Yin Huang

Education

- **Rice University**, Houston, TX, USA

  Ph.D. Candidate, Computational and Applied Mathematics 08/2010 - Present

  - Expected Dissertation Topic: "Inverse problems for wave propagation for nonsmooth coefficients"
  - Relevant courses: Numerical Analysis; Numerical Linear Algebra; Numerical Differential Equations; Optimization; High Performance Computing

- **Shanghai Jiao Tong University**, Shanghai, China

  M.S., Mathematics and Applied Mathematics 09/2006 - 03/2009

  - Dissertation topic: "Comparision of many numerical methods for saddle point system arising from the mixed finite element method of elliptic problems with nonsmooth coefficients"

Research Interests

- Forward and inverse problems for non-homogeneous medium
- Seismic waveform tomography
- Parallel computing

Acoustic transparency theorem for the 1D wave equation with a nonsmooth coefficient

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Outline

- Background
- One-dimensional acoustic transparency Theorem
- Plan of the proof
- Discussion
- Conclusion
- References

1D constant density acoustic wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2} = 0,$$

$$u(z,0) = u_0(z),\ \frac{\partial u}{\partial t}(z,0) = u_1(z).$$

- simplest system with key features (variable wave velocity, reflected waves, ...)
- $z \in \mathbb{R}$ depth. $t \in \mathbb{R}$ time.
- $c = c(z)$ wave velocity , nonsmooth
- $u = u(z,t)$ the acoustic potential at (z,t).
- u_0, u_1 are initial conditions.

Inverse problem

Assume that u solves wave equation. Define $\mathcal{T}_c = \frac{\partial u}{\partial t}(0, t)$.

$$\min_c \|\mathcal{T}_c - data\|$$

Bamberger et al. 1979: if

- c is piecewise constant
- the layers have equal travel time
- Neumann boundary condition on $z = 0$

The solution to the inverse problem: c is unique and depends continuously on *data*.

Relation to acoustic transparency theorem

Acoustic transparency theorem is a step of the proof toward this result: For nonsmooth c, c depends continuously on *data*.

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Bounded variation function

Definition

$c \in BV[0, Z]$ means

$$Var(c) = \max_{\mathcal{P}} \sum_{i=1}^{n} |c(z_i) - c(z_{i-1})| \leq M < +\infty,$$

with $\mathcal{P}$ the set of ordered points in $[0, Z]$, including 0 and Z.

Example: c is piecewise constant with $c(z) = c_i$ if $z \in [z_{i-1}, z_i)$.
$\Delta z_i = z_i - z_{i-1}$.

$$Var(c) = \sum_i |c_i - c_{i-1}| = \sum_i \Delta z_i \frac{|c_i - c_{i-1}|}{\Delta z_i} = \int_0^Z \left| \frac{dc}{dz} \right|$$

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Acoustic energy

Density of the material ρ is constant.

- $-\rho\frac{\partial u}{\partial t}$ is the acoustic pressure,
- $\frac{\partial u}{\partial z}$ is the acoustic particle velocity,

Energy over interval $[0, Z]$ at time t

$$E(t) = \frac{\rho}{2}\int_0^Z \left(\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2\right)(z,t)dz$$

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Acoustic transparency theorem

Theorem

Suppose u_0 *and* u_1 *are zero outside of* $[0, Z]$, $T > \int_0^Z \frac{1}{c}$ *and* $c(z) = c(0)$ *for* $z < 0$. $c \in BV[0, Z]$ *gives*

$$kE(0) \leq \int_{-T}^{T} \left(\frac{\partial u}{\partial t}(0, t) \right)^2 dt \leq KE(0)$$

with $K = \sup_{z \in [0,Z]} c(z)$ *and* $k = K \exp\left(-\hat{r}\, Var(\log c)\right)$, *where* $\hat{r}$ *depends on the supremum and infimum of* c.

Importance of the lower bound:

Otherwise, arbitrarily small part of the energy reaches the surface
$\Rightarrow$ No possibility of stable solution for the inverse problem.

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Discussion of existing and relevant results

- Symes 1983, 1986 provided a proof by flipping the space and time variables
- Cox and Zuazua 1995 proved energy decay for 1D medium of bounded variation by analyzing the corresponding eigenvalue problem which also gives acoustic transparency.
- ...

Why we need a new proof?

- Existing proofs could not be generalized to multi-dimensional problems.
- A function space of velocity c that is both necessary and sufficient is needed.
- A proof to show the necessity and sufficiency of the given function space is needed.

Plan of the proof

- The upper bound is given by integration by parts.
- By a theorem due to Kahane, if $c \in BV[0, Z]$, then for every $n \in \mathbb{N}$ there exists piecewise constant c_n with n jumps so that $\max_{z\in[0,Z]}|c(z) - c_n(z)| \leq \frac{Var(c)}{n}$ (see DeVore 1998).
- Choose piecewise constant approximations c_n, then we have $u_n \to u$ in energy, by Theorem 2.8.2 of Stolk PhD thesis 2000.
- The lower bound for piecewise constant c is from the analysis of reflections and transmissions at jumps in c.

The lower bound we have now depends on the number of layers.

Uniform lower bound is the goal.

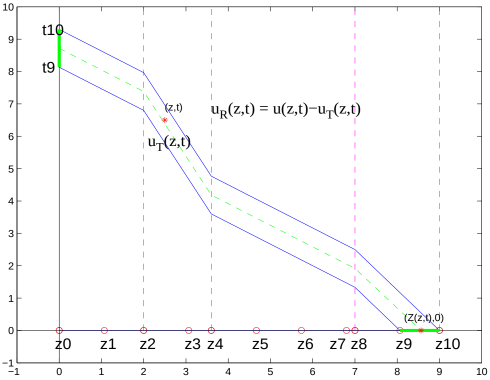
Multi-interfaces with localized initial conditions I

Construct the solution by repeating the reflection and transmission. Assume u_0 and u_1 are zero outside of $[z_{i-1}, z_i]$, with velocity c_i.

- Transmission coefficients $T_k = \frac{2c_k}{c_k + c_{k+1}}$.
- u_{iU} up-going wave in the i-th interval,
- Purely transmitted part of the wave

$$u_{iT}(0, t) = \prod_{k=1}^{i-1} T_k u_{iU}(Z(0, t))$$

Multi-interfaces with localized initial conditions II



- u_0 and u_1 are zero outside of $[z_{N-1}, z_N]$
  $\Rightarrow \; u(0,t) = u_{NT}(0,t)$ for $t \in [t_{N-1}, t_N]$
- $\Rightarrow \prod_{k=1}^{N-1} T_k$ has a possitive lower bound is necessary for acoustic transparency theorem.

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Current result for purely transmitted wave

- If $c \in BV[0, Z]$, product of transmission coefficients

$$\prod_{k=1}^{N-1} T_k \geq \exp\left(-\frac{\hat{r}}{2}\mathrm{Var}(\log c)\right),$$

  with $\hat{r}$ depends on the supremum and infimum of c.
- Let the u_T denote the purely transmitted part of the solution with arbitrary initial conditions

$$\int_{-T}^{T}\left(\frac{\partial u_T}{\partial t}(0,t)\right)^2 dt \geq \sum_{i=1}^{N}\left(\prod_{k=1}^{i-1} T_k\right)^2 \frac{c_i^2}{4c_1}\int_{z_{i-1}}^{z_i}\left((\frac{du_0}{dz})^2 + \frac{u_1^2}{c_i^2}\right) dz$$

Discussion

Bounded variation is not necessary for the product of transmission coefficients to be bounded away from zero.

- $c \in BV[0, Z] \Rightarrow \prod_{k=1}^{N-1} T_k \geq k > 0$. k depends on $Var(\log c)$.
- For N layers material, where N is even, let

$$c = \begin{cases} 1 - \frac{1}{\sqrt{N}} \text{ odd layer} \\ 1 + \frac{1}{\sqrt{N}} \text{ even layer,} \end{cases}$$

  $Var(c) \to +\infty$ as $N \to +\infty$.
- $\prod_{k=1}^{N-1} T_k \geq \exp(-\frac{1}{2})$ with $c \notin BV[0, Z]$.

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Discussion

What is both necessary and sufficient?

- BV-2 is both necessary and sufficient (by Demanet)

$$Var_2(c) = \max_{\mathcal{P}} \left(\sum_{i=1}^{n} (c(x_i) - c(x_{i-1}))^2 \right)^{\frac{1}{2}} \leq M < +\infty,$$

  where $\mathcal{P}$ is the set of all ordered points of $[0, Z]$.

- Approximate BV-2 function c with piecewise constant function
- Define the transmission coefficient T_k
- $c \in$ BV-2 $\Longleftrightarrow \prod_{k=1}^{N-1} T_k \geq k > 0$, k depends on $Var_2(\log c)$.

Conclusion

- Acoustic transparency is necessary for the stable solution of the inverse problem.
- Lower bound of the acoustic transparency thoerem depends on $\prod_{k=1}^{N-1} T_k$ and the number of layers N.
- $\int_{-T}^{T} \left(\frac{\partial u}{\partial t}(0,t)\right)^2 dt$ has a lower bound only if $\prod_{k=1}^{N-1} T_k$ has a positive lower bound.
- $\prod_{k=1}^{N-1} T_k$ has a positive lower bound $\Longleftrightarrow$ $c \in$ BV-2.

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