Source function estimation in extended full waveform inversion

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The Rice Inversion Project (TRIP)

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Overview

Objective

Recover Earth model with seismic waveform inversion

Problems

Unknown source

Local minima

Solution

Variable projection method

Extended modelling concept

Extended modelling concept

Abstract setting for forward map $\mathcal{F} : \mathcal{M} \to \mathcal{D}$

$$F[m] = d$$

F: forward modelling operator
m: model (v, r, w)
d: sampled pressure data at receivers

Extended forward map $\bar{\mathcal{F}} : \bar{\mathcal{M}} \to \mathcal{D}$ [Symes, 2008]

$$\bar{F}[\bar{m}] = d$$

$\bar{F}$: extended forward modelling operator
$\bar{m}$: extended model ($v(\mathbf{x})$, $\bar{r}(\mathbf{x}, \mathbf{h})$, $w(t)$, ...)
d: sampled pressure data at receivers

Linearized acoustic modelling (Born approximation)

p - reference (incident) pressure field

$$\left(\frac{1}{v^2(\mathbf{x})}\frac{\partial^2}{\partial t^2} - \nabla^2\right) p(t,\mathbf{x};\mathbf{x_s}) = w(t)\delta(\mathbf{x}-\mathbf{x_s})$$

δp - scattered (perturbation) pressure field

$$\left(\frac{1}{v^2(\mathbf{x})}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta p(t,\mathbf{x};\mathbf{x_s}) = \frac{2\bar{r}(\mathbf{x},\mathbf{h})}{v^2(\mathbf{x})}\frac{\partial^2 p}{\partial t^2}(t,\mathbf{x};\mathbf{x_s})$$

$w(t)$: source function
v: velocity of seismic waves
$\bar{r}(\mathbf{x},\mathbf{h}) = \frac{\delta\bar{v}(\mathbf{x},\mathbf{h})}{v(\mathbf{x})}$: extended reflectivity ($\bar{v}(\mathbf{x},\mathbf{h})$: extended velocity)
p: pressure field
$\mathbf{x}$: position in earth model
$\mathbf{x}_s$: source location

Green's function

$G(t, \mathbf{x}; \mathbf{x}_s)$ - Green's function, impulse response of the medium

$$\left(\frac{1}{v^2(\mathbf{x})}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(t, \mathbf{x}; \mathbf{x_s}) = \delta(t)\delta(\mathbf{x} - \mathbf{x_s})$$

$$p(t, \mathbf{x}; \mathbf{x}_s) = G(t, \mathbf{x}; \mathbf{x}_s) * w(t)$$
$$\delta p(t, \mathbf{x}_r; \mathbf{x}_s) = \bar{f}[v]\bar{r} * w(t)$$

$$\begin{aligned}\delta p(t, \mathbf{x}_r; \mathbf{x_s}) &= \left[\frac{\partial^2}{\partial t^2}\int d\mathbf{x}\int d\mathbf{h}\frac{2\bar{r}(\mathbf{x}, \mathbf{h})}{v^2(\mathbf{x})} G(t, \mathbf{x} + \mathbf{h}; \mathbf{x}_r) * G(t, \mathbf{x} - \mathbf{h}; \mathbf{x}_s)\right] * w(t) \\ &= \bar{f}[v]\bar{r} * w(t)\end{aligned}$$

Extended full waveform inversion (EFWI)

Abstract setting for Inversion:

$$m = F^{-1}[d]$$

- This inverse problem is large scale and nonlinear.
- Indirect approach: formulate as an optimization problem

Given d, find m that minimizes the output least square (OLS) objective function (Tarantola, Lailly, 1980s to present)

$$min_m J_{OLS}[m,d] = \frac{1}{2}\|F[m]-d\|^2$$

$$min_{v,\bar{r},w} J[v,\bar{r},w,d] = \frac{1}{2}\|\bar{f}[v]\bar{r} * w - d\|^2 + \frac{\alpha^2}{2}\|A\bar{r}\|^2$$

Source function $w(t)$ in EFWI

$$J = \frac{1}{2}\|\bar{f}[v]\bar{r} * w - d\|^2 + \frac{\alpha^2}{2}\|A\bar{r}\|^2$$

Following [Rickett, 2013]'s approach, write in two matrix forms

$$J = \frac{1}{2}\|\bar{F}_g[v,\bar{r}]w - d\|^2 + \frac{\alpha^2}{2}\|A\bar{r}\|^2 \quad (1)$$

$$J = \frac{1}{2}\|W\bar{f}[v]\bar{r} - d\|^2 + \frac{\alpha^2}{2}\|A\bar{r}\|^2 \quad (2)$$

$\bar{F}_g[v,\bar{r}]$: matrix that applies convolutions to w

W: matrix that applies convolution to $\bar{f}[v]\bar{r}$

Source function $w(t)$ in EFWI

1st form

$$J = \frac{1}{2}\|\bar{F}_g[v,\bar{r}]w - d\|^2 + \frac{\alpha^2}{2}\|A\bar{r}\|^2$$

$$\bar{F}_g w = \begin{pmatrix} \bar{f}_0 & 0 & \cdots & 0 & \cdots & 0 \\ \bar{f}_1 & \bar{f}_0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ \bar{f}_{N_w-1} & \bar{f}_{N_w-2} & \cdots & \bar{f}_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{f}_{N_t-1} & \bar{f}_{N_t-2} & \cdots & \bar{f}_{N_t-N_w} & \cdots & \bar{f}_0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N_w-1} \\ \vdots \\ 0 \end{pmatrix}$$

N_w: number of time samples for source wavelet signature
N_t: number of time samples for recorded seismic data

Variable Projection Method (VPM) [Golub and Pereyra, 1973]

$$min_{v,\bar{r},w} J_{OLS}[v,\bar{r},w,d] = \frac{1}{2}\|\bar{F}_g[v,\bar{r}]w - d\|^2 + \frac{\alpha^2}{2}\|A\bar{r}\|^2$$

VPM

- Step 1: eliminate the linear variable (w)
- Step 2: minimize the reduced objective function $(v,\bar{r})$
- Step 3: use the optimal value $(v,\bar{r})$ to solve for w

The constrained equation [Rickett, 2013]

$$w = \bar{F}_g^{+} d = \left(\bar{F}_g^{*}\bar{F}_g\right)^{-1} \bar{F}_g^{*} d$$

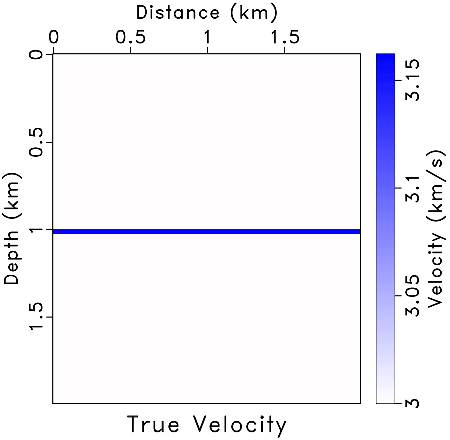
Reduced objective function

$$min_{v,\bar{r}} J_{OLS}[v,\bar{r},d] = \frac{1}{2}\|\bar{F}_g\bar{F}_g^{+} d - d\|^2 + \frac{\alpha^2}{2}\|A\bar{r}\|^2$$

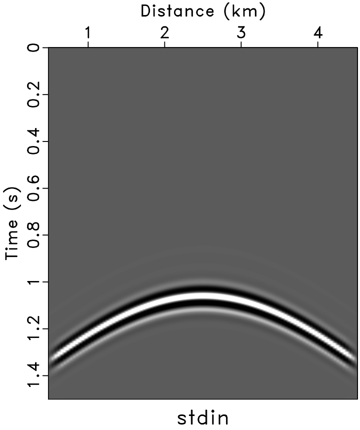
$\bar{F}_g^{+}[v,\bar{r}]$: the pseudo-inverse of $\bar{F}_g[v,\bar{r}]$.

$\bar{F}_g^{*}\bar{F}_g$: dimensions of $N_w \times N_w$, inverted easily with direct methods.

Test

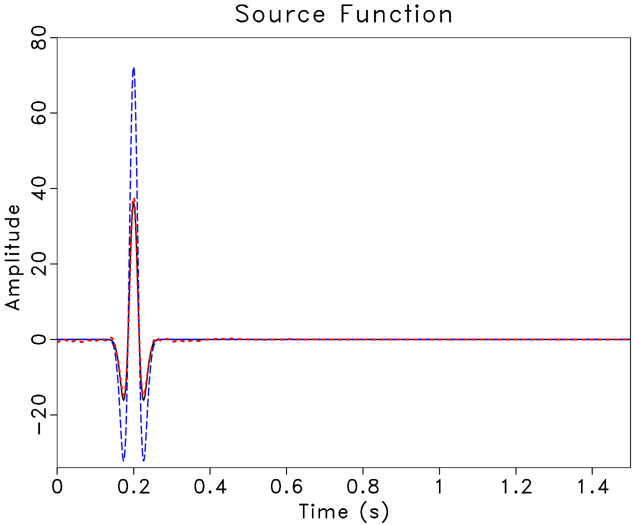


Distance (km)
0
0.5
1
1.5
Depth (km)
0
0.5
1
1.5
3.15
3.1
3.05
3
Velocity (km/s)
True Velocity



Distance (km)
1
2
3
4
Time (s)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
stdin

Inverted source function $w(t)$ (wrong amplitude)



black

true $w(t)$

blue

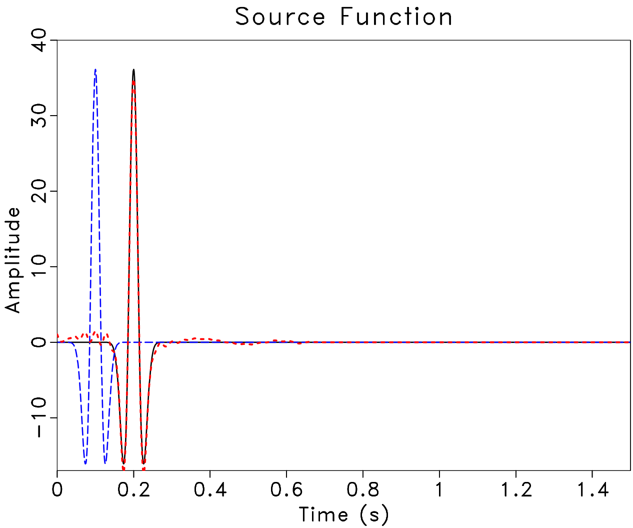
initial $w(t)$

red

inverted $w(t)$

Initial source function with 2 times larger amplitude than the true one. Result from 7 iterations of CG.

Inverted source function $w(t)$ (time shift)



black

true $w(t)$

blue

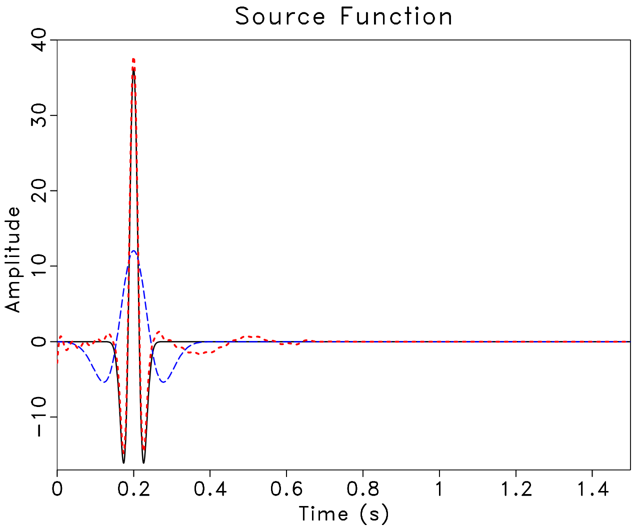
initial $w(t)$

red

inverted $w(t)$

Initial source function with $0.1s$ time shift. Result from 7 iterations of CG.

Inverted source function $w(t)$ (peak frequency)



black

true $w(t)$

blue

initial $w(t)$

red

inverted $w(t)$

Initial source function with peak frequency $5Hz$ (true: $15Hz$). Result from 7 iterations of CG.

Variable Projection Method (VPM)

Reduced objective function

$$min_{v,\bar{r}} J_{OLS}[v,\bar{r},d] = \frac{1}{2}\|\bar{F}_g \bar{F}_g^{+} d - d\|^2 + \frac{\alpha^2}{2}\|A\bar{r}\|^2$$

Advantages [Golub and Pereyra, 2003]:

- The reduced problem has the same minima as the original one
- Eliminate the linear variable

Gradient calculation

Reduced objective function

$$min_{v,\bar{r}} J_{OLS}[v,\bar{r},d] = \frac{1}{2}\|\bar{F}_g[v,\bar{r}]\bar{F}_g^{+}[v,\bar{r}]d - d\|^2 + \frac{\alpha^2}{2}\|A\bar{r}\|^2$$

Nested approach [Symes and Kern, 1994, Almomin and Biondi, 2013]

1. Inner optimize over $\bar{r}$ for each v
2. Outer optimize over v

1. Gradient of J with respect to $\bar{r}$ (nonlinear)

$$\nabla_{\bar{r}} J = \bar{f}[v]^* W[v,\bar{r}]^* d_r + \alpha^2 A^* A\bar{r}$$

data residual: $d_r = \bar{F}_g[v,\bar{r}]\bar{F}_g^{+}[v,\bar{r}]d - d$

Gradient calculation

1. Gradient of J with respect to $\bar{r}$

$$\nabla_{\bar{r}} J = \bar{f}[v]^* W[v, \bar{r}]^* d_r + \alpha^2 A^* A \bar{r}$$

data residual: $d_r = \bar{F}_g[v, \bar{r}] \bar{F}_g^+[v, \bar{r}] d - d$

2. Gradient of J with respect to v

$$\nabla_v J = D\bar{f}^* \left(W[v, \bar{r}]^* d_r, \bar{r}[v] \right)$$

Numerical implementation

- Finite difference `C` code in `Madagascar`
- `TAPENADE` - Online Automatic Differentiation Engine

  derivative $\bar{f}[v]$ and its adjoint $\bar{f}[v]^*$ (dot product test)

  2nd order derivative $\bar{D}f[v]$ and its adjoint $\bar{D}f[v]^*$ (dot product test)
- Parallel computing (MPI)
- PML [Grote and Sim, 2010]

Summary

Methods:

- Variable projection method

  Eliminate the linear variable w

  Reduced objective function

- Extended modelling concept

  Overcome local minima obstacle

Future work

- VPM in RVL
- Source function estimation in IWAVE
- Reduce computational cost

Acknowledgement

Thanks to

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