About Me

Mario Bencomo

- Currently 2^{nd} year graduate student in CAAM department at Rice University.
- B.S. in Physics and Applied Mathematics (Dec. 2010).
- Undergraduate University: University of Texas at El Paso (UTEP).

Discontinuous Galerkin and Finite Difference Methods for the Acoustic Equations with Smooth Coefficients

Mario Bencomo

TRIP Review Meeting 2013

RICE

Problem Statement

- Acoustic Equations (pressure-velocity form):

$$\rho(\boldsymbol{x})\frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x},t)+\nabla p(\boldsymbol{x},t)=0 \tag{1a}$$

$$\frac{1}{\kappa}(\boldsymbol{x})\frac{\partial p}{\partial t}(\boldsymbol{x},t)+\nabla\cdot\boldsymbol{v}(\boldsymbol{x},t)=f(\boldsymbol{x},t) \tag{1b}$$

for $\boldsymbol{x}\in\Omega$ and $t\in[0,T]$, where $\boldsymbol{x}=(x,z)$ and $\Omega=[0,1]^2$.

- Boundary and initial conditions:

$$p=0, \quad \text{on } \partial\Omega\times[0,T]$$

$$p(\boldsymbol{x},0)=p_0(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{v}(\boldsymbol{x},0)=\boldsymbol{v}_0(\boldsymbol{x})$$

Research focus:

- Analyze the computational efficiency of **discontinuous Galerkin (DG)** and **finite difference (FD)** methods in the context of the acoustic equations with **smooth coefficients**.

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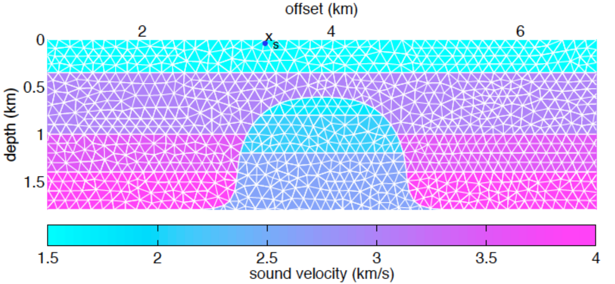
- Analyze the computational efficiency of **discontinuous Galerkin (DG)** and **finite difference (FD)** methods in the context of the acoustic equations with **smooth coefficients**.

Why Smooth Coefficients?

- Relevant for seismic applications: smooth trends in real data!
- Comparison has not been done before!
  Previous work with discontinuous coefficients *(Wang, 2009)*.
  - efficiency of DG over 2-4 FD (dome inclusion)
  - DG resolves discontinuity interface errors



Focus of Talk

1. Discontinuous Galerkin (DG) Method
   - derivation of scheme
   - basis functions
   - reference element
2. Locally conforming DG (LCDG) Method
   - triangulation
   - reference element
   - basis functions
3. Summary and Future Work

Why DG? (Cockburn, 2006; Brezzi et al., 2004; Wilcox et al., 2010)

- explicit time-stepping, after inverting block diagonal matrix
- can handle irregular meshes and complex geometries
- hp-adaptivity

Idea: Approximate solution by piecewise polynomials on partitioned domain.
Example: 1D piecewise linear approximation

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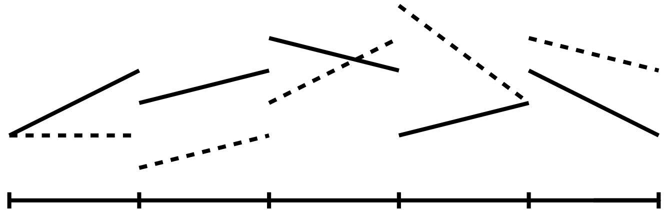

Derivation of DG Semi-Discrete Scheme

Let:

- $\tau \in \mathscr{T}$, for some triangulation $\mathscr{T}$ on Ω
- test function $w \in C^\infty(\Omega)$

Then,

$$\rho \frac{\partial v_x}{\partial t} + \frac{\partial p}{\partial x} = 0 \Longrightarrow \int_\tau \rho \frac{\partial v_x}{\partial t} w \, d\boldsymbol{x} + \int_\tau \frac{\partial p}{\partial x} w \, d\boldsymbol{x} = 0$$

I.B.P. and replace p with **numerical flux** p^* in boundary integral,

$$\int_\tau \rho \frac{\partial v_x}{\partial t} w \, d\boldsymbol{x} - \int_\tau p \frac{\partial w}{\partial x} \, d\boldsymbol{x} + \int_{\partial\tau} p^* w \, n_x \, d\sigma = 0$$

$$\overset{IBP}{\Longrightarrow} \int_\tau \rho \frac{\partial v_x}{\partial t} w \, d\boldsymbol{x} + \int_\tau \frac{\partial p}{\partial x} w \, d\boldsymbol{x} + \int_{\partial\tau} (p^* - p) w \, n_x \, d\sigma = 0 \quad (2)$$

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Nodal Basis Functions

Finite dimensional space $\mathscr{W}_h$:

$$\mathscr{W}_h = \{w : w|_\tau \in \mathbb{P}^N(\tau), \forall \tau \in \mathscr{T}\}$$

Nodal DG: Use nodal basis, i.e.,

$$\mathbb{P}^N(\tau) = \text{span}\{\ell_j^\tau(\boldsymbol{x})\}_{j=1}^{N^*} \quad \forall \tau \in \mathscr{T},$$

where

- Lagrange polynomials $\ell_j^\tau(\boldsymbol{x}_i^\tau) = \delta_{ij}$ for given nodal set $\{\boldsymbol{x}_i^\tau\}_{i=1}^{N^*} \subset \tau$
- $N^* = \frac{1}{2}(N+1)(N+2)$, a.k.a., degrees of freedom per triangular element

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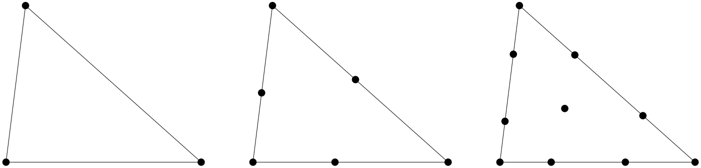
Example of nodal sets $\{\boldsymbol{x}_j^\tau\}_{j=1}^{N^*}$:



(a) $N = 1, N^* = 3$ (b) $N = 2, N^* = 6$ (c) $N = 3, N^* = 10$

DG Semi-Discrete Scheme

Find $v_x, p \in \mathscr{W}_h$ such that

$$\int_\tau \rho \frac{\partial v_x}{\partial t} w \, d\boldsymbol{x} - \int_\tau \frac{\partial p}{\partial x} w \, d\boldsymbol{x} + \int_{\partial \tau} (p^* - p) w \, n_x \, d\sigma = 0$$

for all $w \in \mathscr{W}_h$ and $\tau \in \mathscr{T}$.

Note:

$$v_x \in \mathscr{W}_h \Longrightarrow v_x(\boldsymbol{x}, t)|_\tau = \sum_{j=1}^{N^*} v_x(\boldsymbol{x}_j^\tau, t) \ell_j^\tau(\boldsymbol{x}),$$

where $\{v_x(\boldsymbol{x}_j^\tau, t)\}_{j=1}^{N^*}$ are unknowns. Same for v_z and p, and numerical fluxes v_x^*, v_z^*, p^*.

Nodal Coefficient Vectors

$$\mathbb{R}^{N^*} \quad \begin{cases} \mathbf{v}_x(t) & := & [v_x(\boldsymbol{x}_1^\tau, t), v_x(\boldsymbol{x}_2^\tau, t), \ldots, v_x(\boldsymbol{x}_{N^*}^\tau, t)]^T \\ \mathbf{v}_z(t) & := & \cdots \\ \mathbf{p}(t) & := & \cdots \end{cases}$$

$$\mathbb{R}^{N+1} \quad \begin{cases} \mathbf{v}_{\boldsymbol{n}}^m(t) & := & [v_{\boldsymbol{n}}(\boldsymbol{x}_{m_1}^\tau, t), v_{\boldsymbol{n}}(\boldsymbol{x}_{m_2}^\tau, t), \ldots, v_{\boldsymbol{n}}(\boldsymbol{x}_{m_{N+1}}^\tau, t)]^T \\ \mathbf{p}^m(t) & := & \cdots \\ (\mathbf{v}_{\boldsymbol{n}}^m)^*(t) & := & [v_{\boldsymbol{n}}^*(\boldsymbol{x}_{m_1}^\tau, t), v_{\boldsymbol{n}}^*(\boldsymbol{x}_{m_2}^\tau, t), \ldots, v_{\boldsymbol{n}}^*(\boldsymbol{x}_{m_{N+1}}^\tau, t)]^T \\ (\mathbf{p}^m)^*(t) & := & \cdots \end{cases}$$

where $\mathbf{v}_{\boldsymbol{n}}(\boldsymbol{x}_{m_j}^\tau, t) = n_x^m v_x(\boldsymbol{x}_{m_j}^\tau, t) + n_z^m v_z(\boldsymbol{x}_{m_j}^\tau, t)$, and similar for $\mathbf{v}_{\boldsymbol{n}}^*(\boldsymbol{x}_{m_j}^\tau, t)$.

DG Semi-Discrete Scheme

From

$$\int_\tau \rho \frac{\partial v_x}{\partial t} w \, d\boldsymbol{x} + \int_\tau \frac{\partial p}{\partial x} w \, d\boldsymbol{x} + \int_{\partial\tau} (p^* - p) w \, n_x \, d\sigma = 0,$$

to

$$M \mathscr{R} \frac{d}{dt} \mathbf{v}_x(t) + S^x \mathbf{p}(t) + \sum_{m=1}^{3} n_x^m M^m \left((\mathbf{p}^m)^* - \mathbf{p}^m \right)(t) = 0,$$

where:

mass matrix	$(M)_{ij}$	$:=$	$\int_\tau \ell_i^\tau \ell_j^\tau \, d\boldsymbol{x},$	in $\mathbb{R}^{N^* \times N^*}$
mass matrix	$(M^m)_{ij}$	$:=$	$\int_{e_\tau^m} \ell_i^\tau \ell_{m_j}^\tau \, d\sigma,$	in $\mathbb{R}^{N^* \times (N+1)}$
stiffness matrix	$(S^x)_{ij}$	$:=$	$\int_\tau \ell_i^\tau \frac{\partial \ell_j^\tau}{\partial x} \, d\boldsymbol{x},$	in $\mathbb{R}^{N^* \times N^*}$
ρ-matrix	$(\mathscr{R})_{ij}$	$:=$	$\delta_{ij} \rho(\boldsymbol{x}_j^\tau),$	in $\mathbb{R}^{N^* \times N^*}$

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DG Semi-Discrete Scheme

System of ODE's:

$$\mathscr{R}\frac{d}{dt}\mathbf{v}_x(t) = -D^x\mathbf{p}(t) + \sum_{m=1}^{3} n_x^m L^m\left((\mathbf{p}^m)^* - \mathbf{p}^m\right)(t)$$

for each $\tau \in \mathscr{T}$, where

$$D^x = M^{-1}S^x, \quad L^m = M^{-1}M^m.$$

Similar result for other equations:

$$\mathscr{R}\frac{d}{dt}\mathbf{v}_z(t) = -D^z\mathbf{p}(t) + \sum_{m=1}^{3} n_z^m L^m\left((\mathbf{p}^m)^* - \mathbf{p}^m\right)(t)$$

$$\mathscr{K}^{-1}\frac{d}{dt}\mathbf{p}(t) = \mathbf{f}(t) - D^x\mathbf{v}_x(t) - D^z\mathbf{v}_z(t) - \sum_{m=1}^{3} L^m\left((\mathbf{v}_{\boldsymbol{n}})^* - \mathbf{v}_{\boldsymbol{n}}\right)(t)$$

where

$$(\mathscr{K})_{ij} = \delta_{ij}\kappa(\boldsymbol{x}_j^\tau), \quad (\mathbf{f})_i = \int_\tau f\ell_i^\tau\, d\boldsymbol{x}, \quad D^z = M^{-1}S^z.$$

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DG Semi-Discrete Scheme

Acoustic equations:

$$\rho \frac{\partial}{\partial t} v_x = -\frac{\partial}{\partial x} p$$

$$\rho \frac{\partial}{\partial t} v_z = -\frac{\partial}{\partial z} p$$

$$\frac{1}{\kappa} \frac{\partial}{\partial t} p = f - \nabla \cdot \boldsymbol{v}$$

DG scheme:

$$\mathscr{R} \frac{d}{dt} \mathbf{v}_x(t) = -D^x \mathbf{p}(t) + \sum_{m=1}^{3} n_x^m L^m \left((\mathbf{p}^m)^* - \mathbf{p}^m \right)(t)$$

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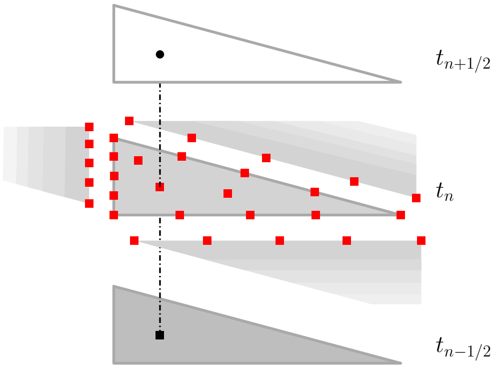
Visualizing DG

After time discretization (leapfrog):

$$\mathbf{v}_x^{n+1/2} = \mathbf{v}_x^{n-1/2} - \Delta t \mathscr{R}^{-1} \left[D^x \mathbf{p}^n + \sum_{m=1}^{3} n_x^m L^m \left((\mathbf{p}^m)^* - \mathbf{p}^m \right)^n \right]$$



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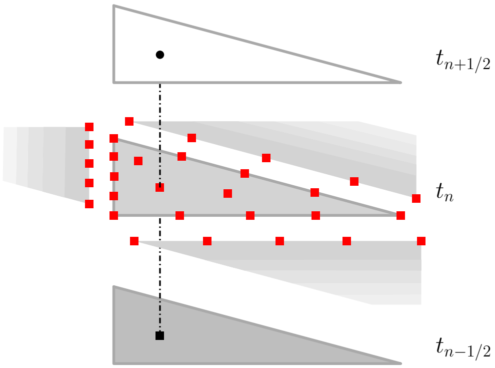

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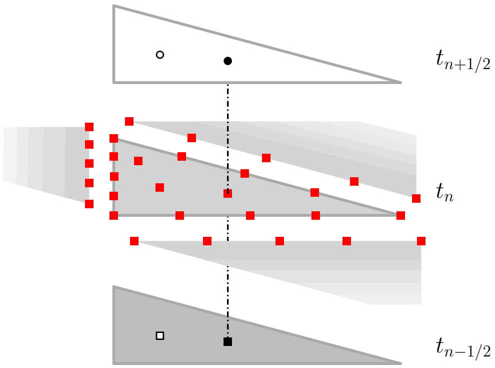

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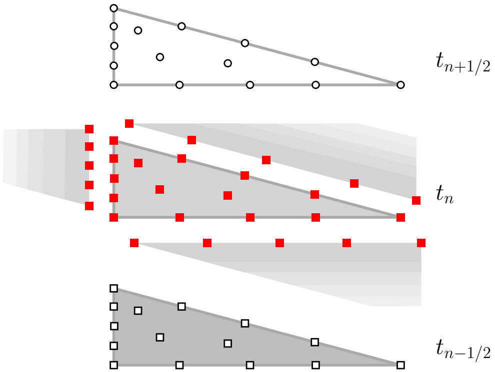
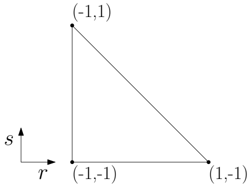


Use of Reference Element in DG

Reference triangle $\hat{\tau}$:



Idea: Carry computations in $\hat{\tau}$

- construct nodal set and basis functions in $\hat{\tau}$:

$$\{\ell_j\}_{j=1}^{N^*} \text{ s.t. } \ell_j(\boldsymbol{r}_i) = \delta_{ij} \text{ for } \{\boldsymbol{r}_i\}_{i=1}^{N^*} \subset \hat{\tau}$$

- mass matrix computations

$$\int_\tau \ell_j^\tau \ell_i^\tau \, d\boldsymbol{x} = J(\tau) \int_{\hat{\tau}} \ell_j \ell_i \, d\boldsymbol{r} \Longrightarrow M = J(\tau)\hat{M}$$

Locally Conforming DG (LCDG) Method for Acoustics

Why LCDG method? (Chung & Engquist, 2006, 2009)

- locally and globally energy conservative
- optimal convergence rate
- explicit time-step

My contribution: nodal DG implementation of LCDG

- basis functions and nodal sets
- use of reference element

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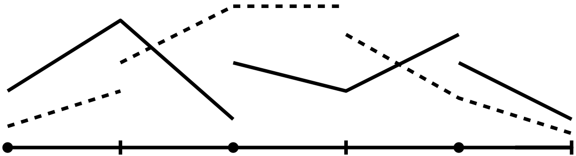
LCDG Idea: Enforce continuity of the pressure field and the normal component of the velocity field in a “staggered” manner.

Example: 1D piecewise linear polynomial approximation



Algorithm: `unif_tri`

1. given N_{sub}, subdivide Ω into $N_{sub} \times N_{sub}$ partition of squares
2. triangulate each square by adding a diagonal from top-left to bottom-right
3. Pick interior point at each triangle and re-triangulate

Figure: Example of `unif_tri` for $N_{sub} = 2$.

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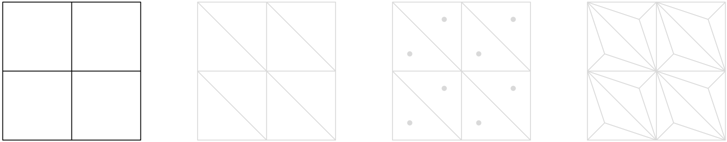

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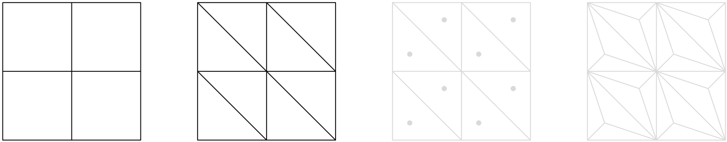

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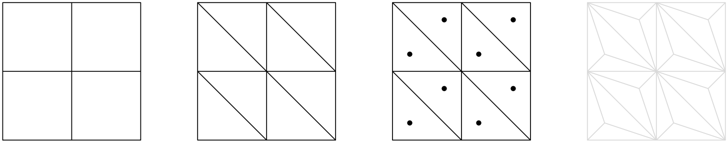

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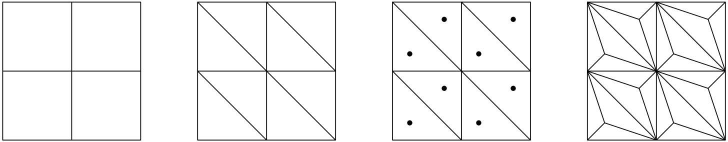

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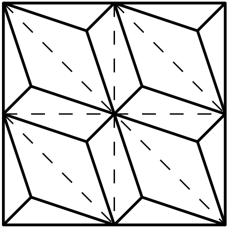
LCDG Approximation Spaces

Pressure space $\mathscr{P}_h$: $q \in \mathscr{P}_h$ if

- $q|_\tau \in \mathbb{P}^N(\tau)$
- q continuous on ***dashed*** edges
- $q = 0$ on $\partial\Omega$

Velocity space $\mathscr{V}_h$: $\boldsymbol{u} \in \mathscr{V}_h$ if

- $\boldsymbol{u}|_\tau \in \mathbb{P}^N(\tau)^2$
- $\boldsymbol{u} \cdot \boldsymbol{n}$ continuous ***dashed*** edges



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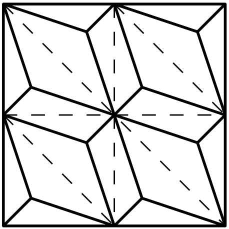

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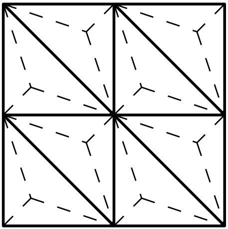
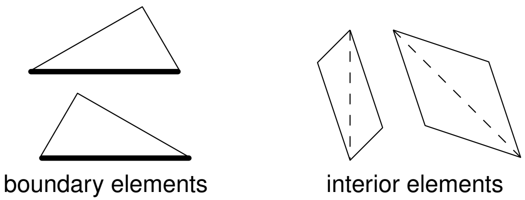
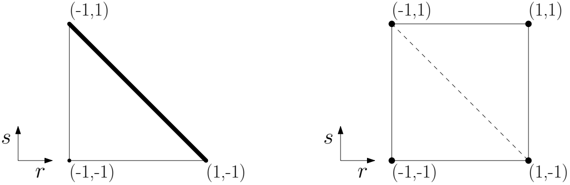


LCDG: Basis for $\mathscr{P}_h$

Two types of pressure elements:

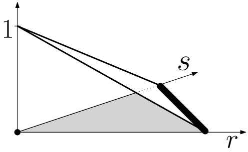


Reference elements:

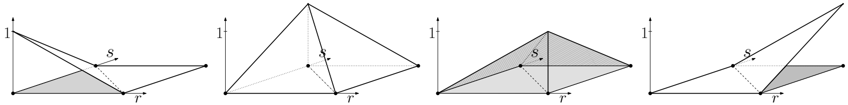


LCDG: Basis for $\mathscr{P}_h$

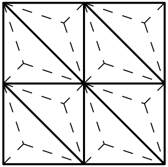
Basis functions for boundary pressure elements ($N = 1$ example):



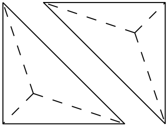
Basis functions for pressure interior elements ($N = 1$ example):



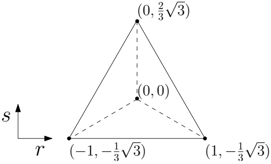
LCDG: Basis for $\mathscr{V}_h$



Velocity elements and respective reference elements:



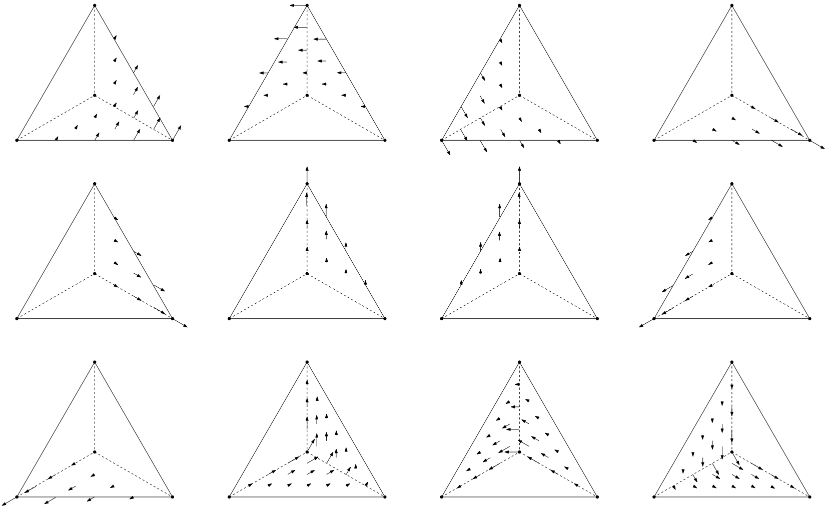
velocity elements



reference element

LCDG: Basis for $\mathcal{V}_h$

Basis functions, $N = 1$ example:



For a triangulation $\mathscr{T}$, find $p \in \mathscr{P}_h$ and $\boldsymbol{v} \in \mathscr{V}_h$ such that

$$\int_\Omega \rho \frac{\partial \boldsymbol{v}}{\partial t} \cdot \boldsymbol{u}\, d\boldsymbol{x} - B_h^*(p, \boldsymbol{u}) = 0$$

$$\int_\Omega \frac{1}{\kappa} \frac{\partial p}{\partial t} q\, d\boldsymbol{x} + B_h(\boldsymbol{v}, q) = \int_\Omega fq\, d\boldsymbol{x}$$

for all $q \in \mathscr{P}_h$ and $\boldsymbol{u} \in \mathscr{V}_h$, where

$$B_h^*(p, \boldsymbol{u}) = -\int_\Omega p\, \nabla \cdot \boldsymbol{u}\, d\boldsymbol{x} + \sum_{e \in \mathscr{E}_p^0} \int_e p\, [\boldsymbol{u} \cdot \hat{\boldsymbol{n}}]\, d\sigma$$

$$B_h(\boldsymbol{v}, q) = \int_\Omega \boldsymbol{v} \cdot \nabla q\, d\boldsymbol{x} - \sum_{e \in \mathscr{E}_v} \int_e \boldsymbol{v} \cdot \hat{\boldsymbol{n}} [q]\, d\sigma.$$

- standard DG
  - motivation
  - basis functions
  - reference element
- LCDG
  - triangulation
  - spaces $\mathscr{P}_h$, $\mathscr{V}_h$
  - reference elements and basis functions

Pending Work and Future Directions

Pending work:

- implementation of LCDG
- absorbing BC *(Chung & Engquist, 2009)*
- time discretization (leapfrog and Runge-Kutta)
- error analysis

Future directions:

- non-uniform triangulation
- 3D acoustics
- elasticity equations

References

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