Linearized Multi-Parameter Inversion

Rami Nammour

Total E&P USA

TRIP 2012 Meeting
March 30, 2012
Outline

• Linearization of the inverse problem
• Normal equations
• Cramer’s rule
• Variable density acoustics (2 parameters)
  • 1D layered example
  • Marmousi with homogeneous background
• Comparison with existing approaches
• Summary and future work
The Acoustic Wave Equation

\[
\frac{1}{\rho(x)c^2(x)} \frac{\partial^2 p(x, t)}{\partial t^2} - \nabla \cdot \frac{1}{\rho(x)} \nabla p(x, t) = f(x, t),
\]

with appropriate IC and BC.

\(c(x):\) velocity, \(\rho(x):\) density, \(p(x):\) pressure, and \(f(x, t):\) source.

The wave equation:

- Predicts the pressure at the surface for all time
- Defines a nonlinear map,

\[S: [\rho, c] \rightarrow p|_{\text{surface}},\]

- Inverse problem: recover \(\rho, c,\) given \(p|_{\text{surface}}\)
Abstraction

Let:

- \( m(x) \): model (consists of p-parameters: impedance, velocity, density,...)
- \( p(x, t) \): state (the solution of the system: pressure,...)

Then, if \( S \) is the Forward Map:

- The Forward Problem:
  \[ S[m] = p|_{\text{surface}} \]

- The Inverse Problem:
  \[ S[m] \approx S^{\text{obs}} \]

Given \( S^{\text{obs}} \), get \( m(x) \)

*Nonlinear and Large Scale!*
Solution depends nonlinearly on coefficients; if we have an approximation \( m_0 \) to the model, **Linearization** is advantageous:

- Write \( m = m_0 + \delta m \)
  - \( m_0 \): Given reference model
  - \( \delta m \): First order perturbation about \( m_0 \)

- Define Linearized Forward Map \( F[m_0] \) (Born Modeling):
  \[
  F[m_0] \delta m = \delta p
  \]

- **Linear** inverse problem:
  \[
  F[m_0] \delta m \approx S^{obs} - S[m_0] := d
  \]
Normal Equations

Interpret as least squares problem: need to solve normal equations

\[ N[m_0] \delta m := F^*[m_0]F[m_0] \delta m = F^*[m_0]d \]

\[ N := F^*[m_0]F[m_0] : \text{Normal Operator (Modeling + Migration)}, \]
\[ b := F^*d : \text{migrated image} \]

- Can only apply \( N \), modeling + migration
- \textit{Large Scale}: millions of equations/unknowns, also \( \delta m \rightarrow N \delta m \) expensive
- Cannot use Gaussian elimination \( \Rightarrow \) need rapidly convergent iteration \( \Rightarrow \) good preconditioner
- Will tell you how to make \( N \) undo itself!
Multi-Parameter Inversion, Toy Problem
Example 1: Layered model, homogeneous background

Figure: $vp$, velocity perturbation

Figure: $dn$, density perturbation
$m_{\text{mig}_1}$, velocity component of the migrated image.

$m_{\text{mig}_2}$, density component of the migrated image.

**Figure:** Migrated images mixing the contributions from density and velocity.
Lessons Learned

• Discontinuities mixed in migrated images (NOT Interpretable!)
• Amplitudes distorted
• Need to:
  • Separate contributions (notoriously hard)
  • Resolve resolution problem
  • Resolve ill conditioning problem (density for v.d.a.)
  • Correct amplitudes after separation
The Trick: Cramer’s Rule

Want to solve

$$N m = b.$$

In this case:

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \end{pmatrix}.$$  

Adjugate (= determinant $\times$ inverse) given by:

$$\text{Adj}(N) = \begin{pmatrix} N_{22} & -N_{12} \\ -N_{12} & N_{11} \end{pmatrix}.$$  

Simple matrix multiplication:

$$\text{Adj}(N) \, N = \begin{pmatrix} N_{22}N_{11} - N_{12}^2 & N_{22}N_{12} - N_{12}N_{22} \\ -N_{12}N_{11} + N_{11}N_{12} & N_{11}N_{22} - N_{12}^2 \end{pmatrix}.$$
• Would be great if we had:

\[ \text{Adj}(N) N = \det(N) I \]

• Diagonal \( \rightarrow \) no mixing

• Caution: operators not numbers!

• But we can say (in some cases)

\[ \text{Adj}(N) N \approx \det(N) I \]

• Who says so?
The Big Guns

• Normal operator is a matrix of pseudodifferential operators:
  • Smooth background model $m_0$ (Beylkin 1985)
  • Scalar wave fields
  • Polarized vector fields (P-P, P-S, S-S). (Beylkin and Burridge, 1989; De Hoop, 2003)

• Pseudodifferential $\rightarrow$ entries of $N$ approximately commute!
Not The End Of The Story

- We never have the entries of $N$
- We can only apply them to data
Another Revelation

\[ \text{Adj}(N) b = J^T N J b. \]  

(1)

Where,

\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

Simply

\[ N \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} = \text{det}(N) \begin{pmatrix} m_2 \\ -m_1 \end{pmatrix} \]

Lesson: apply \( N \) to specific permutations of the right hand side \( b \rightarrow N \) undoes itself!
Application of Adjugate

\[(J^T J m_{mig})_1 \approx \text{det}(N) m_1\]  \[(J^T J m_{mig})_2 \approx \text{det}(N) m_2\]

**Figure:** The application of the adjugate separates the velocity and density contributions.
Dividing by the determinant

• To undo $\det(N)$, apply $N$ to form:

$$N \det(N) \delta m \approx \det(N) N \delta m = \det(N) b$$

(2)

• given $b$ and $\det(N) b$, approximate scaling factor $c$:

$$c = \arg\min_{c \in \Psi DO} \|b - c \det(N) b\|^2$$

(3)

• Approximate solution:

$$\delta m = N^{-1} b \approx N^{-1} c \det(N) b \approx c \det(N) N^{-1} b$$

$$\approx c \det(N) \delta m := \delta m_{inv}$$

(4)
Figure: Scaling of the migrated images by $\det(N)$, used to undo the determinant
Approximate Inverse after Amplitude Correction

Figure: The approximate inverse. The contributions from velocity and density are separated and the amplitudes are corrected.
Figure: Data misfit, versus target data. The inverted model fits 70% of the data.
Marmousi with homogeneous background

Figure: Perturbations to the homogeneous background.

Density perturbation.  

Velocity perturbation.
Migrated images

Migrated image, density component.  

Migrated image, velocity component.

Figure: $b$, the migrated images.
Application of the adjugate

Adjugate application, density component.  

Adjugate application, velocity component.

Figure: $\text{Adj}(N)b$, application of adjugate.
Amplitude correction

Approximate inverse, density component.

Approximate inverse, velocity component.

Figure: Approximate inverse.
Data Fit 45% 

Figure: Data misfit, versus target data. The inverted model fits 45% of the data.
Existing Approaches

- For one parameter:
  - One application of $N$ to approximate inverse
  - Known as *scaling methods*
  - Approximation of inverse: *scaling factor*

- For multiparameters:
  - Amplitude Versus Offset (AVO) variations $\rightarrow$ information about physical parameters
  - Geometric optics calculations $\rightarrow$ asymptotic formulas to approximate $N^{-1}$
  - Minimize objective function:

$$\|F \delta m - d\|^2$$

using Krylov subspace methods
Proposed method:

- Not iterative
- Uses wave equation migration (Reverse Time Migration)
- No geometric optics computations
- Relies only on application of normal operator
- **Novel for multiparameters:** Few applications of $N \rightarrow$ approximate inverse
Summary and Future Work

- Derived preconditioner for linearized multiparameter inverse problem
- Showed 1D and 2D examples → 3D
- Demonstrated on homogeneous background for 2D → smooth
- Applied to v.d.a. → linear elasticity (3 parameters) . . .
THANK YOU!

- Dr. Symes
- TRIP
- Total E&P USA