Linearized Multi-Parameter Inversion

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Outline

- Linearization of the inverse problem
- Normal equations
- Cramer's rule
- Variable density acoustics (2 parameters)
  - 1D layered example
  - Marmousi with homogeneous background
- Comparison with existing approaches
- Summary and future work

The Acoustic Wave Equation

$$\frac{1}{\rho(x)c^2(x)}\frac{\partial^2 p(x,t)}{\partial t^2} - \nabla \cdot \frac{1}{\rho(x)}\nabla p(x,t) = f(x,t),$$

with appropriate IC and BC.

$c(x)$: velocity, $\rho(x)$: density, $p(x)$: pressure, and $f(x,t)$: source.

The wave equation:

- Predicts the pressure at the surface for all time
- Defines a nonlinear map,

$$S : [\rho, c] \rightarrow p|_{surface},$$

- Inverse problem: recover ρ, c, given $p|_{surface}$

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Abstraction

Let:

- $m(x)$: model (consists of p-parameters: impedance, velocity, density,...)
- $p(x,t)$: state (the solution of the system: pressure,...)

Then, if S is the Forward Map:

- The Forward Problem:

$$S[m] = p|_{surface}$$

- The Inverse Problem:

$$S[m] \approx S^{obs}$$

Given S^{obs}, get $m(x)$

Nonlinear and Large Scale !

Linearization

Solution depends nonlinearly on coefficients; if we have an approximation m_0 to the model, Linearization is advantageous:

- Write $m = m_0 + \delta m$
  m_0: Given reference model
  δm: First order perturbation about m_0

- Define Linearized Forward Map $F[m_0]$ (Born Modeling):

$$F[m_0]\delta m = \delta p$$

- Linear inverse problem:

$$F[m_0]\delta m \approx S^{obs} - S[m_0] := d$$

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Normal Equations

Interpret as least squares problem: need to solve normal equations

$$N[m_0]\delta m := F^*[m_0]F[m_0]\delta m = F^*[m_0]d$$

$N := F^*[m_0]F[m_0]$: Normal Operator (Modeling + Migration),
$b := F^* d$: migrated image

- Can only apply N, modeling + migration
- *Large Scale*: millions of equations/unknowns, also $\delta m \rightarrow N\,\delta m$ expensive
- Cannot use Gaussian elimination $\Rightarrow$ need rapidly convergent iteration $\Rightarrow$ good preconditioner
- Will tell you how to make N undo itself!

Multi-Parameter Inversion, Toy Problem

Example 1: Layered model, homogeneous background

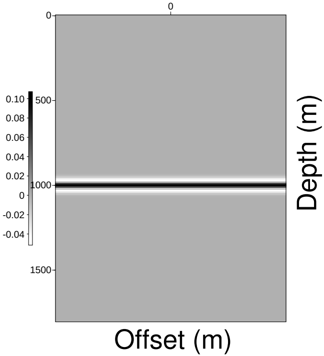


Figure: vp, velocity perturbation

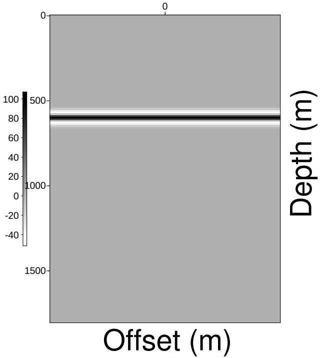
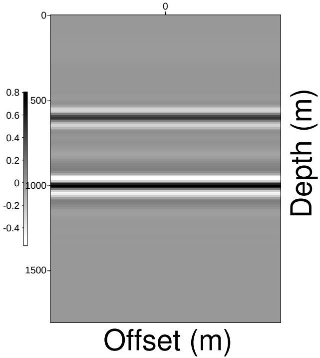
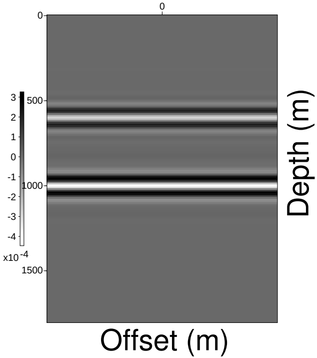


Figure: dn, density perturbation

Migrated Images



m_{mig_1}, velocity component of the migrated image.



m_{mig_2}, density component of the migrated image.

Figure: Migrated images mixing the contributions from density and velocity.

Lessons Learned

- Discontinuities mixed in migrated images (NOT Interpretable!)
- Amplitudes distorted
- Need to:
  - Separate contributions (notoriously hard)
  - Resolve resolution problem
  - Resolve ill conditioning problem (density for v.d.a.)
  - Correct amplitudes after separation

The Trick: Cramer's Rule

Want to solve

$$N\,m = b.$$

In this case:

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \end{pmatrix}.$$

Adjugate (= determinant $\times$ inverse) given by:

$$Adj(N) = \begin{pmatrix} N_{22} & -N_{12} \\ -N_{12} & N_{11} \end{pmatrix}.$$

Simple matrix multiplication:

$$Adj(N)\,N = \begin{pmatrix} N_{22}N_{11} - N_{12}^2 & N_{22}N_{12} - N_{12}N_{22} \\ -N_{12}N_{11} + N_{11}N_{12} & N_{11}N_{22} - N_{12}^2 \end{pmatrix}$$

TOTAL RICE

- Would be great if we had:

$$Adj(N)\,N = det(N)\,I$$

- Diagonal $\rightarrow$ no mixing
- Caution: operators not numbers!
- But we can say (in some cases)

$$Adj(N)\,N \approx det(N)\,I$$

- Who says so?

The Big Guns

- Normal operator is a matrix of pseudodifferential operators:
  - Smooth background model m_0 (Beylkin 1985)
  - Scalar wave fields
  - Polarized vector fields (P-P, P-S, S-S). (Beylkin and Burridge, 1989; De Hoop, 2003)
- Pseudodifferential $\rightarrow$ entries of N approximately commute!

Not The End Of The Story

- We never have the entries of N
- We can only apply them to data

Another Revelation

$$Adj(N)\, b = J^T N J\, b. \quad (1)$$

Where,

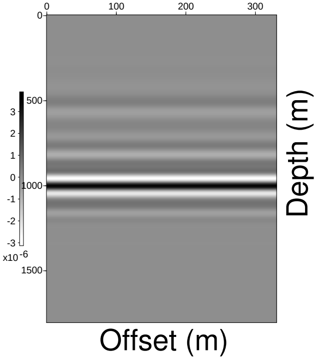
$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Simply

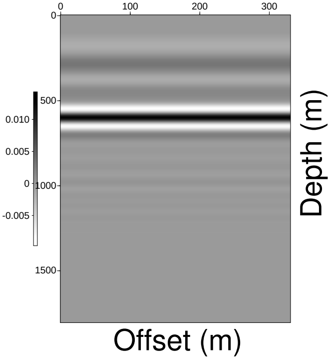
$$N \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} = det(N) \begin{pmatrix} m_2 \\ -m_1 \end{pmatrix}$$

Lesson: apply N to specific permutations of the right hand side $b \rightarrow N$ undoes itself!

Application of Adjugate



$$(J^T N J\, m_{\text{mig}})_1 \approx \det(N)\, m_1$$



$$(J^T N J\, m_{\text{mig}})_2 \approx \det(N)\, m_2$$

Figure: The application of the adjugate separates the velocity and density contributions.

Dividing by the determinant

- To undo $det(N)$, apply N to form:

$$N\,det(N)\,\delta m \approx det(N)\,N\,\delta m = det(N)\,b \tag{2}$$

- given b and $det(N)\,b$, approximate scaling factor c:

$$c = \underset{c \in \Psi DO}{\textsf{argmin}} \| b - c\,det(N)\,b \|^2 \tag{3}$$

- Approximate solution:

$$\begin{aligned} \delta m = N^{-1} b \approx N^{-1}\,c\,det(N)\,b \approx c\,det(N)\,N^{-1} b \\ \approx c\,det(N)\,\delta m := \delta m_{inv} \end{aligned} \tag{4}$$

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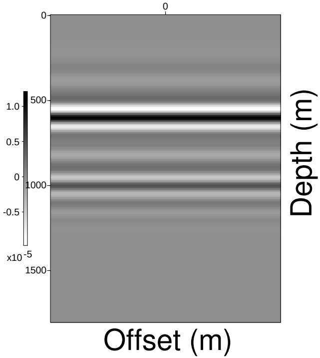
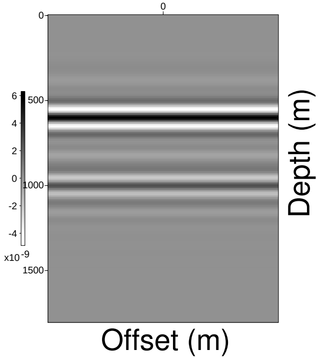


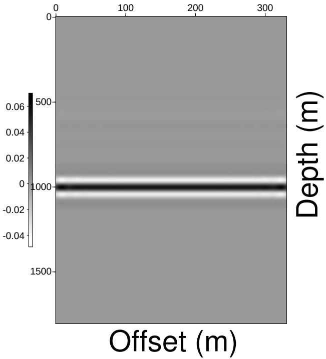
$\det(N)\, m_{\mathrm{mig}_1}$



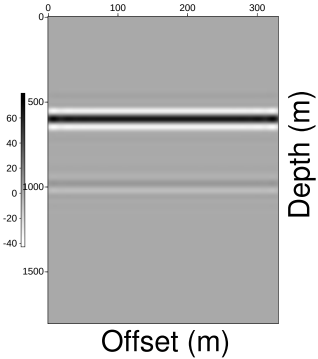
$\det(N)\, m_{\mathrm{mig}_2}$

Figure: Scaling of the migrated images by $\det(N)$, used to undo the determinant

Approximate Inverse after Amplitude Correction



inv_{vp}



inv_{dn}

Figure: The approximate inverse. The contributions from velocity and density are separated and the amplitudes are corrected.

Data Fit 70%

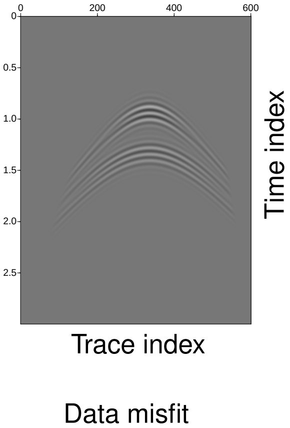
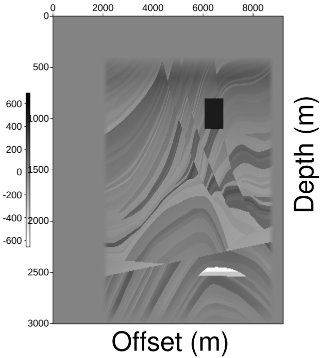
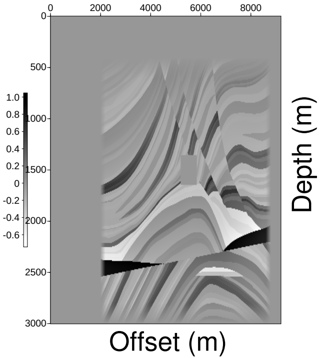


Figure: Data misfit, versus target data. The inverted model fits 70% of the data.

Marmousi with homogeneous background



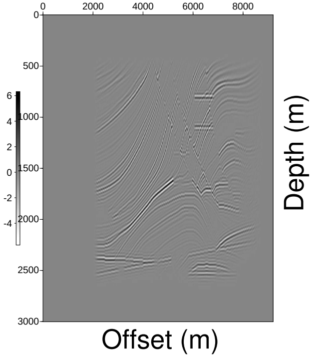
Density perturbation.



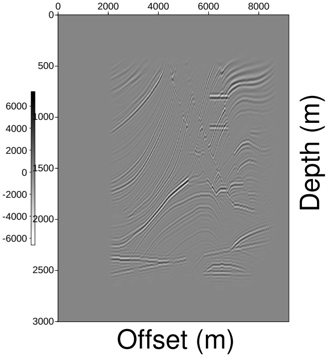
Velocity perturbation.

Figure: Perturbations to the homogeneous background.

Migrated images



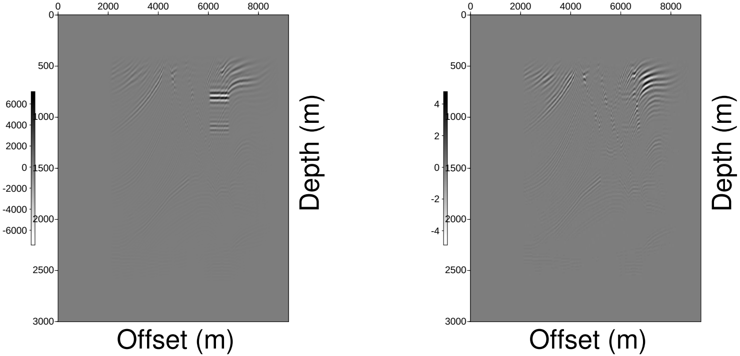
Migrated image, density component.



Migrated image, velocity component.

Figure: b, the migrated images.

Application of the adjugate

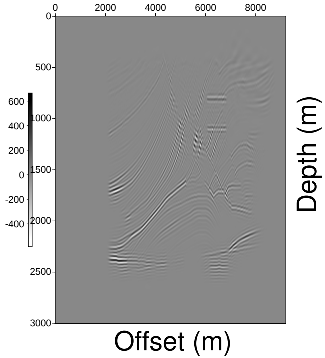


Adjugate application, density component.

Adjugate application, velocity component.

Figure: $Adj(N)\, b$, application of adjugate.

Amplitude correction

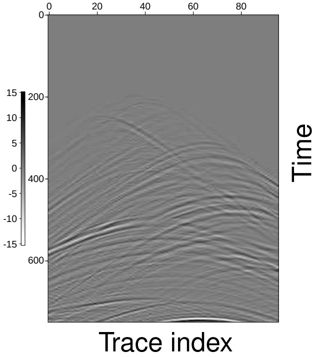


Approximate inverse, density component.

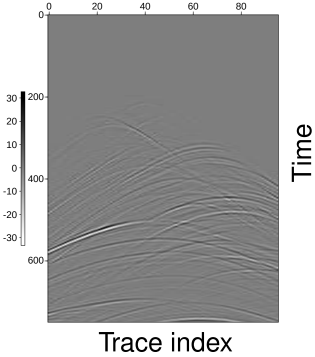
Approximate inverse, velocity component.

Figure: Approximate inverse.

Data Fit 45%



Data misfit



Target Data

Figure: Data misfit, versus target data. The inverted model fits 45% of the data.

Existing Approaches

- For one parmeter:
  - One application of N to approximate inverse
  - Known as *scaling methods*
  - Approximation of inverse: *scaling factor*
- For multiparameters:
  - Amplitude Versus Offset (AVO) variations $\rightarrow$ information about physical parameters
  - Geometric optics calculations $\rightarrow$ asymptotic formulas to approximate N^{-1}
  - Minimize objective function:

$$\|F\,\delta m - d\|^2$$

    using Krylov subspace methods

Proposed method:

- Not iterative
- Uses wave equation migration (Reverse Time Migration)
- No geometric optics computations
- Relies only on application of normal operator
- Novel for multiparameters: Few applications of $N \rightarrow$ approximate inverse

Summary and Future Work

- Derived preconditioner for linearized multiparameter inverse problem
- Showed 1D and 2D examples $\rightarrow$ 3D
- Demonstrated on homogeneous background for 2D $\rightarrow$ smooth
- Applied to v.d.a. $\rightarrow$ linear elasticity (3 parameters) ...

THANK YOU !

- Dr. Symes
- TRIP
- Total E&P USA