**Education**

**Rice University**
Ph.D. in Geophysics, Earth Science

**China University of Petroleum (East China)**
B.S. in Exploration Geophysics

**Research**

- An approximate inverse to extended born modeling operator
- Elastic wave modeling based on the decoupled equation
- Target-oriented wave-equation least-squares migration
An Approximate Inverse to Extended Born Modeling Operator

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The Rice Inversion Project
Annual Review Meeting

April 19, 2013
Outline

1. Set Up of Background
2. Derivation from Ten Kroode’s New Operator Pair
3. Numerical Test and Preliminary Result
4. Summary and Future Plans
Inverse Problem

The Usual Set-up

- \( \mathcal{M} = \) a set of models
- \( \mathcal{D} = \) a Hilbert space of (potential) data
- Forward Map \( \mathcal{F} : \mathcal{M} \rightarrow \mathcal{D} \)

Inverse Problem:

Given \( d \in \mathcal{D} \), find \( m \in \mathcal{M} \) so that \( \mathcal{F}[m] \simeq d \)

Wave equation:

\[
\frac{1}{c^2} \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u(x, t) = \delta(x - s) \delta(t - t_0)
\]
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Wave equation:

\[
\frac{1}{\rho(x) v^2(x)} \frac{\partial^2 u}{\partial t^2}(x, t) - \nabla \cdot \frac{1}{\rho(x)} \nabla u(x, t) = f(x_s, t)
\]
Born Approximation (Linearized Inverse Problem)

Given smooth background velocity $v(x, y, z) = v(x)$, seismic data $d(x_r, t; x_s)$, find oscillatory reflectivity $r(x) = \frac{\delta v(x)}{v(x)}$ to fit the data:

$$F[v] r \approx d$$

Born Modeling (Acoustic Forward Operator $F[v]$)

$$\left( \frac{1}{v^2} - \nabla^2 \right) G = \delta(t) \delta(x - x_s); \quad \left( \frac{1}{v^2} - \nabla^2 \right) \delta u = \frac{2r}{v^2} G$$

$$F[v] r(x_r, t; x_s) = \delta u(x_r, t; x_s)$$

Assumption: Single scattering at points of discontinuity of impedance in the subsurface (No multiple scattering!)
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\]

\[
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\]

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Migration is an approximate solution of this linearized inverse problem
Adjoint of Born Modeling Operator

**Adjoint of Born Modeling = Imaging Operator**

- Migration operator (producing image) is adjoint or transpose of modeling operator (Lailly, Tarantola, Claerbout (80’s)).
- Migration operator can **position reflectors correctly** but with possibly incorrect amplitudes.
  - Due to the symmetry of wave-propagation with respect to time-reversal, migrating with the adjoint operator treats event kinematics correctly, and produces structurally correct images of the subsurface. It is robust to the presence of noise, or missing or inconsistent data.
  - The migration with the adjoint doesn’t treat seismic amplitudes correctly. It focus on kinematics rather than amplitudes, amplitude terms are usually completely ignored, or artificially constructed so that $F^*F \approx I$.
- True amplitude migration is (pseudo) inverse.
Born Modeling

\[ F[v]r(x_r, t; x_s) = \frac{\partial^2}{\partial t^2} \int dx \int d\tau \frac{2r(x)}{v^2(x)} G(x, t - \tau; x_r) G(x, \tau; x_s) \]
Born Modeling

\[ F[v] r(x_r, t; x_s) = \frac{\partial^2}{\partial t^2} \int dx \int d\tau \frac{2r(x)}{v^2(x)} G(x, t - \tau; x_r) G(x, \tau; x_s) \]

The adjoint of \( F \) is a prestack migration operator. It is defined by

\[ \int dx_r dx_s dt (Fr)(x_r, t; x_s) d(x_r, t; x_s) = \int dx r(x)(F^* d)(x) \]
Born Modeling

\[ F[v]r(x_r, t; x_s) = \frac{\partial^2}{\partial t^2} \int dx \int d\tau \frac{2r(x)}{v^2(x)} G(x, t - \tau; x_r) G(x, \tau; x_s) \]

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\[ \int dx_r dx_s dt (Fr)(x_r, t; x_s) d(x_r, t; x_s) = \int dxr(x) (F^* d)(x) \]

Do the integral by parts shows that

\[ F^* d(x) = \frac{2}{v^2(x)} \int dx_s dx_r dt d\tau G(x, t - \tau; x_r) \frac{\partial^2 G(x, \tau; x_s)}{\partial \tau^2} d(x_r, t; x_s) \]
Least Squares Inversion:
Given $d$, find $m$ to minimize

$$J_{LS}[m] = \| F[m] - d \|^2 [ + \text{Regularizing terms} ]$$

Due to the local minima problem, extended model was introduced.

- Definition: the modeling of wavefields is extended to nonphysical models depending on redundant parameters.
- Extended Model $\tilde{F} : \tilde{M} \rightarrow D$ where $\tilde{M}$ is a larger model space $=$ models depending on $x$ and $h$.
Extended Modeling (subsurface common offset):
In integral representation of $F[v]r$, permit $r$ to depend on (half) offset $h$.

When we use it to solve the inverse problem, it needs many iterations. So we need an approximate inverse to do the preconditioning.
\[ \tilde{K}i = \frac{1}{2\pi} \int dxdh \omega e^{-i\omega t} G(x_r, x + h, \omega) \frac{\partial i(x, h)}{\partial z} G(x - h, x_s, \omega) \]

\[ \tilde{I}d = \frac{32}{\pi v^2(x)} \int dx_r dx_s d\omega (-i\omega) \frac{\partial G^*(x + h, x_r, \omega)}{\partial z_r} d(x_r, x_s, \omega) \frac{\partial G^*(x_s, x - h, \omega)}{\partial z_s} \]

**Result 3 (Fons ten Kroode, 2012)**

\( \tilde{K} \) and \( \tilde{I} \) are the Fourier integral operators of order 1 and \(-1\) respectively. There exist order zero pseudo-differential operators \( \Psi_X \) and \( \Psi_Y \), such that

\[ \tilde{I} \circ \tilde{K} = \Psi_X \]

\[ \tilde{K} \circ \tilde{I} = \Psi_Y \]

The operator \( \Psi_X \) acts as identity on focused space-shift-extended images. The operators \( \Psi_Y \) acts as identity on primary reflection data.

(http://iopscience.iop.org/0266-5611/28/11/115013)
\[(\tilde{K}i)(x_s, x_r, t) = \frac{1}{2\pi} \int dx dh d\omega e^{-i\omega t} G(x_r, x + h, \omega) \frac{\partial i(x, h)}{\partial z} G(x - h, x_s, \omega)\]

\[= \int dx dh d\tau G(x + h, t - \tau; x_r) \frac{\partial i(x, h)}{\partial z} G(x - h, \tau; x_s)\]

\[\bar{F}[v] r = \frac{\partial^2}{\partial t^2} \int dx dh d\tau G(x + h, t - \tau; x_r) \frac{2r(x, h)}{v^2(x)} G(x - h, \tau; x_s)\]

If we assume \(i(x, h) = \frac{2r(x, h)}{v^2(x)}\), then we have

\[\bar{F} \circ \frac{\partial}{\partial z} \circ \frac{2r}{v^2} = \frac{\partial^2}{\partial t^2} \circ \tilde{K} \circ i\]
\[ 
\tilde{l} = \frac{32}{\pi v^2(x)} \int dx_r dx_s d\omega (-i\omega) \frac{\partial G^*(x + h, x_r, \omega)}{\partial z_r} d(x_r, x_s, \omega) \frac{\partial G^*(x_s, x - h, \omega)}{\partial z_s} 
\]

\[ = \frac{64}{v^2(x)} \frac{\partial^2}{\partial z_r \partial z_s} \int dx_s dx_r dt d\tau \frac{\partial G(x - h, \tau; x_s)}{\partial \tau} G(x + h, t - \tau; x_r) d(x_s, x_r, t) 
\]

\[ \tilde{F}^* d = \frac{2}{v^2(x)} \int dx_s dx_r dt d\tau \frac{\partial^2 G}{\partial \tau^2} (x - h, \tau; x_s) \partial^2 G(x + h, t - \tau; x_r) d(x_r, t; x_s) 
\]

\[ = -\frac{2}{v^2(x)} \int dx_s dx_r dt d\tau \frac{\partial G(x - h, \tau; x_s)}{\partial \tau} G(x + h, t - \tau; x_r) \frac{\partial d(x_r, t; x_s)}{\partial t} 
\]

\[ - \frac{64}{v^2(x)} \frac{\partial^2}{\partial z_s \partial z_r} \circ \tilde{F}^* \circ \int_t = \tilde{l} \]
If we apply $\tilde{K}$ on $\tilde{I}$, the result is in terms of a ratio of some slowness $(s, s_+, s_-)$ at different places, where $s$, $s_+$, $s_-$ are the slowness at the points $x, x + h, x - h$ respectively. So $s_+, s_-$ are the slowness values at the ends of two different rays of geometric optics.
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- If the rays meet at the same place, i.e. the data focuses. Then $s_+ = s_- = s$, the result will equal to $I$. 
If we apply $\tilde{K}$ on $\tilde{l}$, the result is in terms of a ratio of some slowness $(s, s_+, s_-)$ at different places, where $s, s_+, s_-$ are the slowness at the points $x, x + h, x - h$ respectively. So $s_+, s_-$ are the slowness values at the ends of two different rays of geometric optics.

- If the rays meet at the same place, i.e. the data focuses. Then $s_+ = s_- = s$, the result will equal to $l$.

- If the rays meet at different places, the result will be the ratio of slownesses. It will be bounded by the biggest slowness divided by the smallest slowness, which means identity scaled by a very small range of numbers. Even if the result isn’t identity, it’s not big.
If we apply $\tilde{K}$ on $\tilde{I}$, the result is in terms of a ratio of some slowness $(s, s_+, s_-)$ at different places, where $s, s_+, s_-$ are the slowness at the points $x, x + h, x - h$ respectively. So $s_+, s_-$ are the slowness values at the ends of two different rays of geometric optics.

- If the rays meet at the same place, i.e. the data focuses. Then $s_+ = s_- = s$, the result will equal to $I$.
- If the rays meet at different places, the result will be the ratio of slownesses. It will be bounded by the biggest slowness divided by the smallest slowness, which means identity scaled by a very small range of numbers. Even if the result isn’t identity, it’s not big.

$$\tilde{K} \circ \tilde{I} \sim I$$
From the analysis above, we can get the following results:

\[
\tilde{F} \circ \frac{\partial}{\partial z} \circ \frac{2r}{v^2} = \frac{\partial^2}{\partial t^2} \circ \tilde{K} \circ i
\]

\[
-\frac{64}{v^2(x)} \frac{\partial^2}{\partial z_s \partial z_r} \circ \tilde{F}^* \circ \int_t = \tilde{l}
\]

\[
\tilde{K} \circ \tilde{l} \simeq l
\]
From the analysis above, we can get the following results:

\[
\tilde{F} \circ \frac{\partial}{\partial z} \circ \frac{2r}{\nu^2} = \frac{\partial^2}{\partial t^2} \circ \tilde{K} \circ i
\]

\[
- \frac{64}{\nu^2(x)} \frac{\partial^2}{\partial z_s \partial z_r} \circ F^* \circ \int_t = \tilde{l}
\]

\[
\tilde{K} \circ \tilde{l} \simeq l
\]

\[
\tilde{F} \circ \frac{-64}{\nu^2} \frac{\partial^2}{\partial z_s \partial z_r} \circ F^* \circ \int \int \int_t \simeq l
\]
From the analysis above, we can get the following results:

\[
\begin{align*}
\bar{F} \circ \frac{\partial}{\partial z} \circ \frac{2r}{v^2} &= \frac{\partial^2}{\partial t^2} \circ \bar{K} \circ i \\
-\frac{64}{v^2(x)} \frac{\partial^2}{\partial z_s \partial z_r} \circ \bar{F}^* \circ \int_t = \bar{l} \\
\bar{K} \circ \bar{l} &\simeq l
\end{align*}
\]

Now we can get the approximate inverse

\[
\bar{F}^{-1} \simeq \frac{\partial}{\partial z} \circ \frac{-64}{v^2} \frac{\partial^2}{\partial z_s \partial z_r} \circ \bar{F}^* \circ \int \int \int_t
\]
\[ r(x, h) = \bar{F}^{-1} d \simeq \frac{\partial}{\partial z} \circ \frac{-64}{v^2} \frac{\partial^2}{\partial z_s \partial z_r} \circ \bar{F}^* \circ \int \int \int_t d(x_r, t; x_s) \]

1. **Data Preparation:** Since we need to take derivative to \( z_s, z_r \), we need four data \((d(x_{s1}, x_{r1}), d(x_{s1}, x_{r2}), d(x_{s2}, x_{r1}), d(x_{s2}, x_{r2}))\)

   \[ s_1 \star \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup r_1 \]

   \[ s_2 \star \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup r_2 \]

2. **Integrate the data:** Just do the sums

3. **Do prestack depth migration:** Use normal RTM code

4. **Derivative:** Do differential three times

\[
\frac{\partial^2 i(x, h; x_s, x_r)}{\partial z_s \partial z_r} = \frac{i(x_{s1}, x_{r1}) + i(x_{s2}, x_{r2}) - i(x_{s1}, x_{r2}) - i(x_{s2}, x_{r1})}{\Delta z^2}
\]
Preliminary Result

[Graphs showing Migration Result and Approximate Inverse Result]

Thanks for Yujin's RTM code
If it’s true
- Approximate Inverse
- Get the amplitude right
- It’s not expensive

Possible Problems
- Ten Kroode(2012) → 3D How about 2D?
  - Green Function: 3D form $modified$ 2D form
    (Creation of GRT inversion formula, William Symes, 1998)
- Blow up the low frequency and kill the high frequency.
  (A lot of sums and differences)
- Numerical Errors
Future Plans

- Go through the proof, make any necessary modification for 2D.
- Implement the operator in 2D and 3D
- Replace four migrations with respect to one-way operator
  
  replace $\frac{\partial}{\partial z_r}$ with the one way operator
  
  according to the reciprocal principle, same to $\frac{\partial}{\partial z_s}$

- Apply this operator as a preconditioner
Dr. William Symes
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