Harmonic Coordinate Enhanced Finite Element Space: Basis Representation and Accurate Integral Evaluation

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Background

Usual regular grid solution by FDTD inadequate for variable density acoustics in simple cases (Symes and Vdovina, 2008)

Finite difference modifications to rescue regular grid

- Muir et al. (1992) based on Schoenberg-Muir
- Immersed interface method (Leveque and Li, 1994)

Unstructured mesh family such as finite element method (FEM) and discontinuous Galerkin obvious solutions, but leave behind regular grid efficiency favored for large problems

Owhadi and Zhang (2006) present scale-free method based on change of variables using σ-harmonic map of Alessandrini and Nesi (2001). The 1D case is an old story seen in homogenization (Bensoussan et al., 1978) and inversion (Bamberger et al., 1979).

Previously...

Applied harmonic coordinates to build finite element approximation space in 1D.

Recovered optimal convergence for Dirichlet problem and scalar acoustic wave equation

Showed the approximation space obtained by composition is equivalent to immersed finite element method (IFEM) constructed by Li (1998)

Found that FEM is “okay” for low contrast density

Mass lumped IFEM solution of the acoustic wave equation convergences with optimal rate (constant density case, see Igor's talk)

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Review: Acoustic Wave Equation

Scalar acoustic wave equation with nonconstant density:

$$\frac{1}{\kappa}\frac{\partial^2 p}{\partial t^2} - \nabla \cdot \left(\frac{1}{\rho}\nabla p\right) = f \qquad p \equiv 0, t \ll 0$$

with appropriate boundary & initial conditions.

Solution is continuous even when ρ, κ piecewise constant

Constant density case is has twice weakly differentiable solutions, but jumps in density mean jumps in first derivative of p

Approximation with $\mathbb{P}_1, \mathbb{Q}_1, \ldots$ elements on regular grids is not optimal for general interfaces.

Elements are required that preserve subgrid information to have accurate approximation

Harmonic Coordinates and Effective Media

Effective medium can be established by coordinate change and chain rule:

Suppose F is an invertible coordinate transformation. Let $p(x) = \widetilde{p}(F(x))$, where $\widetilde{p} = \widetilde{p}(y)$, and $y = F(x)$. Then

$$\frac{\partial p}{\partial x_i} = \sum_j \frac{\partial F_j}{\partial x_i} \frac{\partial \widetilde{p}}{\partial y_j} \circ F$$

Spatial term from acoustic wave equation becomes

$$\nabla \cdot \frac{1}{\rho} \nabla p = \sum_j \left[\nabla \cdot \frac{1}{\rho} \nabla F_j \right] \frac{\partial \widetilde{p}}{\partial y_j} \circ F + \sum_{j,k} \left[\frac{1}{\rho} \nabla F_j \cdot \nabla F_k \right] \frac{\partial^2 \widetilde{p}}{\partial y_j \partial y_k} \circ F$$

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Harmonic Coordinates and Effective Media

Set

$$a_{jk} = \left[\nabla F_j \cdot \frac{1}{\rho} \nabla F_k\right] \circ F^{-1}$$

and $\widetilde{\kappa} = \kappa \circ F^{-1}$. Then $\widetilde{p}$ solves

$$\frac{1}{\widetilde{\kappa}} \frac{\partial \widetilde{p}}{\partial t^2} - \sum_{j,k} a_{jk} \frac{\partial^2 \widetilde{p}}{\partial x_j \partial x_k} = f \circ F^{-1}$$

provided F is ρ-harmonic:

$$\nabla \cdot \frac{1}{\rho} \nabla F_j = 0.$$

Harmonic Coordinates

Special case of σ-harmonic map. Suppose F solves the Dirichlet problems

$$\nabla \cdot \left(\frac{1}{\rho} \nabla F_j \right) = 0,$$

with

$$F_j = x_j, \qquad \text{on the domain boundary}, \qquad j = 1, 2.$$

Then F is a coordinate transformation

This is guaranteed in 2D (see Alessandrini, 2001), but pathological cases exist in 3D that prevent bijectiviy (see Owhadi and Zhang, 2006).

Harmonic Coordinates and Effective Media

The acoustic wave equation

$$\frac{1}{\widetilde{\kappa}}\frac{\partial\widetilde{p}}{\partial t^2} - \sum_{j,k} a_{jk}\frac{\partial^2 \widetilde{p}}{\partial y_j \partial y_k} = f \circ F^{-1}$$

is now in *non-divergence form*, which means there is a strong solution, i.e., twice weakly differentiable. Recovered smoothness by coordinate transform!

Recovered smoothness means recovered optimal convergence by finite elements.

Coefficient matrix must satisfy certain conditions to guarantee solvability (Maugeri et al., 2000, and references therein)

Natural to assume this transform can be used to construct finite element approximation space.

Harmonic Coordinates: 1D Example

Let domain $\Omega = [0, 1]$. Suppose constants $\rho_1 \neq \rho_2 \in \mathbb{R}^+$ and $\alpha \in (0, 1)$. Define the density as

$$\rho(x) = \begin{cases} \rho_1, & x < \alpha \\ \rho_2, & x \geq \alpha \end{cases}$$

Then seek the solution F to

$$\frac{d}{dx}\left(\frac{1}{\rho}\frac{dF}{dx}\right) = 0,$$

with $F(0) = 0$ and $F(1) = 1$.

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Harmonic Coordinates: 1D Example

The harmonic map in 1D for single interface is

$$F(x) = \frac{1}{M}\begin{cases}\rho_1 x, & x < \alpha \\ \rho_1\alpha + \rho_2(x-\alpha), & x \geq \alpha\end{cases}$$

where $M = \rho_1\alpha + \rho_2(1-\alpha)$.

Clearly,

- $F(0) = 0$ and $F(1) = 1$,
- F is continuous
- and bijective $\Rightarrow$ invertible

Thus, F is coordinate transformation.

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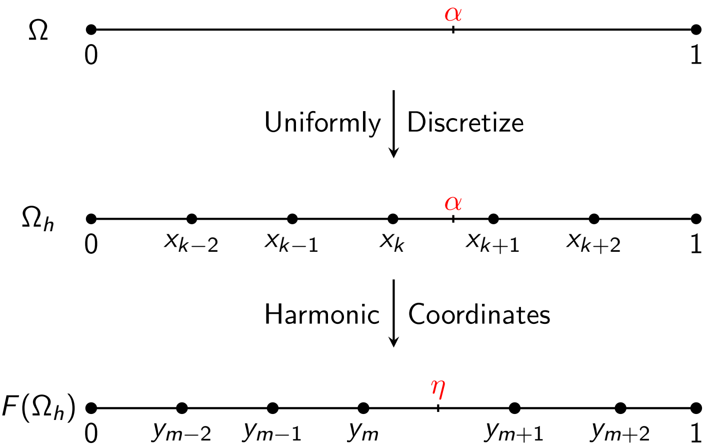
Harmonic Coordinate FEM 1D

To build the basis functions:

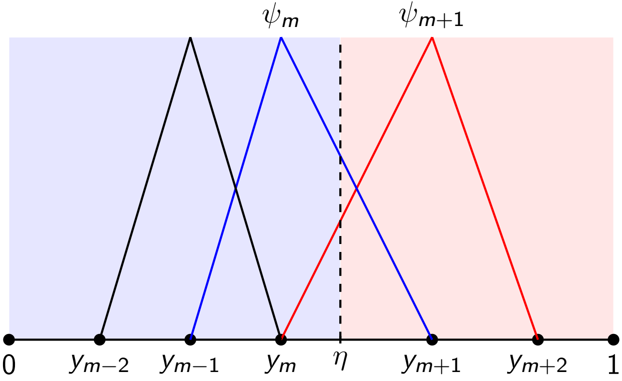
- Uniformly discretize the domain: $\Omega \to \Omega_h$
- Map Ω_h using F to make harmonic grid $F(\Omega_h)$
- Construct the usual nodal basis functions $\phi_j(y)$ in $F(\Omega_h)$
- Pull-back to Ω_h by composition $\phi_j \circ F(x)$

Harmonic Coordinate FEM 1D



Finite Element Basis

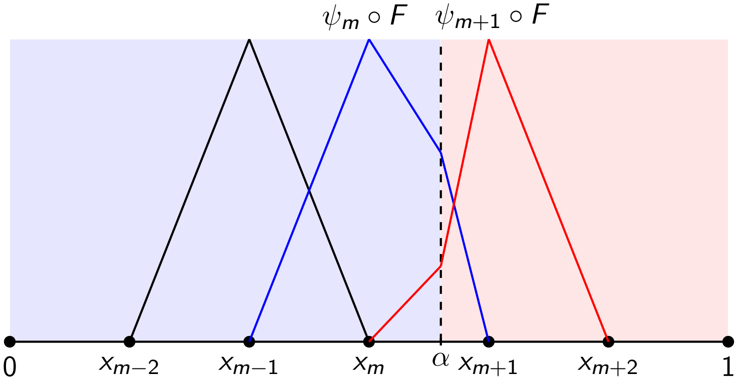
Construct the standard piecewise linear basis $\psi_m(y)$ on the harmonic grid:



The interface in harmonic coordinates is $\eta = F(\alpha)$.

Finite Element Basis

Composing the standard basis $\psi_m(y)$ with the harmonic map F gives "kinky" basis at the interface.

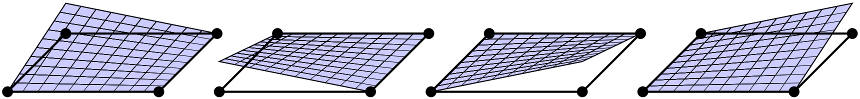


Name the FEM using $\{\psi_m \circ F\}$ as the basis the harmonic coordinate enhanced finite element method (HCE-FEM)

HCE-FEM in 2D

Based on $\mathbb{Q}_1$ finite elements

There are four bilinear basis functions per element



Use the same procedure as 1D to build the composite functions

Major difference: Harmonic map is computed on a very fine grid since no exact solution exists

HCE-FEM in 2D

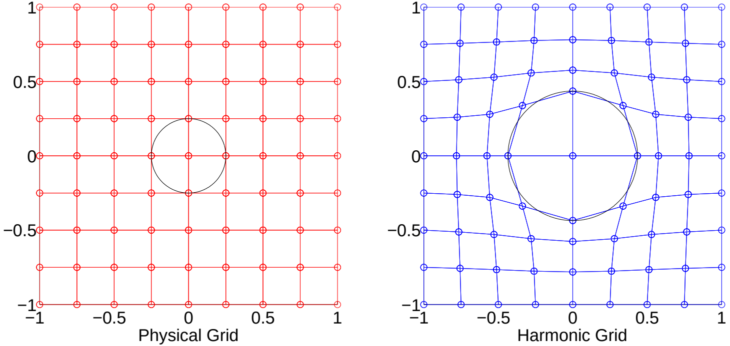
Procedure is the same as the 1D case:

- Domain uniformly discretized using quadrilateral mesh Ω_h
- Map the discrete domain $\Omega_h \to F(\Omega_h)$
- Construct basis functions on harmonic grid quadrilaterals
- Composition $\psi_m \circ F$ provides basis on Ω_h

This is the algorithm proposed by Owhadi and Zhang.

HCE-FEM in 2D

Map the coarse physical grid (left) under the harmonic map to generate a coarse harmonic grid (right). Example has a circular inclusion of radius $r_0 = \pi/12.56$.



Basis functions ψ_m are defined on harmonic grid elements.

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Example: Circular Inclusion 2D

Suppose domain is bi-unit square $[-1,1] \times [-1,1]$ with a circle radius $r_0 = \pi/12.56$ centered at the origin. Find u such that

$$\nabla \cdot \left(\frac{1}{\rho} \nabla u \right) = 9r$$

where $r = \sqrt{x^2 + y^2}$ and

$$\rho(x,y) = \begin{cases} \rho_1, & \text{inside circle,} \\ \rho_2, & \text{outside.} \end{cases}$$

This problem has a closed form solution.

The boundary condition is determined by the solution.

Example: Circular Inclusion 2D

Convergence study with densities $\rho_1 = 10$, $\rho_2 = 1$.

Error estimate proportional h^r where h is the grid size and r is the estimated rate of convergence. The optimal rate for $\mathbb{Q}_1$ finite elements is $r = 2$.

	L_∞ Relative Nodal Error (%)	
h	FEM	HCE-FEM
1/4	3.6244e-001	2.9482e-001
1/8	7.8505e-001	1.1200e-001
1/16	5.7880e-001	3.1088e-002
1/32	3.4700e-001	1.0103e-002
1/64	1.7778e-001	4.9859e-003
r	0.72*	1.52

HCE-FEM: $\|u - u_h\|_\infty \approx Ch^{1.52}$ *Suboptimal!*

* First datum igonored when compuing the rate.

What went wrong?

Suboptimal nodal L_∞ errors indicate nonconforming basis

Turns out basis functions are nonconforming because the support in the physical grid is assumed quadrilateral

Two ways to adjust for this:

- Approximate the harmonic grid elements faithfully
- Approximate the physical support of the basis function accurately

The harmonic grid is where the basis is constructed, so distorted elements would make constructing $\mathbb{Q}_1$ functions impossible.

Toward Accurate Integration

The true basis support in physical coarse grid is distorted because the true harmonic grid elements are approximated by quadrilaterals

Accurate representation of the distorted physical grid elements is needed for accurate integration

How can we accurately represent these integration domains?

New Quadrature: Schematic Explanation

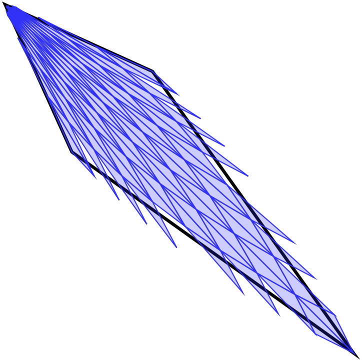
Propose a new quadrature scheme that takes advantage of fine grid used to compute F

Use a “map and mark” procedure to determine basis support in the physical grid:

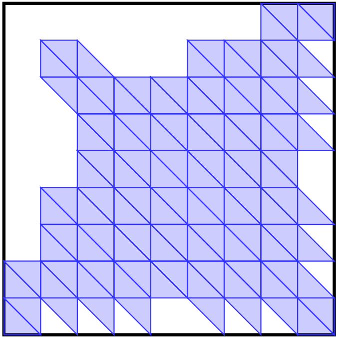
- “Map” fine grid to harmonic coordinates
- “Mark” the fine harmonic grid elements within coarse quadrilaterals in $F(\Omega_h)$

The marked indices are associated one-to-one with physical fine grid elements.

Examples of "Map and Mark"

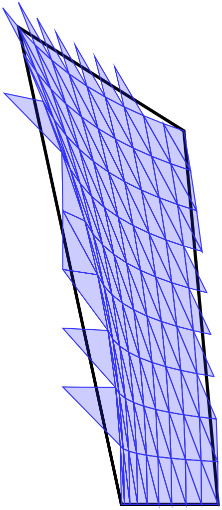


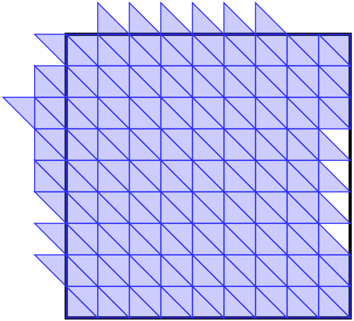
Harmonic Grid Element $F(K)$



K and distorted domain $\widetilde{K}$

Examples of “Map and Mark”





Summary

Identified problem with 2D HCE-FEM

Next step is implementation of the new “map and mark” quadrature scheme

Investigate how accuracy is affected by the solution to the harmonic map subproblem

Interface representation

Elasticity

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