The Conditioning of the Normal Operator for Variable Density Acoustics

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January 29, 2010 / TRIP Annual Meeting

Linear Inverse Scattering

- m: model (material parameters: velocity, impedance, density ...)
- Write $m = m_0 + \delta m$
  m_0: Reference or macro model (given - result of model building, velocity analysis, ...)
  δm: First order perturbation about m_0 (to be found)
- Linear (Born) modeling operator $F[m_0]$, models primary reflections
- Linear inversion: given observed data traces S^{obs}, background traces S_0, find δm so that:

$$F[m_0]\delta m \approx S^{obs} - S_0 := d$$

- Form of AVO analysis
- Component in FWI algorithms

Normal Equations

Interpret as least squares problem: need to solve normal equations

$$N[m_0]\delta m := F^*[m_0]F[m_0]\delta m = F^*[m_0]d$$

$N := F^*[m_0]F[m_0]$: Normal Operator (Modeling + Migration),
$b := F^*d$: migrated image

- *Large Scale*: millions of equations/unknowns, also $\delta m \rightarrow N\,\delta m$ expensive
- Cannot use Gaussian elimination $\Rightarrow$ need rapidly convergent iteration $\Rightarrow$ good preconditioner
- Not narrow band (like Laplace in 2D/3D) $\Rightarrow$ matrix preconditioners ineffective
- Alternative: low order polynomial preconditioner, not obvious

Agenda

- How to build effective low degree polynomial preconditioner

$$N\delta m = b \Rightarrow \delta m \simeq \sum_{i=1}^{p} c_i N^{i-1} b,$$

  - p number of material parameters
  - c_i operators, cheap to apply
  - c_i computable by rapidly converging iteration
- Cost: few Modeling/Migration iterations
- Justification
- Applicability/limitations

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ΨDOs and their symbols

- N is matrix of ΨDOs for *smooth* (non-reflective) m_0 (Beylkin,1985; Rakesh,1988) = operators defined by symbols $a(x,\xi)$

$$Op(a)u(x) = \int\int a(x,\xi)u(y)e^{-i[(x-y).\xi]}\, d\xi\, dy,$$

- $a(x,\xi)$: Scalar function of position x and wavenumber ξ

$$|a(x,\xi)| = \mathcal{O}(|\xi|^m), \quad \text{as } |\xi| \to \infty;$$

$m = ord(a) := ord(Op(a))$

- Calculus of scalar symbols:
  1. $Op(\alpha_1 a_1 + \alpha_2 a_2) = \alpha_1 Op(a_1) + \alpha_2 Op(a_2), \quad \alpha_1, \alpha_2$ scalars
  2. $Op(a_1 a_2) \simeq Op(a_1)Op(a_2) \simeq Op(a_2)Op(a_1)$
  3. $ord(a_1 a_2) = ord(a_1) + ord(a_2)$

  ($\simeq$: difference is lower order ΨDO)

Properties of Normal Operator

- Matrix of pseudodifferential operators, when a polarized signal is scattered uniquely to another polarized signal (P-P, P-S, S-S). (Beylkin and Burridge, 1989; De Hoop, 2003)
- $N = Op(A)$, $A = p \times p$ matrix of scalar symbols

$$Op(A)u(x) = \int \int A(x,\xi)u(y)e^{-i[(x-y).\xi]}\, d\xi\, dy,$$

Polynomial Approximate Inverse

- $A(x, \xi)$ is $p \times p$ matrix: satisfies its own characteristic equation (Cayley-Hamilton):

$$I - \sum_{i=1}^{p} a_i(x,\xi) A^i(x,\xi) = 0,$$

  where $a_i(x,\xi)$ are symbols
- Inverse of $A(x,\xi)$: polynomial of degree $p-1$ in A:

$$I = \left(\sum_{i=1}^{p} a_i(x,\xi) A^{i-1}(x,\xi) \right) A(x,\xi)$$

- Symbol calculus $\Rightarrow \exists$ scalar ΨDOs $\{\bar{c}_1, \ldots, \bar{c}_p\}$ s.t., $\bar{c}_i = Op(a_i)$, $N^\dagger \approx$ "polynomial" of degree $p-1$:

$$I \approx \left(\sum_{i=1}^{p} Op(a_i) op(A^{i-1}) \right) op(A) \approx \left(\sum_{i=1}^{p} \bar{c}_i N^{i-1} \right) N$$

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Solved Problem . . . Not Yet!

- Don't know symbol A of N
- Only have ability to apply N (modeling + migration)
- Not really a polynomial: coefficients are operators!
- Need an independent method to determine coefficient operators, and must be able to apply efficiently

Polynomial Preconditioning

- Don't need $N^\dagger$, only need to solve $Nx = b$, $b = F^*d$
- Approximation of c_i in data adaptive way:

$$\{c_1, \ldots, c_p\} = \underset{c_1,\ldots,c_p \in \Psi DO}{\textit{argmin}} \left\| \left(I - \sum_{i=1}^{p} c_i N^i \right) b \right\|^2 .$$

Know from Cayley-Hamilton that min ≈ 0 (for $c_i = \bar{c}_i$)

- Get approximate solution:

$$x = N^{-1} b \approx N^{-1} \sum_{i=1}^{p} c_i\, N^i\, b \approx \sum_{i=1}^{p} c_i\, N^{i-1}\, b := x_{inv}$$

Approximation of ΨDO

How to represent c_i?

- The action of the ΨDO in 2D (Bao and Symes, 1996):

$$Op(a)\,u(x,z) \approx \int\int a(x,z,\xi,\eta)\hat{u}(\xi,\eta)e^{i(x\xi+z\eta)}\,d\xi\,d\eta$$

  $\hat{u} = \mathcal{F}[u]$.

- Direct Algorithm $O(N^4\log(N))$ complexity ($N = \mathcal{O}(10^3)$)!
- Finite Fourier series of length K:

$$a(x,z,\xi,\eta) \approx \sum_{l=-K/2}^{l=K/2} \hat{a}_l(x,z)e^{il\theta},$$

$$\theta = \arctan\left(\frac{\eta}{\xi}\right)$$

  - Use FFT $\Rightarrow O(KN^2[\log(N)+\log(K)])$
  - K independent of N, depends on smoothness of a
  - θ captures dip-dependence

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Recap

To solve

$$Nx = b,$$

where $N = F^*F$, $b = F^*d$.

Given, $b = F^*d, \ldots, N^p b$

- Represent $c_i = Op(a_i)$
- Compute $\{c_1, \ldots, c_p\} = \underset{c_1,\ldots,c_p \in \Psi DO}{\textit{argmin}} \left\| \left(I - \sum_{i=1}^{p} c_i N^i \right) b \right\|^2.$
- Approximate $x_{inv} := \sum_{i=1}^{p} c_i \, N^{i-1} \, b \approx N^{-1} b = x$

Scaling Methods

- $p = 1$:
  - NO $\approx$ multiplication by a smooth function (Claerbout and Nichols, 1994; Rickett, 2003)
  - Near Diagonal Approximation of NO (Guitton, 2004)
  - Special case (well defined dip): normal operator $\approx$ multiplication by smooth function after composition with power of Laplacian (correction to Claerbout-Nichols - Symes, 2008)
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  - Herrmann et al. (2007) derive a scaling method using *curvelets* to approximate eigenvectors
- $p > 1$:
  - New explanation
  - Old example
  - New conditioning study

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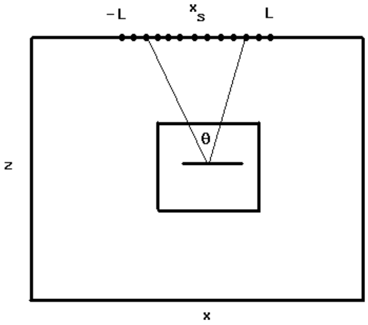
Example: Multi-parameter Case, $p = 2$

Example: Variable density acoustics, impedance and density. Formally the same, solve

$$Nx = b$$

- N is a 2×2 matrix of pseudodifferential operators
- $b = F^* d$ consists of two images, one for each parameter

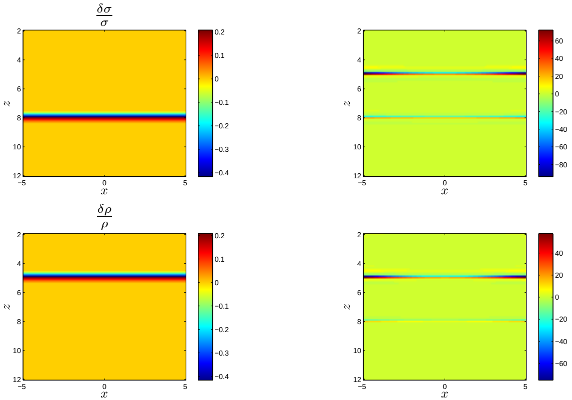
Geometry



N calculated analytically for variable density acoustics, constant background velocity.

The Challenge: Separation

Build model and perturbations, use analytical formula for N to get $b = Nx$



True model: x

Mig images: b

Polynomial Preconditioning

To solve

$$Nx = b,$$

where $N = F^*F$ and $b = F^*d \in \mathit{Range}(N)$.

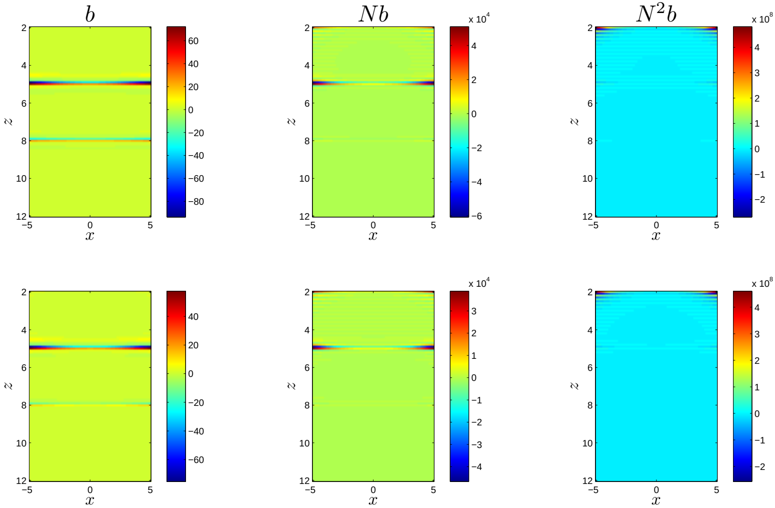
Given b, Nb and N^2b. Compute c_1, c_2:

$$\{c_1, c_2\} = \underset{c_1, c_2 \in \Psi DO}{\mathit{argmin}} \|b - c_1\, Nb - c_2\, N^2 b\|^2.$$

Then,

$$x = N^{-1}b \approx N^{-1}(c_1\, Nb + c_2\, N^2 b) \approx c_1 b + c_2\, Nb := x_{inv}$$

What to expect from N



Conditioning of N

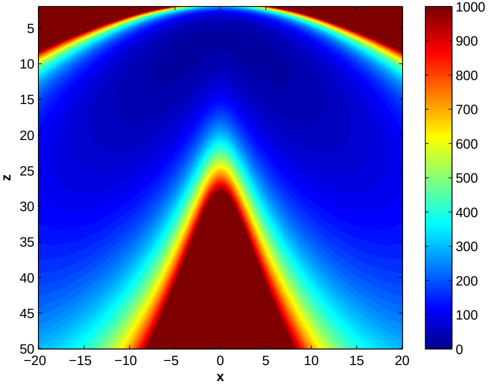


Figure: spatial variation of the condition number of the symbol of N

Preconditioning the Preconditioner

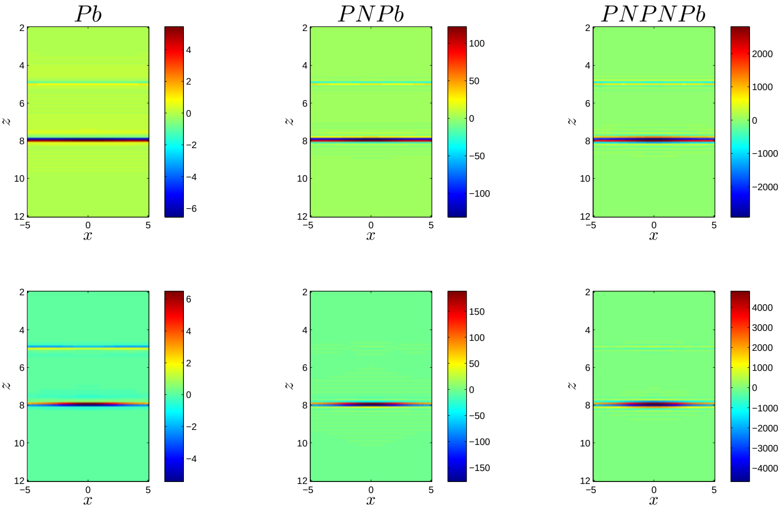
Compute a preconditioner $P \approx N^{-1}$ using full aperture. Then,

- $b \longrightarrow Pb$
- $Nb \longrightarrow PNPb$
- $N^2b \longrightarrow PNPNPb$

Compute polynomial preconditioner:

- $\{c_1, c_2\} = \underset{c_1, c_2 \in \Psi DO}{\textsf{argmin}} \|Pb - c_1\,PNPb - c_2\,PNPNPb\|^2$
- $x = N^{-1}b = (PN)^{-1}Pb \approx c_1\,Pb + c_2\,PNPb := x_{inv}$

Preconditioned Images



Results

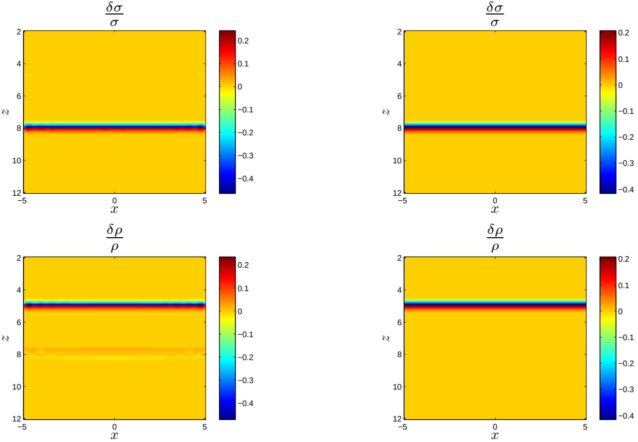


Figure: Comparison between inverted and true image

Conditioning of NP

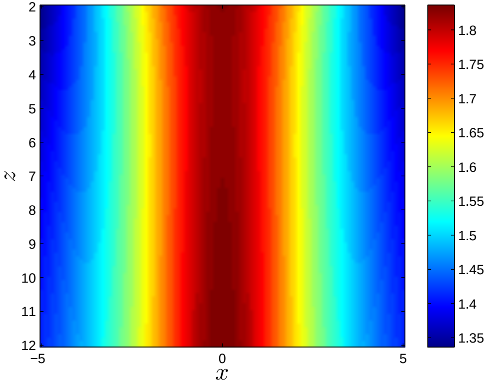


Figure: spatial variation of the condition number of the symbol of NP

Conditioning of symbol of N, continued

- Previous preconditioner specific to problem
- Find more general preconditioner?
- The symbol of $N = op(A)$ for variable density acoustics has the form:

$$A = f(\theta) \begin{pmatrix} 1 & \sin^2(\frac{\theta}{2}) \\ \sin^2(\frac{\theta}{2}) & \sin^4(\frac{\theta}{2}) \end{pmatrix} |\xi|$$

- Opening angle θ: function of position of sources, receiver, and spatial coordinates.
- Ill-conditioning of N captured by the matrix part

Goal: Optimal Weights

- Study conditioning of matrices of the form

$$N = \int_0^{\theta_{max}} d\theta f(\theta) \begin{pmatrix} 1 & \sin^2(\frac{\theta}{2}) \\ \sin^2(\frac{\theta}{2}) & \sin^4(\frac{\theta}{2}) \end{pmatrix}$$

- Minimize the condition number:

$$\kappa = \frac{\lambda_{max}}{\lambda_{min}}, \quad s.t.\ f \geq 0,\ \int_0^{\theta_{max}} f(\theta)\, d\theta = 1$$

- Parametrize in terms of : $S = \lambda_{max} + \lambda_{min} = trace(N)$ and $P = \lambda_{max}\lambda_{min} = det(N)$
- $$\kappa = \frac{S + \sqrt{S^2 - 4P}}{S - \sqrt{S^2 - 4P}}$$

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Reference Case

- Let $f(\theta) = \frac{1}{\theta_{max}}$

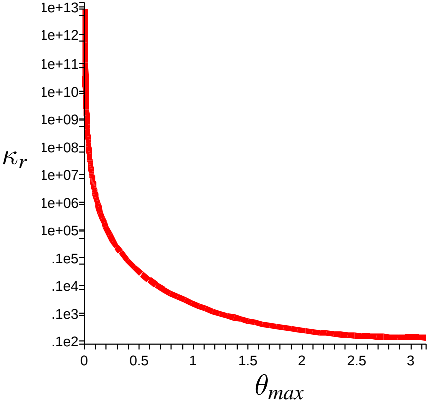

Figure: Condition Number as a function of θ_{max}

Optimal Low Offset/Large Offset Stack

- Look for optimal low offset/large offset stack:

$$f(\theta) = (1-\alpha)\delta(\theta) + \alpha\delta(\theta - \theta_{max}), \quad 0 \leq \alpha \leq 1$$

- Minimizing κ, letting $\beta = \sin^4(\frac{\theta_{max}}{2})$:

$$\alpha = \frac{1}{2+\beta},$$

$$\kappa_{min} = \frac{\beta + 1 + \sqrt{1+\beta}}{\beta + 1 - \sqrt{1+\beta}}$$

- Note: for large offset ($\theta_{max} \to \pi$), small offsets weighted double!

How Much Better?

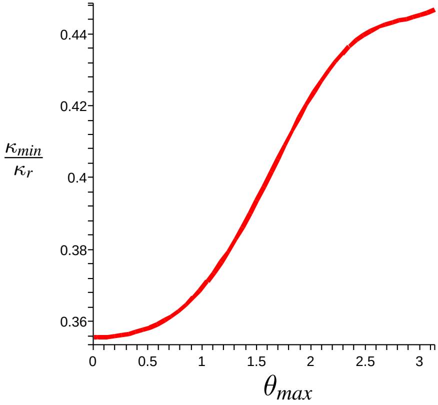

Figure: Ratio of optimal condition number to reference

A Closer Look

For small θ_{max}

- Reference case:
  - $\lambda_{max} = 2 + \mathcal{O}(\theta_{max}^4)$
  - $\lambda_{min} = \frac{\theta_{max}^4}{90} + \mathcal{O}(\theta_{max}^6)$
  - $\kappa_r = \frac{180}{\theta_{max}^4} + \mathcal{O}(\theta_{max}^{-2})$
- Optimal stacks:
  - $\lambda_{max} = 2 + \mathcal{O}(\theta_{max}^4)$
  - $\lambda_{min} = \frac{\theta_{max}^4}{32} + \mathcal{O}(\theta_{max}^8)$
  - $\kappa_{min} = \frac{64}{\theta_{max}^4} + \mathcal{O}(1)$
- Same asymptotics

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First Order Conditions

- First variation of the condition number:

$$\delta\kappa = 0 \Rightarrow 2\frac{\delta S}{S} = \frac{\delta J}{J}$$

- Gives a different parametrization:

$$S^2 = LP \Rightarrow \frac{S^2}{P} = L$$

- With $L \geq 4$,

$$\kappa = \frac{S + \sqrt{S^2 - 4P}}{S - \sqrt{S^2 - 4P}} = \frac{1 + \sqrt{1 - \frac{4}{L}}}{1 - \sqrt{1 - \frac{4}{L}}}$$

- Minimizing $\kappa \Leftrightarrow$ Minimizing $\frac{S^2}{P}$

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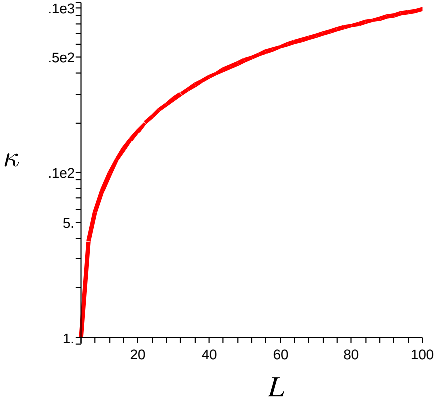

Figure: Condition number as a function of $L = \frac{S^2}{P}$

Future Work

- Derive a class of preconditioners for different geometries
- Precondition a RTM code for variable density acoustics
- Apply for variable density acoustics
- Generalize to linear elasticity
- Extend to 3D

Summary

Multi-parameter case: Polynomial Preconditioning

- Necessity of preconditioning for success
- Apply to variable density acoustics
- Intrinsic ill-conditioning in variable density acoustics
- Linear elasticity . . .

Acknowledgments

- Dr. Eric Dussaud
- Dr. Fuchun Gao
- TRIP sponsors
- NSF grant: DMS 0620821

THANK YOU !

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