

Harmonic Coordinate Finite Element Method for Acoustic Waves

Xin Wang

The Rice Inversion Project

Mar. 30th, 2012

Outline

Introduction

Harmonic Coordinates FEM

Mass Lumping

Numerical Results

Conclusion

Motivation

Scalar acoustic wave equation

$$\frac{1}{\kappa} \frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho} \nabla u = f$$

with appropriate boundary, initial conditions

Typical setting in seismic applications:

- ▶ heterogeneous κ, ρ with low contrast $O(1)$
- ▶ model data κ, ρ defined on regular Cartesian grids
- ▶ large scale \Rightarrow waves propagate $O(10^2)$ wavelengths; solutions for many different f
- ▶ f smooth in time

Motivation

For piecewise constant κ, ρ with interfaces

- ▶ FDM: first order interface error, time shift, incorrect arrival time, no obvious way to fix (Brown 84, Symes & Vdovina 09)
- ▶ Accuracy of standard FEM (eg specFEM3D) relies on adaptive, interface fitting meshes
- ▶ Exception: FDM derived from mass-lumped FEM on regular grid for **constant density acoustics** has 2nd order convergence even with interfaces (Symes & Terentyev 2009)
- ▶ Aim of this project: modify FD/FEM with full accuracy for variable density acoustics

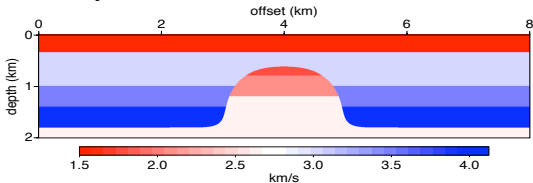


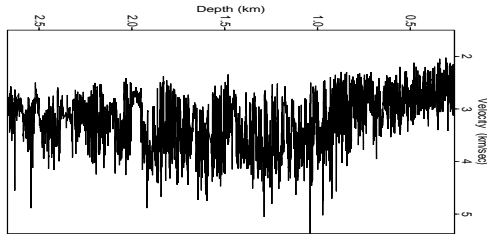
Figure: Velocity model

Motivation

For highly oscillating coefficient (rough medium)

e.g., coefficient varies on scale 1 m \Rightarrow accurate regular FD simulations of 30 Hz waves may require 1 m grid though the corresponding wavelength is about 100 m at velocity of 3 km/s

Can we create an accurate FEM on coarse (wavelength) grid?



A log of compressional wave velocity from a well in West Texas, supplied to TRIP by Total E&P USA and used by permission.

Motivation

Ideal numerical method:

- ▶ high order
- ▶ no special meshing, i.e., on regular grids
- ▶ works for problems with heterogeneous media

Owhadi and Zhang 2007: 2D harmonic coordinate finite element method, sub-optimal convergence on regular grids

Binford 11: showed using the true support recovers the optimal order of convergence on triangular meshes for 2D static interface problems, and mesh of diameter h^2 is used to approximate the harmonic coordinates

Result of project: modification of Owhadi and Zhang's method with full 2nd order convergence for variable density acoustic WE

Outline

Introduction

Harmonic Coordinates FEM

Mass Lumping

Numerical Results

Conclusion

Harmonic Coordinates

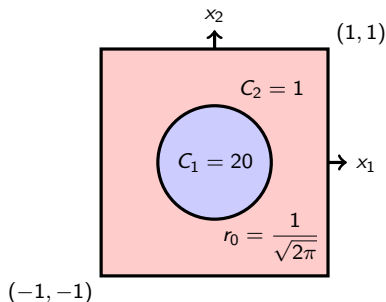
Global C -harmonic coordinates \mathbf{F} in 2D, its components $F_1(x_1, x_2), F_2(x_1, x_2)$ are weak sols of

$$\nabla \cdot C(x)\nabla F_i = 0 \quad \text{in } \Omega$$

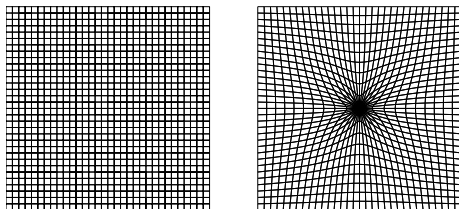
$$F_i = x_i \quad \text{on } \partial\Omega$$

$\mathbf{F} : \Omega \rightarrow \Omega$ C -harmonic coordinates

e.g.,



Harmonic Coordinates



- ▶ physical regular grid $(x_1, x_2) = (jh_x, kh_y)$ (left),
- ▶ harmonic grid $(F_1, F_2) = (F_1(jh_x, kh_y), F_2(jh_x, kh_y))$ (right)

Harmonic Coordinate FEM

Workflow of HCFEM:

- 1 prepare a regular mesh on physical domain, \mathcal{T}^H ;
- 2 approximate \mathbf{F} on a fine mesh \mathcal{T}^h by \mathbf{F}_h
- 3 construct the harmonic triangulation $\tilde{\mathcal{T}}^H = \mathbf{F}_h(\mathcal{T}^H)$;
- 4 construct the HCFE space
 $S^H = \text{span}\{\tilde{\phi}_i^H \circ \mathbf{F}_h : i = 0, \dots, N^h\}$, where
 $\tilde{S}^H = \text{span}\{\tilde{\phi}_i^H : i = 0, \dots, N^h\}$ is Q_1 FEM space on
harmonic grid $\tilde{\mathcal{T}}^H$;
- 5 solve the original problem by Galerkin method on S^H .

\Rightarrow solve n (≤ 3) harmonic problems to obtain harmonic coordinates \mathbf{F} ; resulting stiffness and mass matrices of HCFEM have the same sparsity pattern as in standard FEM

Harmonic Coordinate FEM

Galerkin approximation $u^h \in S^h$ (harmonic coordinate FE space) to u (weak solution of $-\nabla \cdot C \nabla u = f$) has optimal error:

$$\iint_{\Omega} |\nabla u - \nabla u^h|^2 dx = O(h^2)$$

Why this is so: see WWS talk in 2008 TRIP review meeting

Harmonic Coordinate FEM

For wave equation:

$$\frac{1}{\kappa} \frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho} \nabla u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$

Dirichlet BC, $u \equiv 0, t < 0, f \in L^2([0, T], L^2(\Omega))$

Assume f is finite bandwidth \Rightarrow Galerkin approximation u^h in HCFE space S^h on mesh of diam h has optimal order approximation in energy

$$e[u - u^h](t) = O(h^2), \quad 0 \leq t \leq T$$

$$\text{with } e[u](t) := \frac{1}{2} \left(\iint_{\Omega} dx \left| \frac{1}{\sqrt{\kappa}} \frac{\partial u}{\partial t} \right|^2 + \left| \frac{1}{\sqrt{\rho}} \nabla u \right|^2 \right) (t)$$

1D Illustration

1D elliptic interface problem

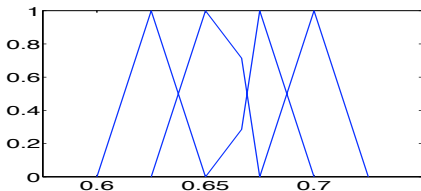
$$(\beta u_x)_x = f \quad 0 \leq x \leq 1, \quad u(0) = u(1) = 0$$

$f \in L^2(0,1)$, β has discontinuity at $x = \alpha$

$$\beta(x) = \begin{cases} \beta^- & x < \alpha \\ \beta^+ & x > \alpha \end{cases}$$

displacement u is continuous as well as normal stress βu_x at α

1D 'linear' HCFE basis:



Outline

Introduction

Harmonic Coordinates FEM

Mass Lumping

Numerical Results

Conclusion

FEM Discretization for Acoustic Waves

For acoustic wave equation:

$$\frac{1}{\kappa} \frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho} \nabla u = f$$

FE space $S = \text{span}\{\psi_j(\mathbf{x})\}_{j=0}^{N_h}$, FE solution $u_h = \sum_{j=0}^{N_h} u_j(t)\psi_j(\mathbf{x})$.

FEM semi-discretization:

$$M^h \frac{d^2 U^h}{dt^2} + N^h U^h = F^h$$

$$U^h(t) = [u_0, \dots, u_{N_h}]^T, F_i^h = \int_{\Omega} f \psi_i \, dx, M_{ij}^h = \int_{\Omega} \frac{1}{\kappa} \psi_i \psi_j \, dx,$$

$$N_{ij}^h = \int_{\Omega} \frac{1}{\rho} \nabla \psi_i \cdot \nabla \psi_j \, dx$$

Mass Lumping

2nd order time discretization:

$$M^h \frac{U^h(t + \Delta t) - 2U^h(t) + U^h(t - \Delta t)}{\Delta t^2} + N^h U^h(t) = F^h(t)$$

\Rightarrow every time update involves solving a linear system $M^h U^h = \text{RHS}$

Replace M^h by a diagonal matrix \tilde{M}^h ,

$$\tilde{M}_{ii}^h = \sum_j M_{ij}^h$$

Can achieve optimal rate of convergence if the solution u is smooth (e.g., $H^2(\Omega)$)

My thesis has the details for validation of mass lumping

Outline

Introduction

Harmonic Coordinates FEM

Mass Lumping

Numerical Results

Conclusion

Implementation and Computation

Implementation: based on **deal.II**, a C++ program library targeted at the computational solution of partial differential equations using adaptive finite elements

- ▶ built-in quadrilateral mesh generation and mesh adaptivity
- ▶ various finite element spaces, DG
- ▶ interfaces for parallel linear system solvers (eg PETSc)
- ▶ data output format for quick view (eg paraview, opendx, gnuplot, ps)

Computation: using DAVinCI@RICE cluster, 2304 processor cores in 192 Westmere nodes (12 processor cores per node) at 2.83 GHz with 48 GB of RAM per node (4 GB per core).

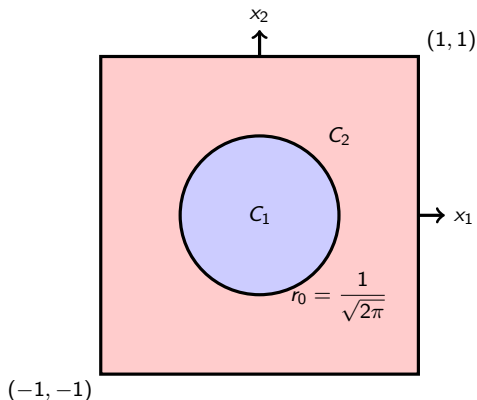
Elliptic BVP - Square Circle Model

$$-\nabla \cdot C(x)\nabla u = -9r \quad \text{in } \Omega$$

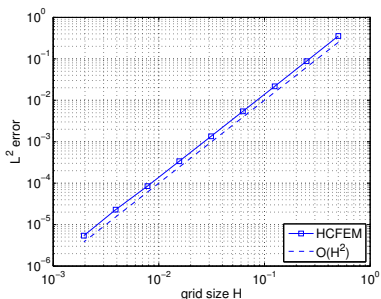
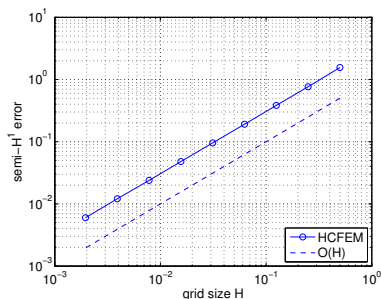
$$\text{where } r = \sqrt{x^2 + y^2}$$

For piecewise const $C(x)$ shown in the figure below, analytical solution:

$$u = \frac{1}{C(x)}(r^3 - r_0^3)$$

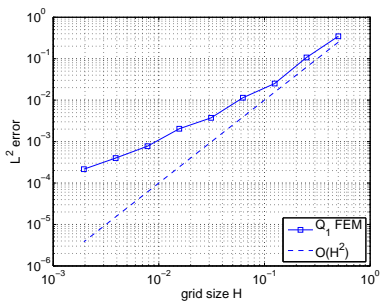
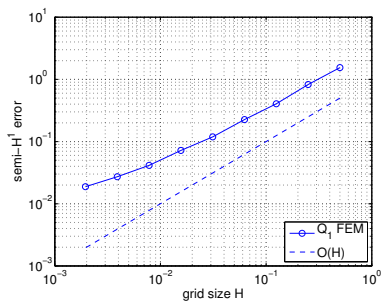


High Contrast: $C_1 = 20, C_2 = 1$



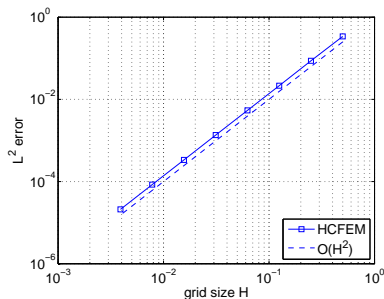
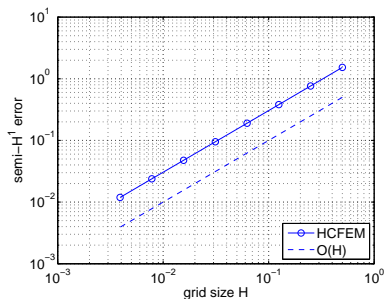
- ▶ HCFEM is applied on the physical grid of diameter H
- ▶ Harmonic coordinates are approximated on the locally refined grid, in which the grid size is $O(h)$ ($h = H^2$) near interfaces.

High Contrast: $C_1 = 20$, $C_2 = 1$



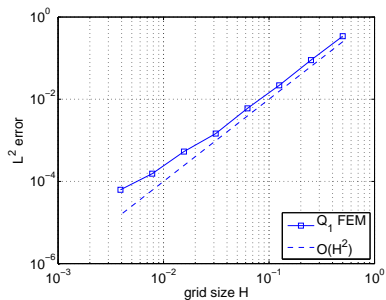
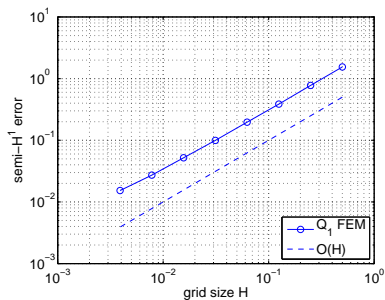
- Standard FEM is applied on the physical grid of diameter H

Low Contrast: $C_1 = 2, C_2 = 1$



- ▶ HCFEM is applied on the physical grid of diameter H
- ▶ Harmonic coordinates are approximated on the locally refined grid, in which the grid size is $O(h)$ ($h = H^2$) near interfaces.

Low Contrast: $C_1 = 2, C_2 = 1$



- Standard FEM is applied on the physical grid of diameter H

2D Acoustic Wave Tests

Acoustic wave equation:

$$\kappa^{-1} \frac{\partial^2 u}{\partial t^2} - \nabla \cdot \left(\frac{1}{\rho} \nabla u \right) = 0$$
$$u(x, 0) = g(x, 0), \quad u_t(x, 0) = g_t(x, 0)$$

with $g(x, t) = \frac{1}{r} f \left(t - \frac{r}{c_s} \right)$ and

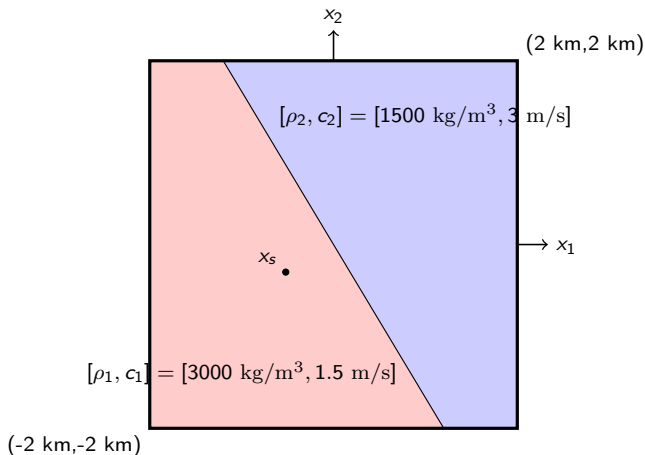
$f(t) = \left(1 - 2(\pi f_0 (t + t_0))^2 \right) e^{-(\pi f_0 (t + t_0))^2}$, f_0 central frequency,

$$c_s = \sqrt{\frac{\kappa(x_s)}{\rho(x_s)}}, \quad t_0 = \frac{1.45}{f_0}$$

The following examples similar to those in Symes and Terentyev, SEG Expanded Abstracts 2009

Dip Model

Central frequency $f_0 = 10$ Hz, $x_s = [-300\sqrt{3}$ m, -300 m]



Dip Model

Q_1 FEM solution, regular grid quadrature (= FDM) - this is equivalent to using ONLY the node values on the regular grid to define mass, stiffness matrices



Entire domain		
h	e	p
7.8 m	3.62e-1	-
3.9 m	9.30e-2	1.96
1.9 m	2.31e-2	2.01

Figure: $T = 0.75$ s

Dip Model

Q_1 FEM solution, accurate quadrature for mass and stiffness matrices' computation,

Q_1 FEM with accurate quadrature is also (2,2) FDM but with different coefficients - this is Igor's result (constant density acoustics).



Entire domain		
h	e	p
7.8 m	3.56e-1	-
3.9 m	9.13e-2	1.96
1.9 m	2.27e-2	2.01

Dip Model

HCFEM solution



Entire domain		
h	e	p
7.8 m	3.63-1	-
3.9 m	9.36e-2	1.96
1.9 m	2.34e-2	2.01

Figure: $T = 0.75$ s

Dome Model

central frequency $f_0 = 15$ Hz, $x_s = [3920 \text{ m}, 3010 \text{ m}]$

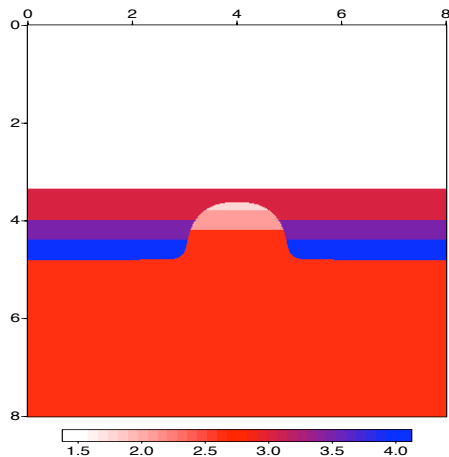


Figure: Velocity model

Dome Model

Difference between HCFEM solution on regular grid ($h = 7.8125$ m) and FEM solution on locally refined grid, same time stepping

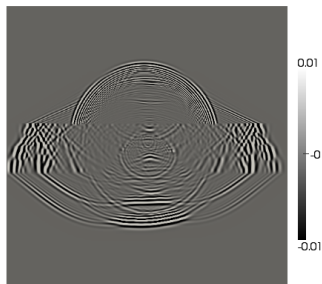


Figure: $T = 1.3$ s

Dome Model

Difference between FEM solution on regular grid ($h = 7.8125$ m) and FEM solution on locally refined grid, same time stepping

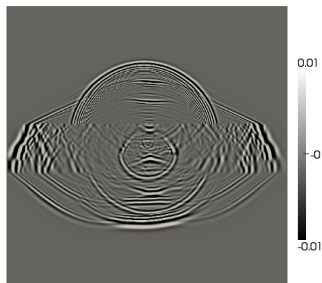


Figure: $T = 1.3$ s

Outline

Introduction

Harmonic Coordinates FEM

Mass Lumping

Numerical Results

Conclusion

Conclusion

2D harmonic coordinate finite element method (HCFEM) on regular grids achieves second order convergence rate for static and dynamic acoustic interface problems.

For dip model: HCFEM and mass lumping is at least as good as Q1 FEM with accurate quadrature, when density contrasts are low (typical of seismic). Both seem to get rid of stairstep diffractions (more or less). More refined analysis shows HCFEM somewhat more accurate.

For dome model: HCFEM closer to refined-grid FEM when same (very short) time steps taken

Future work:

- ▶ Fill in the theoretical gaps,
- ▶ Extensions, eg higher order, DG, elasticity.