Gradient artifacts in space-shift differential semblance

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TRIP Annual Review, 2010
Agenda

Background

Ray Theory

The Fix
Space-shift Differential Semblance

Space-shift gather / HOCIG via shot record migration: (IEI, Biondi, Sava, Fomel,...)

\[ I(x, z, h) = \sum_{x_s} \int dt\ S(x - h, z, t; x_s)R(x + h, z, t; x_s) \]

\( S = \) source wavefield, \( R = \) receiver wavefield - computed anyhow (depth extrapolation, two-way plus time reversal,...)

2D for convenience only!

Differential semblance MVA objective, in simplest form:

\[ J[v] = \int \int \int dxdzdh\ h^2 \ |I(x, z, h)|^2 \]

Shen’s thesis 04, others (Shen & coauthors 03, 05, 07, Shen & S. 08, Kabir 07, Fei 09, 10)
Space-shift Differential Semblance

Upshot, it works, but...

Gradient tends to oscillate horizontally - side lobes inhibit convergence (Biondi 08, Fei 10, Vyas 10)

This talk: where the oscillations come from, and how they might be removed.

Goal: redefine the gradient so that it is *still a gradient*, but suppress oscillations.

Approach: explicit ray theory computation suggests a fix
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Ray-theoretic Expression of SSDS

Assume band-unlimited data (Green’s function) - then data $d(x_s, x_r, t)$ and image $I(x, z, h)$ related by

$$I(x, z, h) = \int \int \int \int d(x_s) d(x_r) dt d\tau$$

$$d(x_s, x_r, t) G(x_r, x + h, z, t - \tau) G(x_s, x - h, z, \tau)$$

$$= (F^T d)(x, z, h)$$

$F = \text{extended Born modeling operator}$
Ray-theoretic Expression of SSDS

Simplest form: assume

- rays obey “DSR” condition - no significant amount of energy on rays turning horizontal.
- rays also obey “simple ray geometry” condition - no multipathing

Also: ignore amplitudes, (more or less) uniform factors of frequency (powers of Laplacian) throughout talk.

Then \( G(x_s, x, z, t) \approx \delta(t - T(x_s, x, z)) \) where \( T(x_s, x, z) = \) (unique) traveltime \( (x_s, z_s) \rightarrow (x, z) \)

More: \( F^T, F \) are invertible in a large region of phase space containing most reflection energy (see de Hoop - Stolk - S. 2009)
Ray-theoretic Expression of SSDS

Rewrite using *image space* and *data space* dot products

\[ J[\nu] = \langle hI, hI \rangle_I = \langle F^T d, h^2 F^T d \rangle_I \]

\[ = \langle d, Fh^2 F^T d \rangle \]

\[ = \int \int \int dx_s dx_r dt \ d(x_s, x_r, t) \int \int \int dk_s dk_r d\omega \exp(i(\omega t + k_s x_s + k_r x_r)) \bar{H}^2(x_s, x_r, t, k_s, k_r, \omega) \hat{d}(k_s, k_r, \omega) \]

*Egorov’s Thm*: symbol \( \bar{H} \) = function in data phase space which maps, under ray tracing, to \( h^2 \) in image phase space.
Ray-theoretic Expression of SSDS

\( \bar{H} \) homogenous of degree 0 in phase space coords, so write in terms of function \( H \) of phase angles \( \theta_s, \theta_r \):

\[
\omega \sin \theta_r = k_r, \quad \omega \sin \theta_s = k_s
\]

- receiver ray; starts at \( x_r, z_r \), takeoff angle \( \theta_r \):
  \[
  z \mapsto X(x_r, \theta_r, z), \quad T(x_r, \theta_r, z)
  \]
- source ray; starts at \( x_s, z_s \), takeoff angle \( \theta_s \):
  \[
  z \mapsto X(x_s, \theta_s, z), \quad T(x_s, \theta_s, z)
  \]
- two-way time condition: \( t = T(x_s, \theta_s, z) + T(x_r, \theta_r, z) \) - determines \( z(x_s, x_r, t, \theta_s, \theta_r) \)

\[
H(x_s, x_r, t, \theta_s, \theta_r) = X(x_r, \theta_r, z) - X(x_s, \theta_s, z),
\]

\[
z = z(x_s, x_r, t, \theta_s, \theta_r)
\]
Homogeneous medium, velocity $v$:

\[
\begin{align*}
    z(x_s, x_r, t, \theta_s, \theta_r) &= \frac{vt}{\sec \theta_r + \sec \theta_s} \\
    H(x_s, x_r, t, \theta_s, \theta_r) &= x_r - x_s + z(\tan \theta_r - \tan \theta_s)
\end{align*}
\]
Computing the gradient

Key observation: only $v$-dependent quantity in oscillatory integral for $J[v]$ is symbol $H$.

$$\delta H = \delta X(x_r, ...) - \delta X(x_s, ...) + \frac{\partial X}{\partial z}(x_r, ..) \delta z - \frac{\partial X}{\partial z}(x_s, ..) \delta z$$

Compute $\delta z$ by differentiating $t = T(x_s, \theta_s, z) + T(x_r, \theta_r, z)$ implicitly, use ray perturbation equations - obtain

$$\delta H = \int_0^z dz' (z - z') \left[ V_r \cdot \nabla \frac{\delta v}{v}(z', x_r + z' \tan \theta_r) + V_s \cdot \nabla \frac{\delta v}{v}(z', x_s + z' \tan \theta_s) \right]$$

$V_s, V_r =$ messy functions of $\theta_s, \theta_r$. 
Computing the gradient

\[ \delta H = \int_0^Z dz' \ (z - z') \left[ V_r \cdot \nabla \frac{\delta v}{v} (z', x_r + z' \tan \theta_r) 
\right. \\
\left. + V_s \cdot \nabla \frac{\delta v}{v} (z', x_s + z' \tan \theta_s) \right] \]

- **tomographic:** $\delta H = \text{integral along ray pair, like traveltime perturbation}$
- **sensitive to oscillations:** unlike traveltime perturbation, involves $\nabla \delta v$
Computing the gradient

Assess effect on gradient at “wrong” velocity: assume that $d$ is Born data for “target” velocity $v^*$, reflectivity $r(z_d, x_d)$. Ignoring amplitude and frequency factors,

$$d(x_s, x_r, t) = \int \int dx_d dz_d \delta(t - T^*(x_r, x_d, z_d) - T^*(x_s, x_d, z_d))r(z_d, x_d)$$

Insert into expression for $J$, get

$$J[v] = \int \int \int \int dx_d dz_d dx'_d dz'_d r(z_d, x_d)r(z'_d, x'_d)K(z_d, x_d; z'_d, x'_d)$$

in which $K$ represented by same integral as $J$ above with $t = T^*(x_r, x_d, z_d) + T^*(x_s, x_d, z_d))$ in expression for $H$, and oscillatory phase $\Phi$.

Important: $\delta J$ rep’ed by same “double diffraction” integral with $\delta H$ in place of $H$. 
Computing the gradient

Use stationary phase to eliminate 4 of 11 integrals, ignore resulting frequency and amplitude factors, obtain

\[ \delta J[v]\delta v = \int \int \int \int dx_d dz_d dx_d' dz_d' r(z_d, x_d) r(z_d', x_d') \delta K(z_d, x_d; z_d', x_d') \]

\[ \delta K = \int \int \int d\theta_s d\theta_r d\omega e^{i\Phi} A \]

\[ \times \int_{Bz_d}^{Bz_d} dz' (Bz_d - z') \left[ V_r \cdot \nabla \frac{\delta v}{v} (z', x_d + z_d \tan \Theta_r + z' \tan \theta_r) \right. \]

\[ + \left. V_s \cdot \nabla \frac{\delta v}{v} (z', x_d + z_d \tan \Theta_s + z' \tan \theta_s) \right] \]

\[ \Phi = (x_d - x_d') \Phi_x + (z_d - z_d') \Phi_z, \Phi_x, \Phi_z, B, A = \text{messy functions of} \]

\[ \theta_r, \Theta_r = \arcsin \left( \frac{v}{v^*} \sin \theta_r \right), ... \]
Computing the gradient

Can extract explicit multiple integral expression for $\nabla J$ from this formula - but not so enlightening

integral along ray is trivial if $\delta v$ oscillates in perp direction:

$$\delta v(z, x) = \chi(z, x)e^{ik(x - z \tan \theta_s)} \Rightarrow$$

- $\nabla \delta v \simeq k(-\tan \theta_s, 1)^T \delta v$ for large $k$
- $\delta v(z', x_d + z_d \tan \Theta_s + z' \tan \theta_s) = \chi(...e^{ik(x_d + z_d \tan \Theta_s)} \quad \text{approx. independent of ray coord } z' \text{ for large } k$
- so integral along ray $\simeq O(k)\delta v$
- remains approximately true if $\theta_s$ perturbed, sim. for $\theta_r - O(k)$ growth for near-horizontal oscillation
Computing the gradient

Upshot: \( \delta J[v] \delta v = O(k) \) if \( \partial_x \delta v = O(k) \)

\( \Rightarrow \) x-Fourier components of \( \nabla J[v] \) must be large (Plancherel)

\( \Rightarrow \) gradient must generally (square-integrable \( r \)) oscillate in near-horizontal directions, as observed

Finer analysis; if reflectivity \( r(x_d, z_d) \) is smooth in \( x_d \), then can integrate by parts to absorb growth - however at \( x \)-direction singularities this is impossible, leading to vertical diffraction side-lobes observed in numerics (Biondi 08, Fei 10, Vyas 10).

Mathematical expression: for general (non-smooth) \( r \), band-unlimited data, gradient does not exist!
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Proposed Remedy

Conventionally: “The Gradient” = Riesz representer of derivative via $L^2$ inner product ("continuous dot product") and discrete approximations

Non-existence of gradient not a new phenomenon - conventional reflection traveltime tomography gradient does not exist, either! (Delprat-Jannaud & Lailly, GJR 1993).

Morally: rate of change of objective (traveltime misfit, DS,...) depends on derivatives of velocity perturbation, really only makes sense for smooth $v, \delta v$

⇒ must use inner product / norm that controls derivatives
Proposed Remedy

Natural family of norms for this application: $L^2$ Sobolev family

$$\|\delta v\|_k^2 = \int dx [\delta v (I - \sigma^2 \nabla^2)^k \delta v]$$

Comparison: “ordinary” gradient $\nabla_0 J$, Sobolev $k$-norm gradient $\nabla_k J$

$$\nabla_k J = (I - \sigma^2 \nabla^2)^{-k} \nabla_0 J$$

obtain $k$ gradient from ordinary gradient by application of smoothing operator - large horizontal oscillations suppressed - isotropic smoothing applies also to VOCIG-based DS
Proposed Remedy

Program:

- construct DS ("ordinary") gradient computation via RTM
- implement Helmholtz operators powers in rect. geom. using FFTs, sparse matrix methods
- compute $k$-norm gradient, use in optimization (for 2D, $k=2$)