Harmonic Coordinate Finite Element Method for Acoustic Waves

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Outline

Motivation

Variable density acoustic wave equation

$$\frac{1}{\kappa}\frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla u = f$$

with appropriate boundary, initial conditions

Typical setting in seismic applications:

- heterogeneous κ, ρ with low contrast $O(1)$
- model data κ, ρ defined on regular Cartesian grids
- large scale $\Rightarrow$ waves propagate $O(10^2)$ wavelengths; solutions for many different f
- f smooth in time (band-limited)

Motivation

For piecewise constant κ, ρ with interfaces

- FDM: first order interface error, time shift, incorrect arrival time, no obvious way to fix (Brown 84, Symes & Vdovina 09)
- Accuracy of standard FEM (eg specFEM3D) relies on adaptive, interface fitting meshes
- Exception: FDM derived from mass-lumped FEM on regular grid for **constant density acoustics** has 2nd order convergence even with interfaces (Symes & Terentyev 2009)

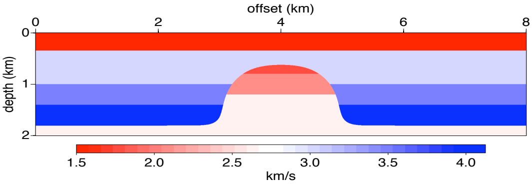


Figure : Velocity model

Motivation

Aim of this project: design approximation method for acoustic wave equation with

- provable optimal (2nd) order convergence
- regular (non-fitted) grids
- practical error control
- computational complexity similar to standard FD/FE methods per time step (perhaps after setup phase)

Review

Transfer-of-approximation FEM (Symes and Wang, 2011) works (2nd order), and has complete theoretical backing, but hopelessly inefficient

Owhadi and Zhang 2007: create new elements by *composing* standard linear elements on triangular mesh with harmonic coordinate map to create new, regular grid elements. Sub-optimal convergence due to element truncation.

Binford 2011: full, un-truncated elements $\Rightarrow$ optimal (2nd) order convergence on triangular meshes for 2D static interface problems

This paper: harmonic coordinate FEM ("HCFEM") on rectangular regular mesh with bilinear ("Q_1") elements.

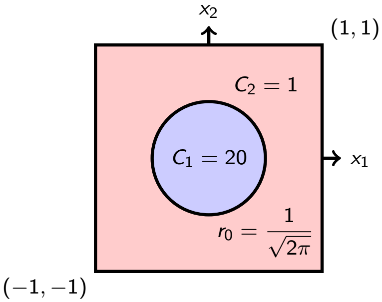
Harmonic Coordinates

Global C-harmonic coordinates $\mathbf{F}$ in 2D, its components $F_1(x_1, x_2), F_2(x_1, x_2)$ are solns of

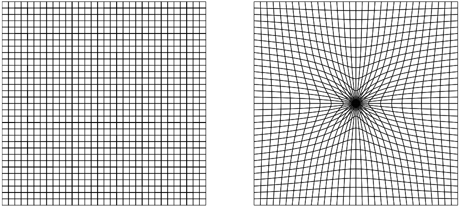
$$\nabla \cdot C(x) \nabla F_i = 0 \quad \text{in } \Omega$$
$$F_i = x_i \quad \text{on } \partial\Omega$$

$\mathbf{F} : \Omega \to \Omega$ C-harmonic coordinates

e.g.,



Harmonic Coordinates



- physical regular grid $(x_1, x_2) = (jh_x, kh_y)$ (left),
- harmonic grid $(F_1, F_2) = \Big(F_1(jh_x, kh_y), F_2(jh_x, kh_y)\Big)$ (right)

Harmonic Coordinate FEM

Workflow of HCFEM:

1. prepare a regular mesh on physical domain, $\mathscr{T}^H$;
2. approximate $\mathbf{F}$ on a fine mesh $\mathscr{T}^h$ by $\mathbf{F}_h$
3. construct the harmonic triangulation $\widetilde{\mathscr{T}}^H = \mathbf{F}_h(\mathscr{T}^H)$;
4. construct the HCFE space
   $S^H = \mathrm{span}\{\tilde{\phi}_i^H \circ \mathbf{F}_h : i = 0, \cdots, N^h\}$, where
   $\tilde{S}^H = \mathrm{span}\{\tilde{\phi}_i^H : i = 0, \cdots, N^h\}$ is isoparametric bilinear (Q_1) FEM space on harmonic grid $\widetilde{T}^H$;
5. solve the original problem by Galerkin method on S^H.

Harmonic Coordinate FEM

- Solving n (≤ 3) harmonic problems to obtain harmonic coordinates
- HCFEM works and as efficient (after setup) as standard FEM
- Accuracy control for HC construction: refine grid to diameter $h = O(H^2)$ at interface.

1D Illustration

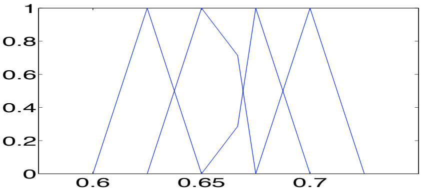
1D elliptic interface problem

$$(\beta u_x)_x = f \quad 0 \le x \le 1, \qquad u(0) = u(1) = 0$$

β has discontinuity at $x = 2/3$

$$\beta(x) = \begin{cases} \beta_0 = 1, & x < 2/3 \\ \beta_1 > 1, & x > 2/3 \end{cases}$$

1D 'linear' HCFE basis:



Mass Lumping

2nd order time discretization with HCFEM:

$$M^h \frac{U^h(t+\Delta t) - 2U^h(t) + U^h(t-\Delta t)}{\Delta t^2} + N^h U^h(t) = F^h(t)$$

$\Rightarrow$ every time update involves solving a linear system $M^h U^h =$RHS

Replace M^h by a diagonal matrix $\tilde{M}^h$,

$$\tilde{M}^h_{ii} = \sum_j M^h_{ij}$$

Theoretical justification: lumped mass solution is just as accurate as the consistent mass solution, can achieve optimal rate of convergence

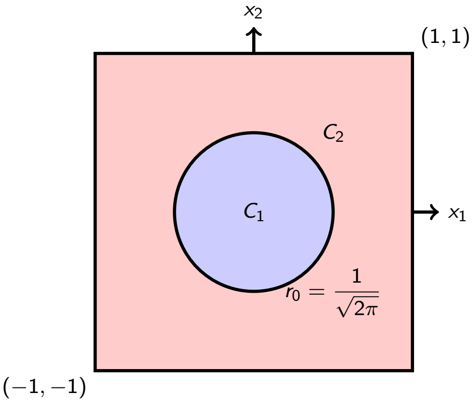
Static acoustic problem - Square Circle Model

$$-\nabla \cdot C(x)\nabla u = -9r \quad \text{in } \Omega$$

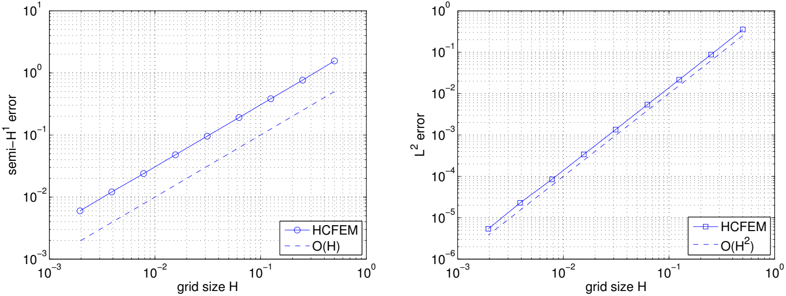
where $r = \sqrt{x^2 + y^2}$

For piecewise const $C(x)$ shown in the figure below, analytical solution:

$$u = \frac{1}{C(x)}(r^3 - r_0^3)$$

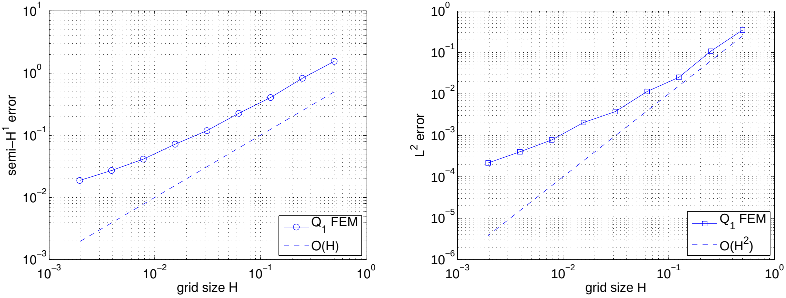


High Contrast: $C_1 = 20, C_2 = 1$



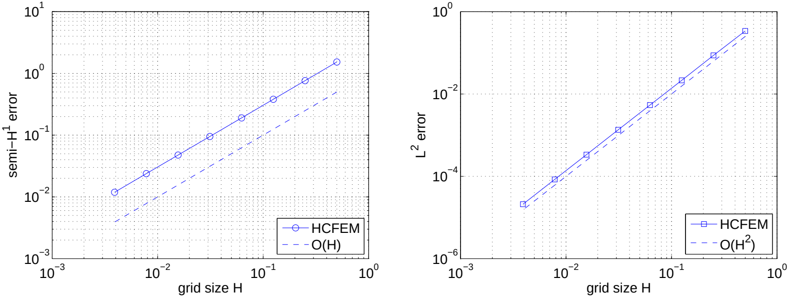
- HCFEM is applied on the physical grid of diameter H
- Harmonic coordinates are approximated on the locally refined grid, in which the grid size is $O(h)$ $(h = H^2)$ near interfaces.

High Contrast: $C_1 = 20, C_2 = 1$



- Standard FEM is applied on the physical grid of diameter H

Low Contrast: $C_1 = 2, C_2 = 1$



- HCFEM is applied on the physical grid of diameter H
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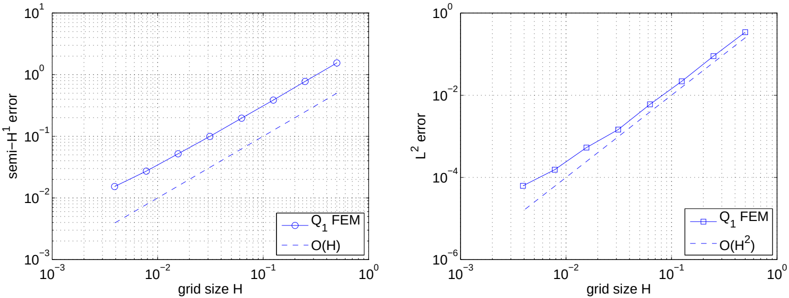
Low Contrast: $C_1 = 2, C_2 = 1$



- Standard FEM is applied on the physical grid of diameter H

2D Acoustic Wave Tests

Acoustic wave equation:

$$\kappa^{-1}\frac{\partial^2 u}{\partial t^2} - \nabla\left(\frac{1}{\rho}\nabla u\right) = 0$$

$$u(x,0) = g(x,0), \quad u_t(x,0) = g_t(x,0)$$

with $g(x,t) = \frac{1}{r} f\left(t - \frac{r}{c_s}\right)$ and

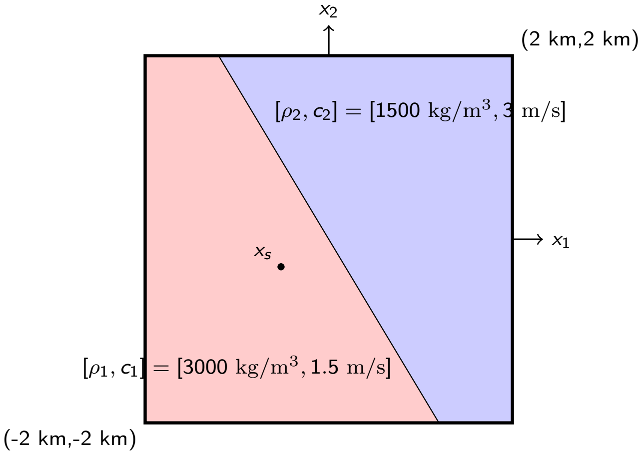
$f(t) = \left(1 - 2\left(\pi f_0\left(t + t_0\right)\right)^2\right) e^{-\left(\pi f_0\left(t+t_0\right)\right)^2}$, f_0 central frequency,

$c_s = \sqrt{\frac{\kappa(x_s)}{\rho(x_s)}}$, $t_0 = \frac{1.45}{f_0}$

The following examples similar to those in Symes and Terentyev, SEG Expanded Abstracts 2009

Dip Model

Central frequency $f_0 = 10$ Hz, $x_s = [-300\sqrt{3}\ \mathrm{m}, -300\ \mathrm{m}]$



Dip Model

Q_1 FEM solution, regular grid quadrature ($=$ FDM) - this is equivalent to using ONLY the node values on the regular grid to compute mass, stiffness matrices

Figure : $T = 0.75$ s

Dip Model

Q_1 FEM solution - accurate quadrature for mass and stiffness matrices' computation,

Figure : Q_1 FEM sol, $T = 0.75$ s

Figure : HCFEM sol, $T = 0.75$ s

Dip Model

RMS error and estimated convergence rate over the region within the red box. The Q_1 FEM here is the one with accurate quadrature for mass and stiffness matrices

RMS error			
h	7.8125 m	3.90625 m	1.953125 m
Q_1 FEM	4.23e-1	1.49e-1	5.72e-2
HCFEM	2.79e-1	7.64e-2	1.94e-2
convergence rate			
h	7.8125 m	3.90625 m	1.953125 m
Q_1 FEM	-	1.51	1.38
HCFEM	-	1.87	1.97

Dome Model

central frequency $f_0 = 15$ Hz, $x_s = [3920\ \mathrm{m}, 3010\ \mathrm{m}]$

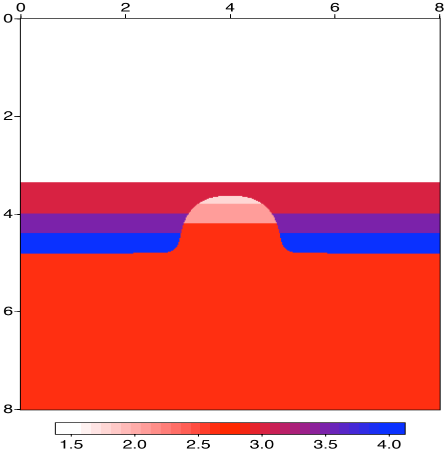


Figure : Velocity model

Dome Model

Difference between HCFEM solution on regular grid ($h = 7.8125$ m) and FEM solution on locally refined grid, same time stepping

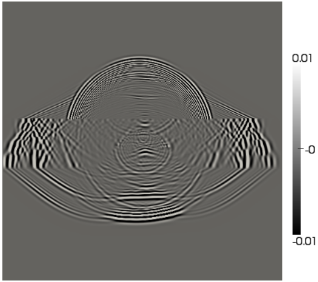


Figure : $T = 1.3$ s

Dome Model

Difference between FEM solution on regular grid ($h = 7.8125$ m) and FEM solution on locally refined grid, same time stepping

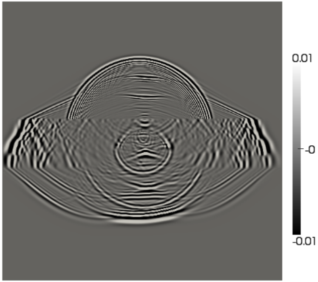


Figure : $T = 1.3$ s

Dome Model

Difference plots

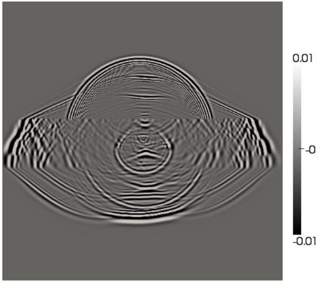


Figure : FEM sol, $T = 1.3$ s

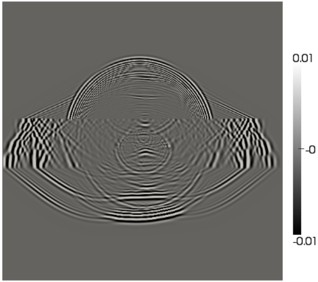


Figure : HCFEM sol, $T = 1.3$ s

Discussion

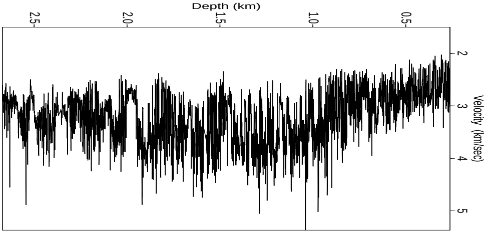
- For dip model: HCFEM and mass lumping roughly as accurate as as good as Q_1 FEM with accurate quadrature, when density contrasts are low (typical of seismic). Both seem to get rid of stairstep diffractions (more or less). More refined analysis shows HCFEM somewhat more accurate.
- For dome model: HCFEM closer to refined-grid FEM when same (very short) time steps taken

Discussion

"Texture": fine scale heterogeneity everywhere (reality?)

e.g., coefficient varies on scale 1 m ⇒ accurate regular FD simulations of 30 Hz waves may require 1 m grid though the corresponding wavelength is about 100 m at velocity of 3 km/s

HCFEM also 2nd order convergent for this type of heterogeneity, but no practical method to control accuracy of HC computation



v_p log from well in West Texas [thanks: Total E&P]

Conclusion

2D HCFEM based on bilinear (Q_1) elements on regular grid achieves second order convergence rate for static and dynamic acoustic interface problems.

Practical method of local grid refinement for HC accuracy control

Mass-lumped Q_1 Galerkin methods (both FEM and HCFEM) have same stencil and computational cost per time step as standard centered FD method, but much improved accuracy

For small density contrasts, standard Q_1 FEM with accurate quadrature and mass lumping is usable to working accuracy, and much cheaper than HCFEM.

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