Implementation of PML in the Depth-oriented Extended Forward Modeling

Lei Fu, William W. Symes

The Rice Inversion Project (TRIP)

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Outline

Introduction

Objective

Numerically solving the wave propagation problem in a bounded region.

Solution

Absorbing Boundary Conditions (ABCs)
Perfectly Marched Layers (PML)

History of PML method

- 1994, Berenger, **split-field PML** for use with Maxwell's equations.
- 1996, Gedney, **uniaxial PML** or **UPML**, described as an artificial anisotropic absorbing material.
- 2010, Marcus J. Grote, "Efficient PML for the wave equation". (fewer auxiliary variables)

Formulation

Denote $\hat{u}$ as the Laplace transform of u

$$\hat{u}(\mathbf{x}, s) = \int_0^\infty e^{st} u(\mathbf{x}, t) dt \tag{1}$$

Outside of the computational domain Ω, $\hat{u}$ then satisfies the Helmholtz equation,

$$s^2 \hat{u} = \frac{\partial}{\partial x_1}\left(c^2 \frac{\partial \hat{u}}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(c^2 \frac{\partial \hat{u}}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(c^2 \frac{\partial \hat{u}}{\partial x_3}\right) \tag{2}$$

Formulation

- Next, we introduce the coordinate transformation

$$x_i \mapsto \tilde{x}_i := x_i + \frac{1}{s}\int_0^{x_i} \zeta_i(x)dx, i = 1, 2, 3 \tag{3}$$

  where ζ_i is the damping profile.
- Partial differentiation:

$$\frac{\partial}{\partial \tilde{x}_i} = \frac{s}{s+\zeta_i}\frac{\partial}{\partial x_i} \tag{4}$$

Formulation

- If $\hat{u}$ satisfies the modeified Helmholtz equation, then

$$s^2\hat{u} = \frac{\partial}{\partial\tilde{x_1}}\left(c^2\frac{\partial\hat{u}}{\partial\tilde{x_1}}\right) + \frac{\partial}{\partial\tilde{x_2}}\left(c^2\frac{\partial\tilde{u}}{\partial\tilde{x_2}}\right) + \frac{\partial}{\partial\tilde{x_3}}\left(c^2\frac{\partial\hat{u}}{\partial\tilde{x_3}}\right) \quad (5)$$

- Then, by replacing partial derivatives $\frac{\partial}{\partial\tilde{x_i}}$ by $\frac{\partial}{\partial x_i}$, and transforming back to time domain, we get the PML modified wave equation:

$$u_{tt}+(\zeta_1+\zeta_2+\zeta_3)u_t+(\zeta_1\zeta_2+\zeta_2\zeta_3+\zeta_3\zeta_1)u = \nabla\cdot(c^2\nabla u)+\nabla\cdot\Phi-\zeta_1\zeta_2\zeta_3\psi$$

$$\Phi_t = \Gamma_1\Phi + c^2\Gamma_2\nabla u + c^2\Gamma_3\nabla\psi \quad (6)$$

$$\psi_t = u$$

Formulation

$$\Gamma_1 = \begin{bmatrix} -\zeta_1 & 0 & 0 \\ 0 & -\zeta_2 & 0 \\ 0 & 0 & -\zeta_3 \end{bmatrix}, \Gamma_2 = \begin{bmatrix} \zeta_2 + \zeta_3 - \zeta_1 & 0 & 0 \\ 0 & \zeta_3 + \zeta_1 - \zeta_2 & 0 \\ 0 & 0 & \zeta_1 + \zeta_2 - \zeta_3 \end{bmatrix},$$

$$\Gamma_3 = \begin{bmatrix} \zeta_2\zeta_3 & 0 & 0 \\ 0 & \zeta_3\zeta_1 & 0 \\ 0 & 0 & \zeta_1\zeta_2 \end{bmatrix}$$

Formulation

The choice of damping profiles $\zeta_i(x_i)$:

$$\zeta_i(x_i) = \bar{\zeta}_i \left[\frac{|x_i - a_i|}{L_i} - \frac{sin\left(\frac{2\pi|x_i - a_i|}{L_i}\right)}{2\pi} \right], a_i \leq |x_i| \leq a_i + L_i \tag{7}$$

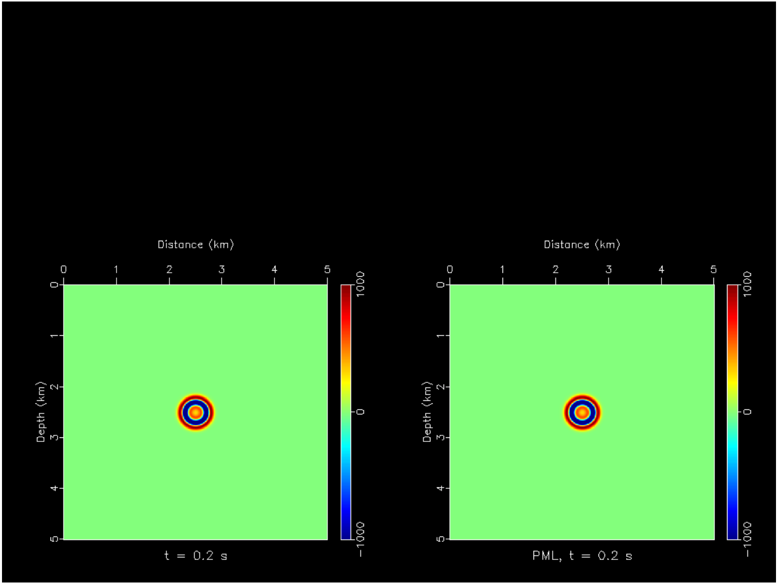
The relative reflection, R, is given by

$$\bar{\zeta}_i = \frac{c}{L_i} log\left(\frac{1}{R}\right) \tag{8}$$

Numerical experiment

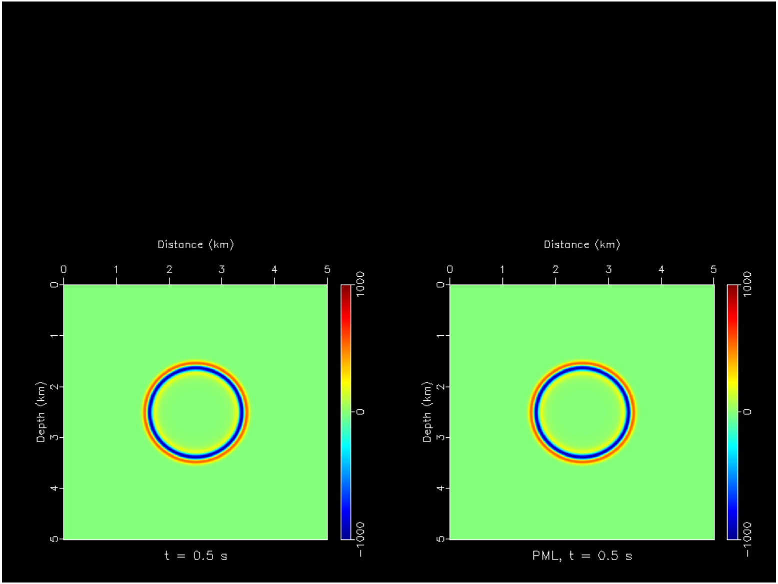
- Constant velocity $c = 3$
- Grid spacing: $\Delta x = \Delta z = 0.01$
- Domain size: 5×5
- Thickness of PMLs: $L = 0.5$
- Source: Ricker wavelet at the center.($f_{peak} = 10Hz$)

t=0.2 s



Distance (km)
0
1
2
3
4
5
Depth (km)
1000
0
−1000
t = 0.2 s
Distance (km)
Depth (km)
PML, t = 0.2 s

t=0.5 s



Distance (km)
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3
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5
Depth (km)
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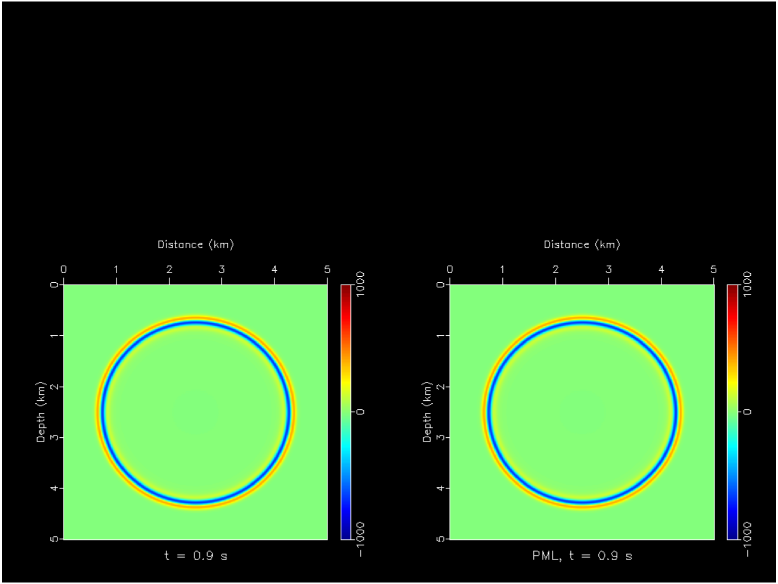
0
-1000
t = 0.5 s
Distance (km)
PML, t = 0.5 s

t=0.9 s


Distance (km)
0
1
2
3
4
5
Depth (km)
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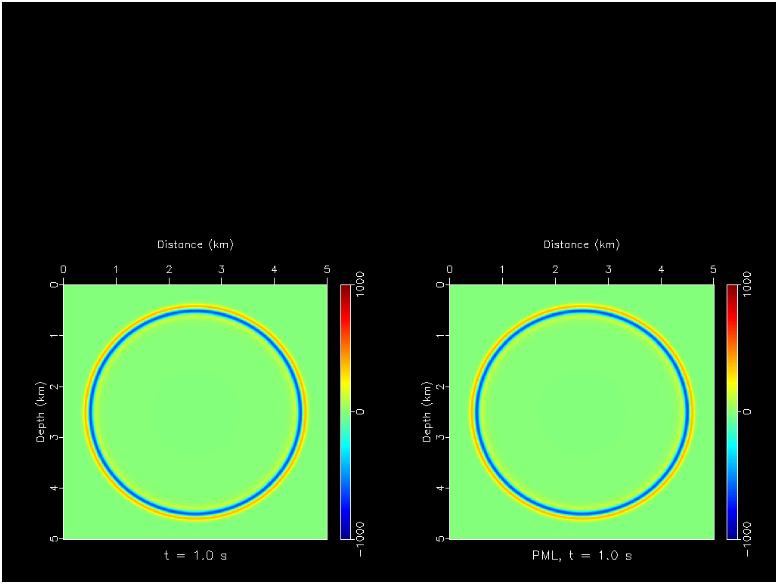
−1000
t = 0.9 s
Distance (km)
Depth (km)
PML, t = 0.9 s

t=1.0 s



Distance (km)
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1
2
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4
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Depth (km)
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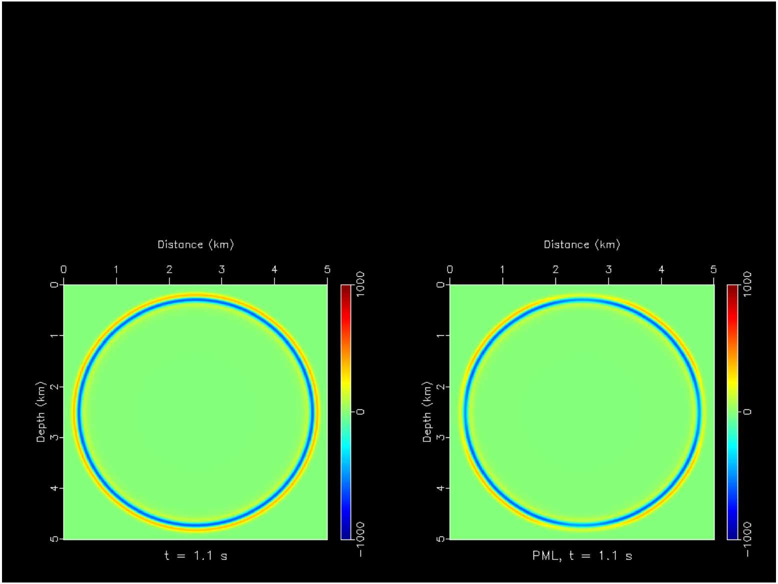
-1000
t = 1.0 s
Distance (km)
PML, t = 1.0 s

t=1.1 s



Distance (km)
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1
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Depth (km)
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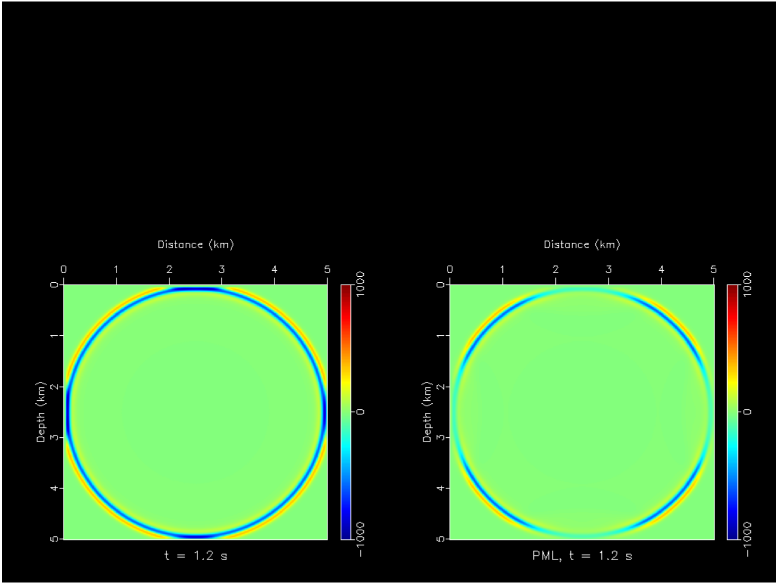
t = 1.1 s
Distance (km)
PML, t = 1.1 s

t=1.2 s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0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
1000
0
−1000
t = 1.2 s
Distance (km)
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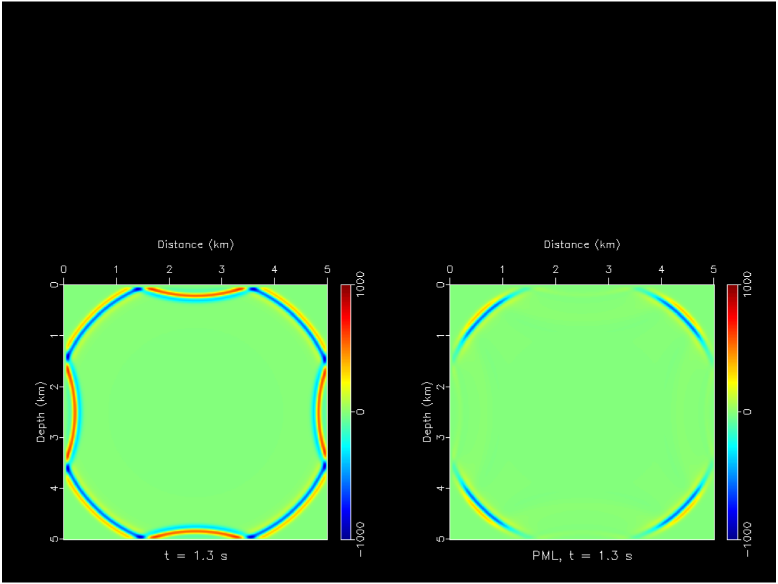
Depth (km)
0 1 2 3 4 5
1000
0
−1000
PML, t = 1.2 s

t=1.3 s


Distance (km)
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Depth (km)
0 1 2 3 4 5
1000
0
−1000
t = 1.3 s
Distance (km)
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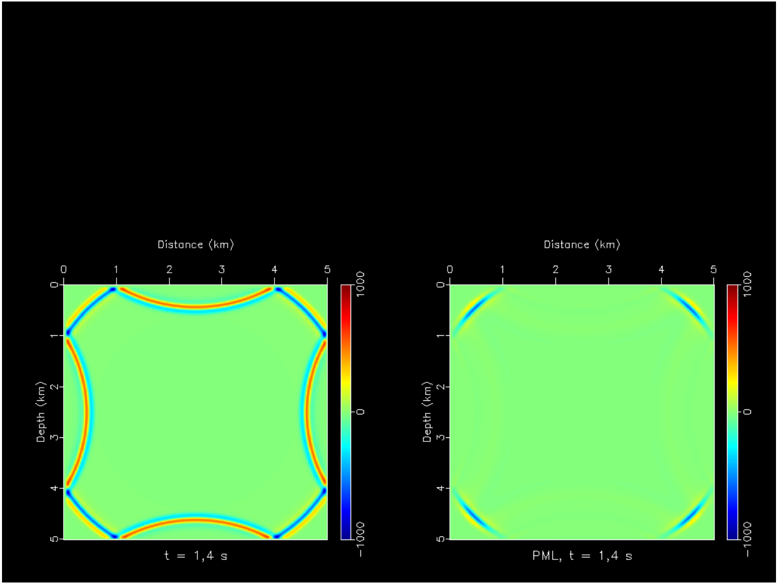
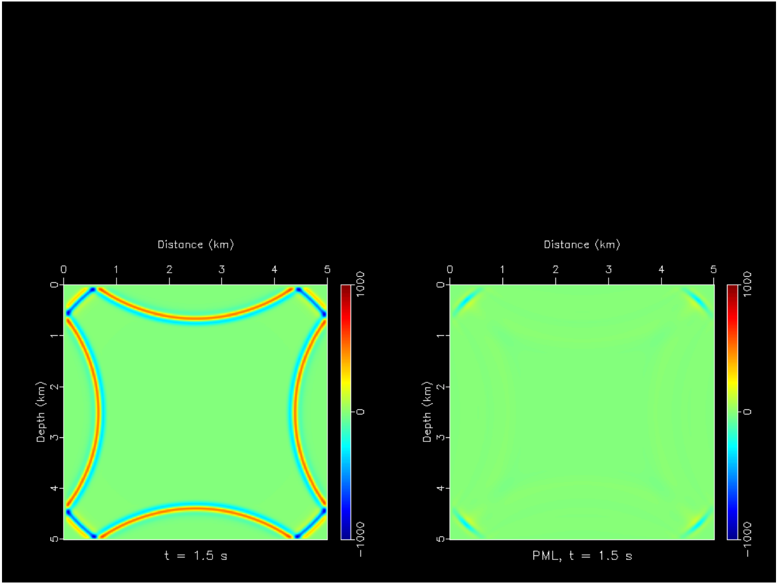
Depth (km)
0 1 2 3 4 5
1000
0
−1000
PML, t = 1.3 s

t=1.4 s

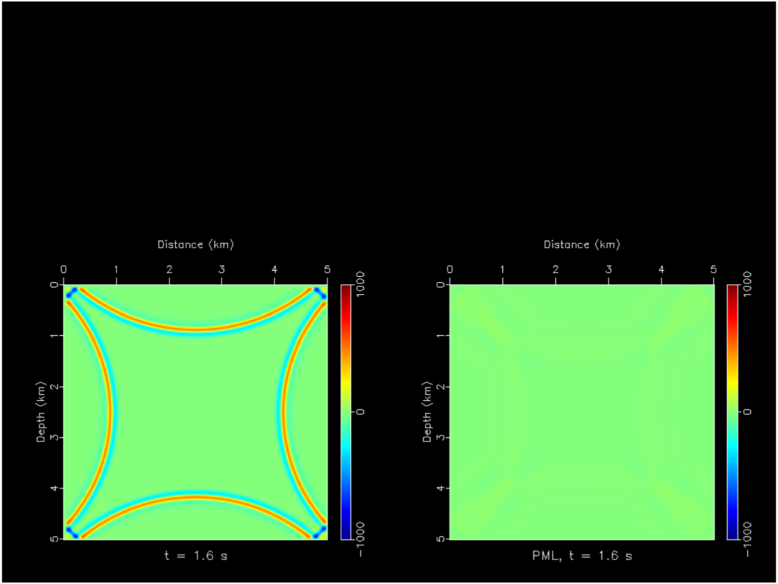


Distance (km)
0
1
2
3
4
5
Depth (km)
1000
0
-1000
t = 1,4 s
Distance (km)
PML, t = 1,4 s

t=1.5 s

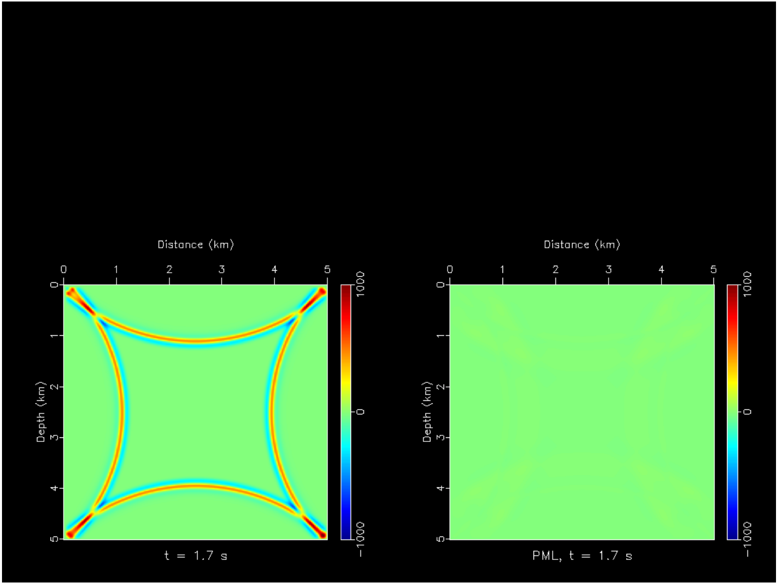

Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
1000
0
−1000
t = 1.5 s
Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
1000
0
−1000
PML, t = 1.5 s

t=1.6 s



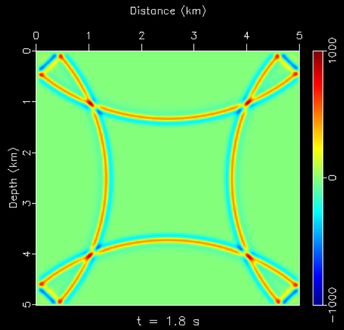
Distance (km)
0
1
2
3
4
5
Depth (km)
0
1
2
3
4
5
1000
0
-1000
t = 1.6 s
Distance (km)
0
1
2
3
4
5
Depth (km)
0
1
2
3
4
5
1000
0
-1000
PML, t = 1.6 s

t=1.7 s



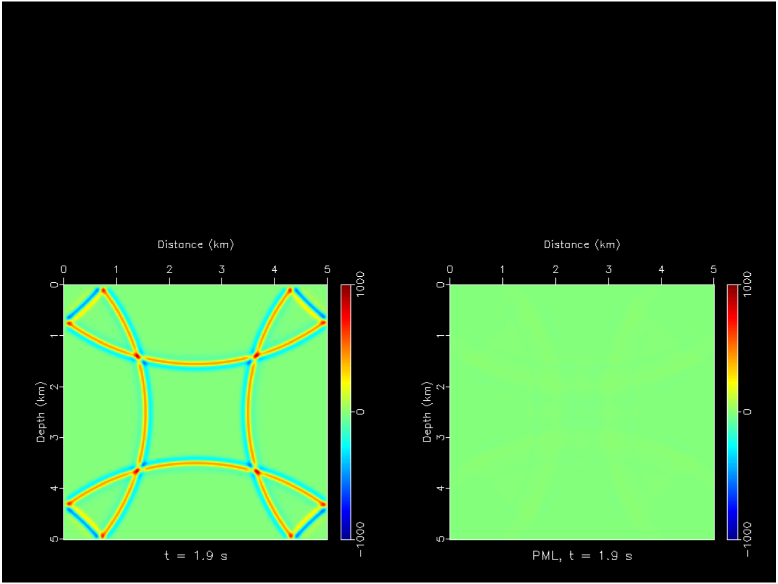
Distance (km)
0
1
2
3
4
5
Depth (km)
1000
0
-1000
t = 1.7 s
Distance (km)
Depth (km)
PML, t = 1.7 s

t=1.8 s

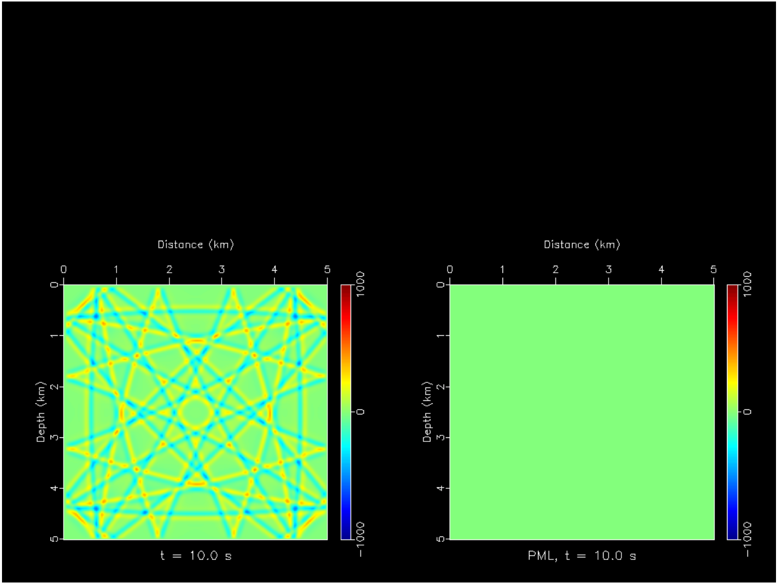


Distance (km)
0
1
2
3
4
5
Depth (km)
1000
0
−1000
t = 1.8 s
Distance (km)
Depth (km)
PML, t = 1.8 s

t=1.9 s

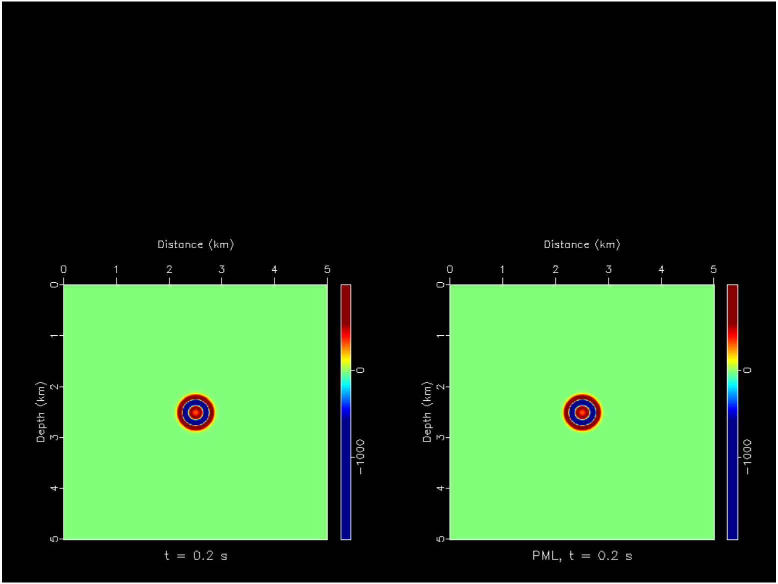

Distance (km)
0
1
2
3
4
5
Depth (km)
1000
0
−1000
t = 1.9 s
Distance (km)
PML, t = 1.9 s

t=10.0 s

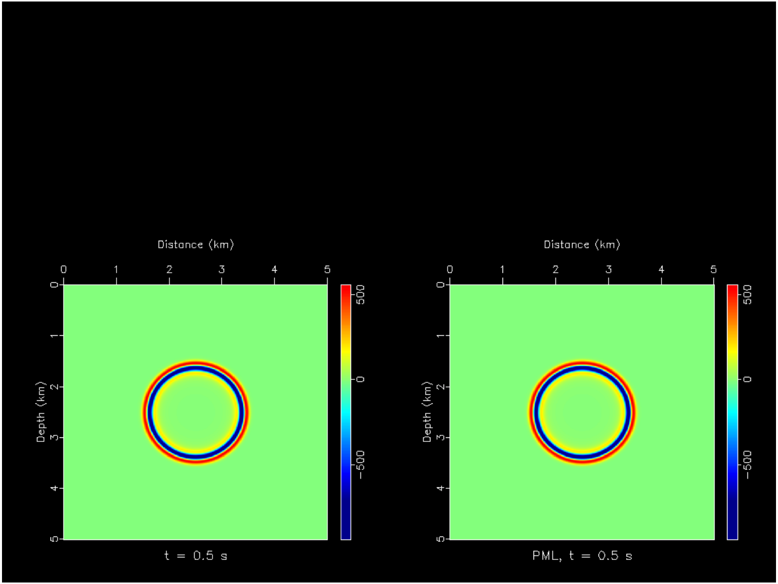


Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
1000
0
-1000
t = 10.0 s
Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
1000
0
-1000
PML, t = 10.0 s

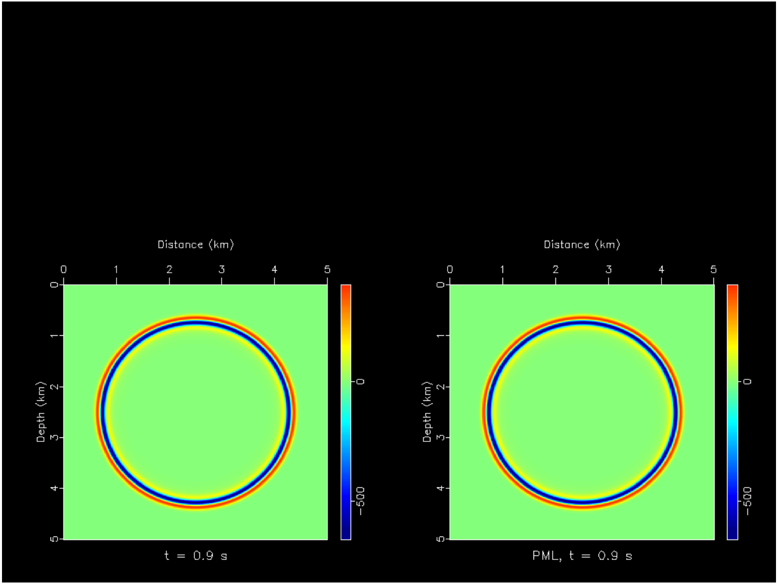
t=0.2 s


Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
0
−1000
t = 0.2 s
Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
0
−1000
PML, t = 0.2 s

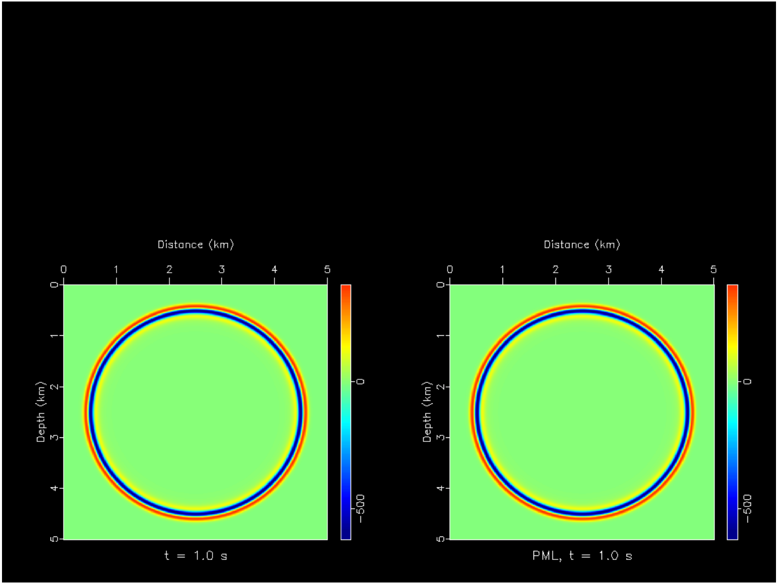
t=0.5 s


Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
500
0
−500
t = 0.5 s
Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
500
0
−500
PML, t = 0.5 s

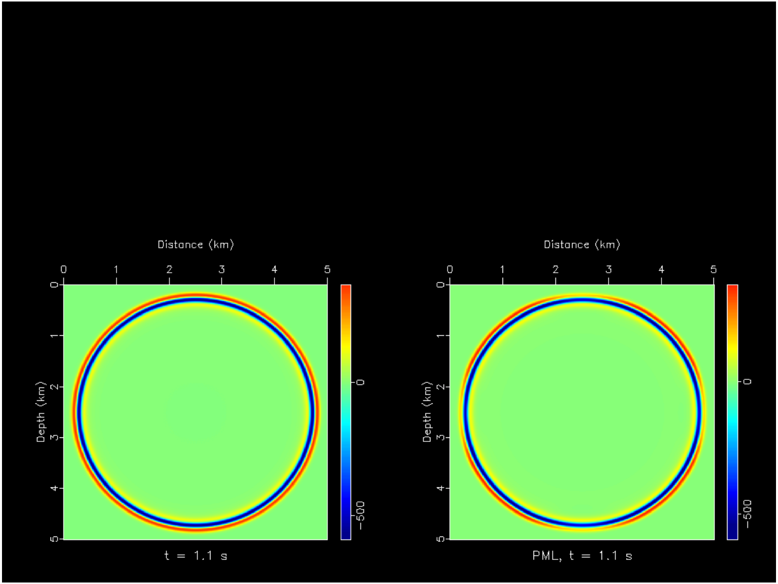
t=0.9 s


Distance (km)
0
1
2
3
4
5
Depth (km)
0
1
2
3
4
5
0
-500
t = 0.9 s
Distance (km)
0
1
2
3
4
5
Depth (km)
0
1
2
3
4
5
0
-500
PML, t = 0.9 s

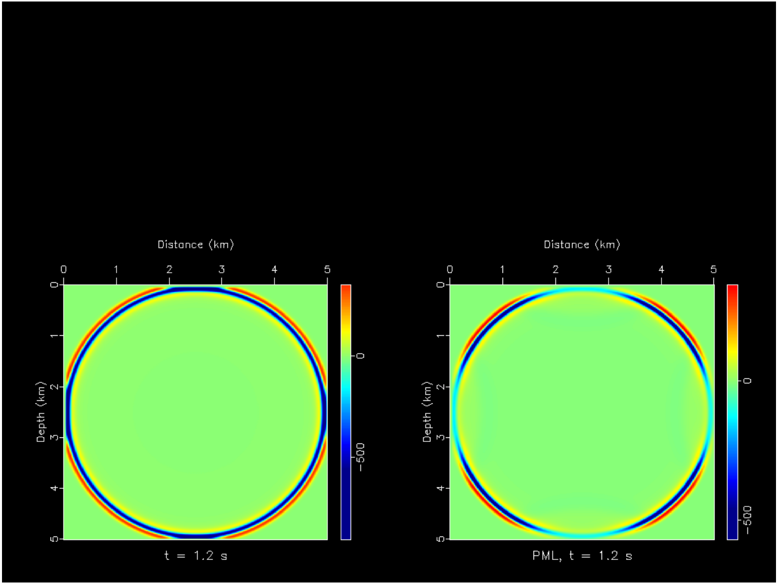
t=1.0 s


Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
0
−500
t = 1.0 s
Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
0
−500
PML, t = 1.0 s

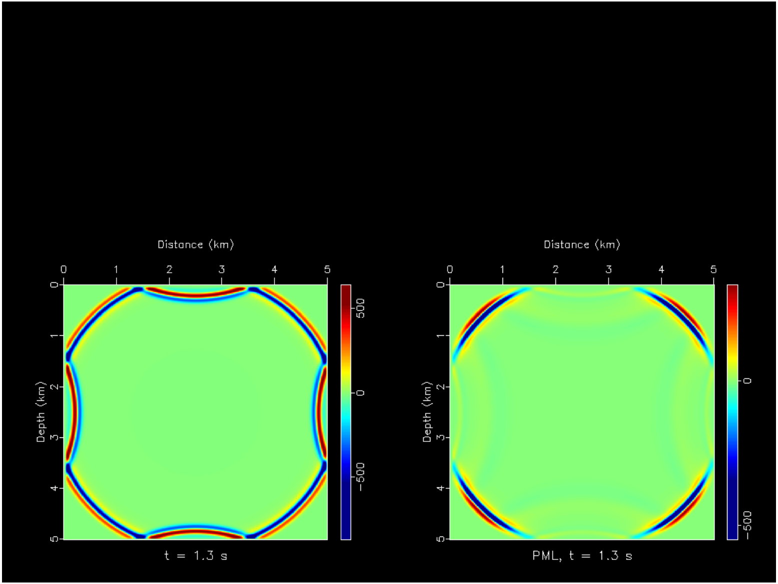
t=1.1 s


Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
0
−500
t = 1.1 s
Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
0
−500
PML, t = 1.1 s

t=1.2 s

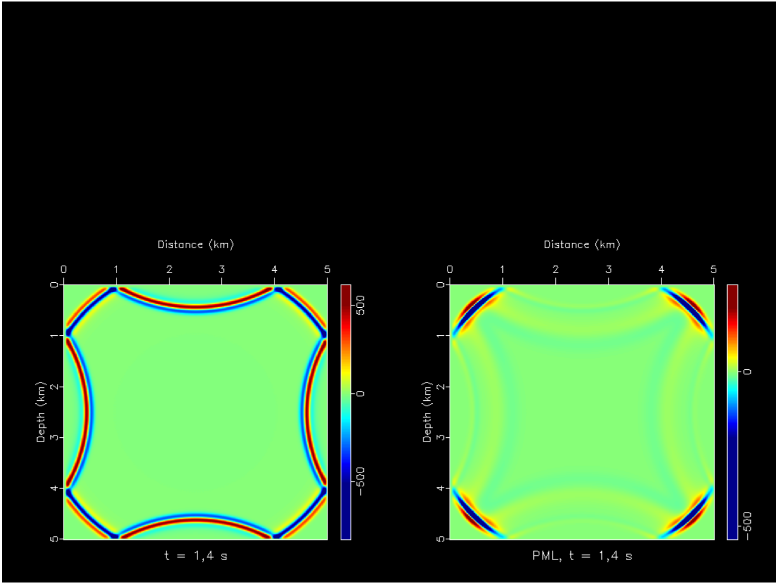

Distance (km)
0
1
2
3
4
5
Depth (km)
0
-500
t = 1.2 s
Distance (km)
Depth (km)
0
-500
PML, t = 1.2 s

t=1.3 s

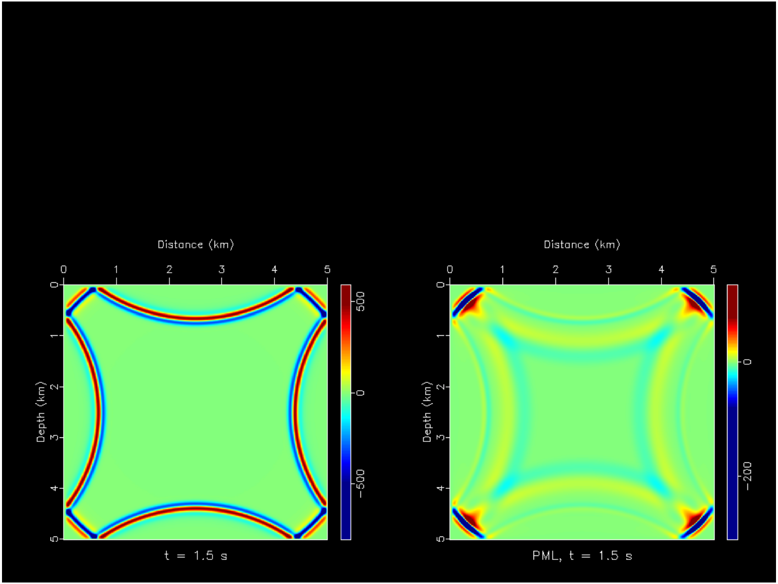


Distance (km)
0
1
2
3
4
5
0
1
2
3
4
5
Depth (km)
500
0
−500
t = 1.3 s
Distance (km)
0
1
2
3
4
5
0
1
2
3
4
5
Depth (km)
0
−500
PML, t = 1.3 s

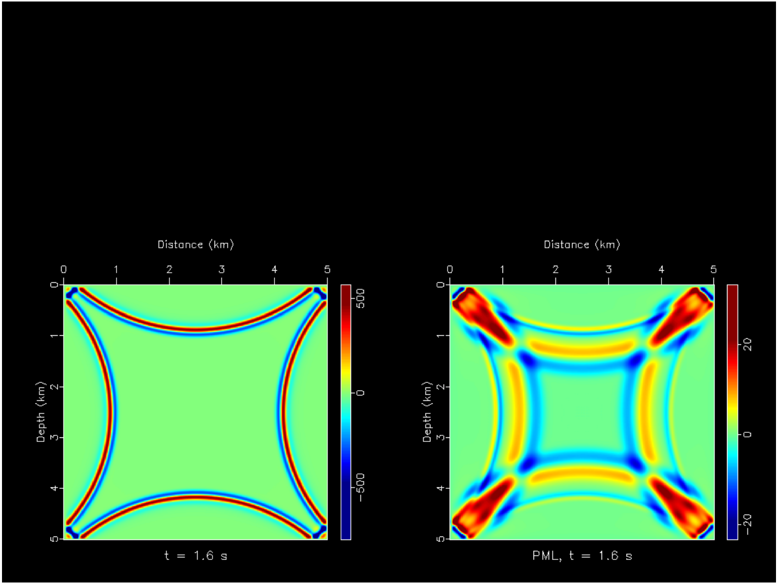
t=1.4 s


Distance (km)
0
1
2
3
4
5
Depth (km)
500
0
−500
t = 1,4 s
Distance (km)
PML, t = 1,4 s

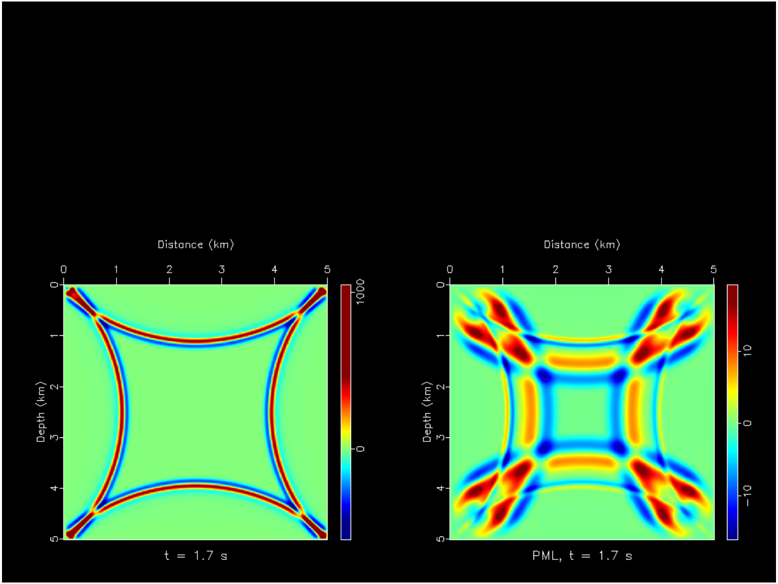
t=1.5 s


Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
500
0
−500
t = 1.5 s
Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
0
−200
PML, t = 1.5 s

t=1.6 s


Distance (km)
0
1
2
3
4
5
Depth (km)
500
0
−500
t = 1.6 s
Distance (km)
Depth (km)
20
0
−20
PML, t = 1.6 s

t=1.7 s


Distance (km)
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Depth (km)
0 1 2 3 4 5
1000
0
t = 1.7 s
Distance (km)
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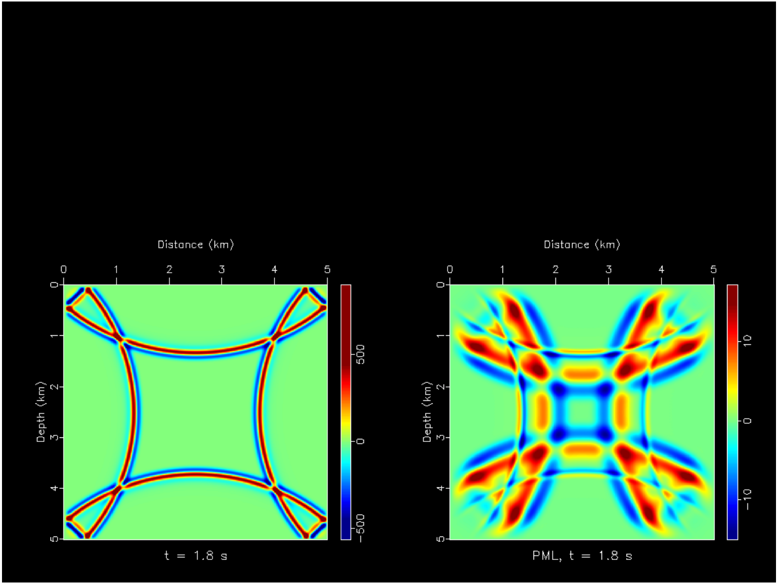
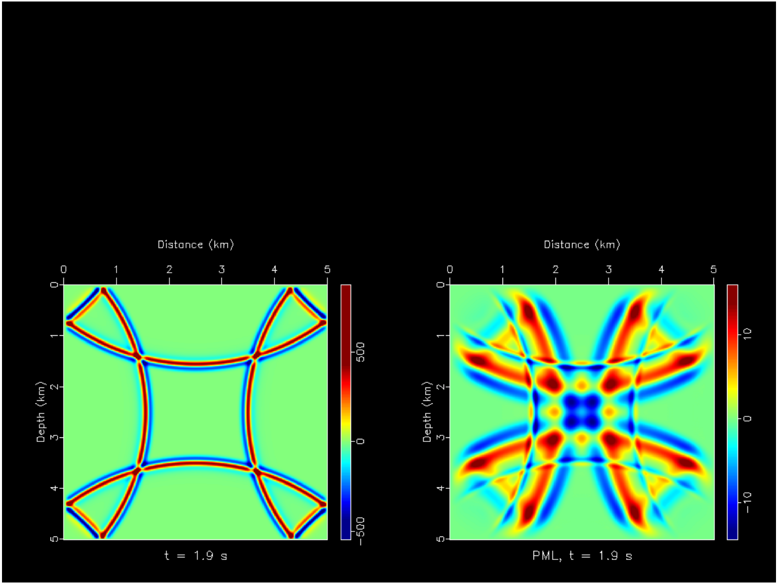
Depth (km)
0 1 2 3 4 5
10
0
−10
PML, t = 1.7 s

t=1.8 s



Distance (km)
0
1
2
3
4
5
0
1
2
3
4
5
Depth (km)
500
0
−500
t = 1.8 s
Distance (km)
0
1
2
3
4
5
0
1
2
3
4
5
Depth (km)
10
0
−10
PML, t = 1.8 s

t=1.9 s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2
3
4
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Depth (km)
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0
−500
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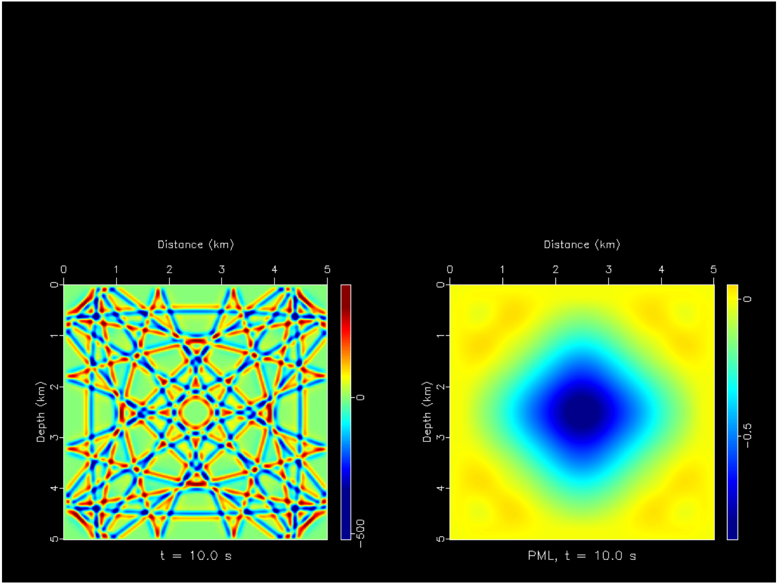
Distance (km)
PML, t = 1.9 s
10
−10

t=10.0 s


Distance (km)
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Depth (km)
0 1 2 3 4 5
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−500
t = 10.0 s
Distance (km)
0 1 2 3 4 5
Depth (km)
0 1 2 3 4 5
0
−0.5
PML, t = 10.0 s

Summary

- Implement PML in depth-oriented extended forward modeling.
- The PML does suppress the reflections exfficiently.
- Simplicity and the fewer auxiliary variables.
- Furture work
  - R vs. $\bar{\zeta}$, Δx_i, θ, ...
  - Choice of $\zeta(x)$ function
  - ABCs vs. PML
  - $2D \rightarrow 3D$

Acknowledgement

Thanks to

DR. WILLIAM SYMES,

TRIP SPONSORS AND MEMBERS