Discontinuous Galerkin Time Domain Methods for Acoustics and Comparison with Finite Difference Time Domain Methods

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Trip Annual Meeting

Jan 29, 2010
Outline

1. Problems and methods

2. Numerical results
   - DGTD and FDTD Comparison
   - Interface error

3. Low-storage curvilinear DGTD
   - Numerical experiments

4. Summary and future plan
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4 Summary and future plan
Seismic wave equations

- **acoustic wave equations (pressure-velocity)** read

\[
\rho(x) \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0
\]

\[
\frac{1}{\kappa(x)} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = f(x, t; x_s)
\]

- **elastic wave equations (pressure-stress)** read

\[
\rho \frac{\partial v_i}{\partial t} = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \quad \Leftarrow \text{conservation of momentum}
\]

\[
\frac{\partial \sigma_{ij}}{\partial t} = \sum_{k, l} c_{ijkl} \left( \frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right) \quad \Leftarrow \text{Hooke's law}
\]

\[ p = \text{acoustic pressure}, \; \sigma_{ij} = \text{stress tensors}, \; \mathbf{v} = \{v_i\} = \text{particle velocity}, \; \rho(x) = \text{mass density}, \; \kappa(x) = \text{bulk modulus}, \; c_{ijkl}(x) = \text{elastic tensor coefficients}, \; f_i(x, t) = \text{body force} \]
2D acoustic wave equations (pressure-velocity)

\[
\frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} + B \frac{\partial q}{\partial z} = [0, 0, w(t)\delta(x - x_s)]^T (\text{+i.c.'s, b.c.'s})
\]

\[
q = \begin{bmatrix} u \\ v \\ p \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1/\rho(x) \\ 0 & 0 & 0 \\ \kappa(x) & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\rho(x) \\ 0 & \kappa(x) & 0 \end{bmatrix}
\]

- easy to prepare parameters \((\rho, \kappa)\)
- analytic solutions for convergence tests are accessible, e.g., homogeneous medium, two-layer media
- fewer implementation issues \(\Rightarrow\) objective comparison
Numerical methods (time domain wave-field)

**FDTD methods**

✓ industry standard
✓ easy implementation
✓ desirable balance: efficiency and accuracy

two approaches:
- conventional-grid: lead to numerical instabilities for material parameters with high contrast discontinuities
  reference: Alford et al., ‘74
- staggered-grid: better numerical performance and widely used recently; may lead to interface error
  reference: Virieux, ‘84, ‘86
  e.g., 4 grids for 3D pressure-velocity acoustic wave equations
Numerical methods (time domain wave-field)

**DGTD methods**

- specialize in hyperbolic PDEs
- capable of handling complex geometries (boundaries and interfaces)
- explicit semi-discrete form for time dependent PDEs
- high order accuracy: \( o(h^{p+1}) \) for Cartesian grids, \( o(h^{p+1/2}) \) for general grid

**reference:** Lesaint and Raviart, ‘74; Johnson and Pitkaranta, ‘86

- very active in computational electromagnetic and fluid dynamics communities

**reference:** Cockburn and Shu, ‘89; Warburton, ‘99; Käser and Dumbser, ‘06
goal of this project

- apply FDTD and DGTD methods to seismic modeling problems
  - DGTD: implementation based on MIDG developed by Warburton
  - FDTD: staggered-grid FDTD code *iwave* by Terentyev, Vdovina and Symes

- make comparison as objective as possible: measure the CPU time and/or the number of floating point operations for two solutions to roughly have the same accuracy
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DGTD and FDTD Comparison

true solutions not accessible ⇒ Richardson extrapolation for error estimation

- assuming the numerical solution $D(h)$ differs from the analytic solution $\bar{D}$ by $E(h) = Ch^p + O(h^{p+1})$, then

$$E(h) \simeq \frac{D(2h) - D(h)}{2^p - 1}$$

- $p$ can be estimated by having $E(2h)$,

$$p \simeq \log_2 \frac{E(2h)}{E(h)}$$

hardware and system (courtesy of Dr. Warburton)

- single precision floating point
- 2.66GHz Intel Core2 Quad Q9450 CPU
- Linux 2.6.18 kernel
- GNU C compiler version 4.1.2
Square-circle model

- computation domain: $[-500 \, \text{m}, 500 \, \text{m}] \times [-500 \, \text{m}, 500 \, \text{m}]$
- radius of the circle: 125 m
- inside the circle: $\rho = 1000 \, \text{kg/m}^3$, $c = 1000 \, \text{m/s}$
- outside the circle: $\rho = 1500 \, \text{kg/m}^3$, $c = 2000 \, \text{m/s}$
- a point source “▽” at $(0, 250 \, \text{m})$
- 41 receivers “△” at the depth $-250 \, \text{m}$, from $-400 \, \text{m}$ to $400 \, \text{m}$ at interval of $20 \, \text{m}$.
- time span: $[0, 2s]$, traces sampled at temporal interval of $5 \, \text{ms}$. 

![Diagram of square-circle model with computation domain and point source and receivers marked.]
Comparison for square-circle model

### 2-4 staggered-grid FDTD

<table>
<thead>
<tr>
<th>recv location (m)</th>
<th>RMS error (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 m</td>
<td>Grid size 5 m</td>
</tr>
<tr>
<td>2.5 m</td>
<td>Grid size 2.5 m</td>
</tr>
<tr>
<td>2.5</td>
<td>Grid size 2.5</td>
</tr>
</tbody>
</table>

- 33.2 GFLOP, 19 sec
- 2.5 m grid, CFL = 0.4
- 3% RMS error

### DGTD, $N = 4$

<table>
<thead>
<tr>
<th>recv location (m)</th>
<th>RMS error (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 m ~ 28 m</td>
<td>Grid size 12 m ~ 28 m</td>
</tr>
<tr>
<td>6 m ~ 14 m</td>
<td>Grid size 6 m ~ 14 m</td>
</tr>
</tbody>
</table>

- 2465 GFLOP, 760 sec
- Grid size range 6 ~ 14 m, 12,572 eles
- 2% RMS error
2D dome model

- point source with Ricker pulse at (3300 m, 40 m)
- significant energy at 30 Hz or a wavelength of 50 m
- receiver at (2300, 20)m depth, 3 sec

Figure: material wave speed
Comparison for 2D dome model

<table>
<thead>
<tr>
<th>Grid size</th>
<th>2-4 staggered-grid FDTD</th>
<th>DGTD (N = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.25 m</td>
<td>0.625 m</td>
</tr>
<tr>
<td>0.7-1.1 s</td>
<td>4.64%</td>
<td>1.65%</td>
</tr>
<tr>
<td>1.1-1.3 s</td>
<td>12.30%</td>
<td>5.54%</td>
</tr>
<tr>
<td>1.5-1.7 s</td>
<td>20.06%</td>
<td>9.45%</td>
</tr>
<tr>
<td>1.9-2.1 s</td>
<td>28.64%</td>
<td>13.92%</td>
</tr>
<tr>
<td># GFLOP</td>
<td>1.03e+4</td>
<td>8.22e+4</td>
</tr>
<tr>
<td>Time</td>
<td>4125 s</td>
<td>32778 s</td>
</tr>
</tbody>
</table>

By extrapolation

- DGTD, N=2: 2.24e+4 GFLOP for 5% RMS error (2nd order)
- 2-4 staggered-grid FDTD: 1.65e+6 GFLOP on 0.23 m grid for the same accuracy (1st order)
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Interface error

Symes & Terentyev & Vdovina ’08 report, using staggered FDTD

- Richardson extrapolation estimates of relative RMS errors for
  3D dome model ($\Delta x = 5$ m) in various window:
  - 0.7-1.1 s: 20%
  - 1.1-1.3 s: 51%
  - 1.5-1.7 s: 88%
  - 1.9-2.1 s: 120%(!)

E.g., Brown ’84, Symes & Vdovina ’09

Numerical error associated with staggered-grid FDTD of wave propagation in heterogeneous media

- Higher order component corresponding to the truncation error
- A first-order error due to the misalignment between numerical grids and material interfaces (time shift)
Interface error

high order convergence rate are expected for DGTD by

- fitting grid points with the interfaces, i.e., interfaces approximated by segments
  - full order convergence for straight-line interfaces
  - 2nd order convergence for curved interfaces \((N > 1)\)

- local mesh refinement (h-adaptation) near the interfaces, i.e., decreasing the time shift effect

reference: *SPICE project*

interface fitting mesh for dome model
Plane Wave: Ricker’s wavelet

- two different material in \([0, 1800 \ m] \times [-15 \ m, 15 \ m]\)
- interface at \(x = 900 \ m\)
- misaligned mesh:

traces of the true and numerical solutions at 500 \( m \), \( h = 10 \ m \),

- first peak \( \rightarrow \) trace of the direct wave
- second peak \( \rightarrow \) trace of the reflected wave
Plane Wave: Ricker’s wavelet

traces of the true and numerical solutions at $500 \, m, h = 10 \, m$,

![Graph showing true and numerical solutions with traces of direct and reflected waves.](image)

$N = 4$
Interface fitting Mesh Example

- using interface fitting mesh

Traces of the true and numerical solutions at 500 m, \( h = 10 \) m

Trace of the direct wave

Trace of the reflected wave

\( N = 1 \)
Interface fitting Mesh Example

- using interface fitting mesh

traces of the true and numerical solutions at 500 m, \( h = 10 \) m

trace of the direct wave

trace of the reflected wave

\[ N = 2 \]
Local Refined Mesh Example

- using local refined mesh grid

traces of the true and numerical solutions at 500 m, $h = 10$ m

trace of the direct wave

trace of the reflected wave

$N = 1$
Local Refined Mesh Example

- using local refined mesh grid

traces of the true and numerical solutions at 500 m, $h = 10$ m

trace of the direct wave

trace of the reflected wave

$N = 2$
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Low-storage curvilinear DGTD

**motivation**

- DGTD on straight-sided eles ⇒ a sub optimal 2nd order convergence rate when geometry representation not complement the accuracy of schemes
- applications with curvilinear geometry, e.g., material interfaces, boundaries
- DGTD has the flexibility to go beyond straight-sided eles
Curvilinear element

- straight-sided triangle $T$: $\forall x \in T \Rightarrow$ the image of a point $(r, s) \in D = \{(r, s)| -1 \leq r, s; r + s \leq 0\}$ under the linear affine transform,

$$x = -\frac{(r + s)}{2}x_1 + \frac{(1 + r)}{2}x_2 + \frac{(1 + s)}{2}x_3$$

$x_i, i = 1, 2, 3$: vertices of $T$

- curvilinear triangle $\tilde{T}$: $\forall x \in \tilde{T} \Rightarrow$ the image under an isoparametric transform

$$x = \sum_j x_jl_j(r, s)$$

$\{l_j\}$: interpolating Lagrange polynomials on $D$

$\{x_j\}$: interpolating points on $\tilde{T}$
Steps to form curvilinear elements

reference: Hesthaven and Warburton, ‘08

- identify element edges that need to be curved
- reallocate the vertices and facial interpolating points on the curved material interfaces and/or boundaries
- blend the facial deformation on edges into the interior interpolating points through Gordon-Hall blending of face node deformation

reference: Gordon and Hall, ‘73
CurviDG formulation

numerical solutions on curvilinear element $\tilde{T}_k$

$$v(x(r, s), t)|_{\tilde{T}_k} = \sum_j v_j^k(t) l_j(r, s)$$

$$p(x(r, s), t)|_{\tilde{T}_k} = \sum_j p_j^k(t) l_j(r, s)$$

symmetric DG variational equations read

$$\rho_k \sum_j (l_i, l_j) \tilde{T}_k \frac{\partial v_j^k}{\partial t} = \sum_j (\nabla_{x,z} l_i, l_j) \tilde{T}_k \ p_j^k - (l_i, \tilde{n} p^*) \frac{\partial \tilde{T}_k}{\partial t}$$

$$\frac{1}{\kappa_k} \sum_j (l_i, l_j) \tilde{T}_k \frac{\partial p_j^k}{\partial t} = - \sum_j (l_i, \nabla_{x,z} l_j) \tilde{T}_k \cdot v_j^k - (l_i, \tilde{n} \cdot (v^* - v^-)) \frac{\partial \tilde{T}_k}{\partial t}$$
Trouble for storage

the mass matrix $M^k$,

$$M_{ij}^k = \int_{\tilde{T}_k} l_i(r, s) l_j(r, s) \, dx \, dz = \int_D l_i(r, s) l_j(r, s) J^k(r, s) \, dr \, ds$$

the Jacobian $J^k(r, s) = \left| \frac{\partial \mathbf{x}}{\partial r} \times \frac{\partial \mathbf{x}}{\partial s} \right|$ is no longer constant,

- compute $M_{ij}^k$ on the fly $\Rightarrow$ slowdown

- store $M_{ij}^k \Rightarrow$ storage scaled as $K_c \frac{(N + 1)^2(N + 2)^2}{4}$

$K_c$: number of curvilinear elements

$\frac{(N + 1)(N + 2)}{2}$: number of interpolation points
Weighting the variational spaces

weighting approximation space proposed by Warburton

\[ V_h^J = \bigoplus_k \text{span} \left\{ \frac{l_j(r, s)}{\sqrt{J^k(r, s)}} \right\} \]

numerical solution in \( V_h^J \)

\[
\begin{align*}
\mathbf{v}(\mathbf{x}(r, s), t)|_{\tilde{T}_k} &= \sum_j \tilde{v}_j^k(t) \frac{l_j(r, s)}{\sqrt{J^k(r, s)}} \\
p(\mathbf{x}(r, s), t)|_{\tilde{T}_k} &= \sum_j \tilde{p}_j^k(t) \frac{l_j(r, s)}{\sqrt{J^k(r, s)}}
\end{align*}
\]

the mass matrix

\[
M_{ij}^k = \int_D \frac{l_i(r, s)}{\sqrt{J^k}} \frac{l_j(r, s)}{\sqrt{J^k}} J^k(r, s) \, dr \, ds = \int_D l_i(r, s) l_j(r, s) \, dr \, ds = M_{ij}
\]

\[\Rightarrow \text{ without storage trouble}\]
\[
\rho_k \sum_j M_{ij} \frac{\partial \tilde{v}_j}{\partial t} = \sum_j \left( \nabla_{x,z} l_i, l_j \right)_D \tilde{p}_j^k - \sum_j \left( l_i, \frac{l_j}{2} \nabla_{x,z} \log(J^k) \right)_D \tilde{p}_j^k \\
- \left( \frac{l_i}{\sqrt{J^k}}, \mathbf{n} \tilde{p}^* \right) \partial \tilde{T}_k \\
\frac{1}{\kappa_k} \sum_j M_{ij} \frac{\partial \tilde{p}_j^k}{\partial t} = \sum_j \left( l_i, \nabla_{x,z} l_j \right)_D \cdot \tilde{v}_j^k + \sum_j \left( l_i, \frac{l_j}{2} \nabla_{x,z} \log(J^k) \right)_D \cdot \tilde{v}_j^k \\
- \left( \frac{l_i}{\sqrt{J^k}}, \mathbf{n} \cdot (\tilde{v}^* - \tilde{v}^-) \right) \partial \tilde{T}_k \\
e.g., \\
\left( l_i, \frac{l_j}{2} \nabla_{x,z} \log(J^k) \right)_D = \sum_n \omega_n^c l_i(r_n^c, s_n^c) l_j(r_n^c, s_n^c) \nabla_{x,z} \log(J^k)(\mathbf{x}(r_n^c, s_n^c))
\]
\[
\{(r_n^c, s_n^c)\}_n, \{\omega_n^c\}_n: \text{ cubature nodes and weights on the reference element } D
\]
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Square-circle model

nodal distribution of curvi eles of degree 8 near the circular region
Square-circle model

- curvilinear DG, $N=8 \Rightarrow$ optimal convergence rate

<table>
<thead>
<tr>
<th>Grid Size Range</th>
<th>RMS Error (Percent)</th>
<th>Estimated Convergence Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>25 m ~ 56 m</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 m ~ 28 m</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Grid Size Range 25 m ~ 56 m**

- RMS error (percent)
- Recv location (m)

**Grid Size Range 12 m ~ 28 m**

- RMS error (percent)
- Recv location (m)
2D dome model

- 301 receivers at the depth 20 m with offset 100 ∼ 6100 m at interval 20 m
- point source with Ricker pulse at (3300 m, 40 m)
- curvilinear DG, N=5 ⇒ almost optimal convergence rate except the boundary and the source nearby
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Summary

**general realization:**

- staggered-grid FDTD + interface error $\Rightarrow$ 1st order convergence rate

- DGTD, $N > 1 +$ 'good' mesh $\Rightarrow$ 2nd convergence rate (sub-optimal)

- curviDG + 'perfect' mesh $\Rightarrow$ optimal convergence rate
Summary

**Comparison observation:**

- FDTD for 'simple' models well resolves the interface error with less computation cost, e.g., square-circle model

- High order DGTD schemes have substantial advantage on 'complex' models and large time span, e.g., 2D dome model

- Realistic models: need to explore for DGTD with advanced techniques, e.g.,
  - Mesh generation from models defined on Cartesian grid
  - Local mesh refinement + local time stepping
Future plan

- linear elastic wave equations

- seismic interface problems, e.g., water (acoustic) and solid (elastic) interface

- effective and efficient numerical methods for these problems, e.g., FD, FEM/DG
Thank You
Q&A