- Conclusion and current work
Introduction

Goal:
From a lot of reflection data, get the model of the earth.

Methods:
Seismic inversion (Gauthier et al., 1986; Santosa & Symes, 1989; Virieux & Operto, 2009, etc.)

- Over-determined: highly redundant in the observed data;
- Local minima: cycle skipping.

Migration velocity analysis (Yilmaz, Seismic Data Analysis, Chapter 10; Biondi, 3D Seismic Imaging, Chapter 11 & 12)

- Part of conventional workflow
- Give correct position of reflectors
Model extension (Symes & Kern 1994; Symes 2008; Sun 2012; Biondi & Almomin 2012; Zhang & Biondi 2012)

- $M = \{ m(x) \}$ Physical model space, velocity, density, bulk modulus, ...
- $\tilde{M} = \{ m(x, h) \}$ Extended model space, $M \subset \tilde{M}$.
- $h$ extended coordinate: shot position, offset, subsurface offset, or scattering angle, ...
Extended Forward Modeling

\[ \bar{F} : \bar{M} \mapsto D \text{ extended forward map} \]

\[ \bar{F}[\bar{m}] = d. \]

Ex: acoustic constant density, shot coordinate extension with \( d = u(x_r, t, x_s) \) and \( \bar{m} = c(x, x_s)^2 \)

\[ \frac{\partial^2 u(x, t, x_s)}{\partial t^2} - c(x, x_s)^2 \Delta u(x, t, x_s) = f(x, t, x_s). \]
Extended Born Modeling

\[ D\bar{F} : M \times \bar{M} \mapsto D \text{ extended Born map} \]

\[ D\bar{F}[m_l]\delta \bar{m} = \delta d. \]

Ex: acoustic constant density, shot coordinate extension with \( \delta d = \delta u(x_r, t, x_s) \), \( m_l = c(x)^2 \) and \( \delta \bar{m} = \delta c(x, x_s)^2 \)

\[
\frac{\partial^2 \delta u(x, t, x_s)}{\partial t^2} - c(x)^2 \Delta \delta u(x, t, x_s) = \delta c(x, x_s)^2 \Delta u(x, t, x_s).
\]
Linearized Extended Waveform Inversion

Model separation: \( \tilde{m} \approx m_l + \delta \tilde{m} \).

- \( m_l \) smooth background model, physical.
- \( \delta \tilde{m} \) reflectivity, extended.

Linearized extended waveform inversion: given reflection data \( \delta d \in D \), find \( m_l, \delta \tilde{m} \) so that

\[
D \tilde{F}[m_l] \delta \tilde{m} \approx \delta d.
\]

(Symes & Carazzone, 1991; Kern & Symes, 1994; Chauris & Noble, 2001; Mulder & ten Kroode, 2002; Shen & Symes, 2008; Symes 2008.)
Reduced Objective Function (ROF)

Define

\[ J[m_l, \delta \tilde{m}] = \frac{1}{2} \| D\tilde{F}[m_l] \delta \tilde{m} - \delta d \|^2 + \frac{\alpha^2}{2} \| A \delta \tilde{m} \|^2 \]

- An annihilator, \( A \delta m = 0 \) for all \( \delta m \in M \).
- \( A = \frac{\partial}{\partial x_s} \) for shot coordinate model extension.

Reduced objective function (Symes & Kern 1994)

\[ \tilde{J}[m_l] = \min_{\delta \tilde{m}} J[m_l, \delta \tilde{m}] \]

with

\[ \delta \tilde{m}[m_l] = (D\tilde{F}[m_l]^T D\tilde{F}[m_l] + \alpha^2 A^T A)^{-1} D\tilde{F}[m_l]^T \delta d. \]

An example of variable projection method (van Leeuwen & Mulder 2009)
Image Stability of Normal Operator

Normal operator

\[ N[m_l] = DF[m_l]^T DF[m_l]. \]

- \( N[m_l] \) essentially pseudodifferential operator.
- \( N[m_l] \) is smooth function of \( m_l \).

Show \( N[m_l] \delta m \) with different \( m_l \) and fixed \( \delta m \)

\[ N[m_l] \delta m \approx L^{\frac{n-1}{2}} P \delta m \]

with \( L \) the Laplacian operator and \( P \) acts as multiplication.

Symes, 2008 Approximate linearized inversion by optimal scaling of prestack depth migration
Image Stability of Normal Operator
Image Stability of Normal Operator

\[ N[m_l] \delta m \approx L^{\frac{n-1}{2}} P \delta m \text{ with } m_l = \sigma m_{Sm}. \]
Image Stability of Normal Operator

\[ N[m_I] \delta m \approx L^{\frac{n-1}{2}} P \delta m \text{ with } m_I = \sigma m_{Sm}. \]
Image Stability of Normal Operator

\[ N[m_l] \delta m \approx L^{\frac{n-1}{2}} P \delta m \text{ with } m_l = \sigma m_{Sm}. \]
Image Stability of Normal Operator

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Image Stability of Normal Operator

\[ N[m_l] \delta m \approx L^\frac{n-1}{2} P \delta m \text{ with } m_l = \sigma m_{Sm}. \]
Scan Tests of Reduced Objective Function

Reduced objective function

\[
\tilde{J}[m_l] = \min_{\delta \tilde{m}} J[m_l, \delta \tilde{m}] = \frac{1}{2} \| D \tilde{F}[m_l] \delta \tilde{m} - \delta d \|^2 + \frac{\alpha^2}{2} \| A \delta \tilde{m} \|^2.
\]

with \( \delta \tilde{m}[m_l] \) the solution of "extended least squares migration (LSM)" (called PICLI method in Ehinger and Lailly, 1993)

\[
(D \tilde{F}[m_l]^T D \tilde{F}[m_l] + \alpha^2 A^T A) \delta \tilde{m}[m_l] = D \tilde{F}[m_l]^T \delta d.
\]

Ehinger and Lailly, 1993, Prestack imaging by coupled linearized inversion: SPIE Proceedings
Marmousi Model

\[ m_{Sm} \]

\[ \delta m \]

**Acquisition geometry:**

- 60 shots starting at 3km, with 100 meters spacing
- 96 receivers are placed behind each shot, with 25 meters spacing
- 200 meters between the first receiver and a shot
- Shots are 12 meters below the sea surface
- Receivers are 8 meters below the sea surface
\[ \delta d = DF[m_i] \delta m \]
Image Gathers of RTM

\[ D_F[m_l]^T \delta d \]
Image Gathers of Extended LSM

\[(D\tilde{F}[m_l]^T D\tilde{F}[m_l] + \alpha^2 A^T A)\delta \tilde{m} = D\tilde{F}[m_l]^T \delta d.\]

\[\alpha = 0.01\]
Image Gathers of Extended LSM

\[(D\bar{F}[m_l]^T D\bar{F}[m_l] + \alpha^2 A^T A)\delta \bar{m} = D\bar{F}[m_l]^T \delta d.\]

\[\alpha = 0.1\]
Smoothness and Unimodality of ROF

Scan test of $\tilde{J}[m_I]$ along line segment

$$m_I = \sigma m_{sm}$$

with $\alpha = 0.01, 0.1, 1.0$ and $\sigma \in [0.6, 1.4]$. 

![Graph showing the value of the objective function for different values of $\alpha$ and $\sigma$.]
Scan test of $\tilde{J}[m_I]$ along line segment

$$m_I = (1 - \sigma)m_{sm} + \sigma m_0$$

with $\alpha = 0.01, 0.1, 1.0$, $\sigma \in [-0.4, 0.6]$ and $m_0(x) = 1500\text{m/s}$. 
Conclusion

- Numerical results suggest image stability of the normal operator;
- Extended reflectivity from least squares migration along shot coordinate resemble each other closely for large $\alpha$;
- Scan tests along line segments show that the reduced objective function has large basin of attraction to the global minimum and is smooth for certain $\alpha$;
Work in progress I

Gradient computation of the reduced objective function

**Reduced objective function**

\[
\tilde{J}[m_l] = \min_{\delta \bar{m}} J[m_l, \delta \bar{m}] = \frac{1}{2} \| D\bar{F}[m_l] \delta \bar{m} - \delta d \|^2 + \frac{\alpha^2}{2} \| A \delta \bar{m} \|^2.
\]

with \( \delta \bar{m}[m_l] \) the solution of "extended least squares migration (LSM)"

\[
(D\bar{F}[m_l]^T D\bar{F}[m_l] + \alpha^2 A^T A) \delta \bar{m}[m_l] = D\bar{F}[m_l]^T \delta d.
\]

**Gradient** of the reduced objective function (Symes & Kern 1994):

\[
\nabla \tilde{J}[m_l] = D^2 \bar{F}[m_l]^T [\delta \bar{m}[m_l], D\bar{F}[m_l] \delta \bar{m} - \delta d]
\]

with \( D^2 \bar{F}[m_l]^T \) the tomographic operator (Biondi and Almomin, 2013).
Use Lagrange multiplier method to compute optimal $\alpha$ automatically

Constrained optimization problem:

$$
\min_{\delta \tilde{m}} \quad \frac{1}{2} \| A \delta \tilde{m} \|^2 \\
\text{subject to} \quad \| D\tilde{F}[m_l] \delta \tilde{m} - \delta d \|^2 \leq \epsilon^2
$$

with $\epsilon$ the noise level of data.

Equivalent to our ”extended least squares migration”

$$
\min_{\delta \tilde{m}} \frac{1}{2} \| D\tilde{F}[m_l] \delta \tilde{m} - \delta d \|^2 + \frac{\alpha^2}{2} \| A \delta \tilde{m} \|^2
$$
Apply preconditioner to accelerate the convergence rate of extended least squares migration

Optimal scaling preconditioner

From

\[ \mathcal{N}(m_l) \approx L^{\frac{n-1}{2}} P \]

get

\[ (\mathcal{N}(m_l))^{-1} \approx P^\dagger L^{\frac{n-1}{2}} \]

with \( P^\dagger = (P + \epsilon I)^{-1} \) for a positive \( \epsilon \).

Symes, 2008 Approximate linearized inversion by optimal scaling of prestack depth migration
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