Discontinuous Galerkin methods (DG) for waves and comparison with Finite Difference methods

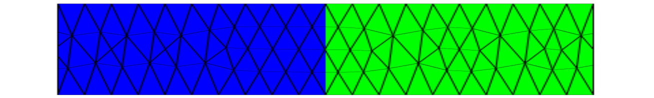
Xin Wang

The Rice Inversion Project

TRIP Annual Meeting 2009

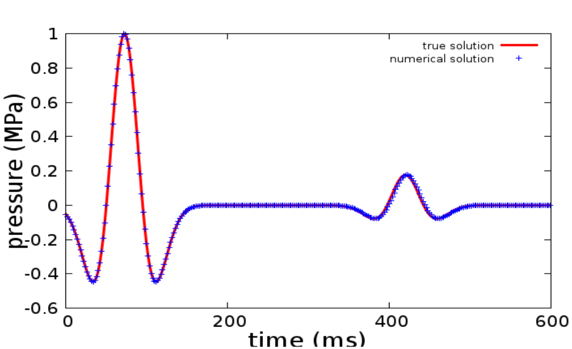
Highlights of this project

- discontinuous Galerkin method (DG) for acoustic wave equation in heterogeneous material (Hesthaven-Warburton) comparison between DG and finite difference method (FD)
- validity and convergence tests in 2-D
  - plane waves
  - point source wave propagation
- unstructured mesh techniques
  mesh misalignment can cause numerical error, e.g., Ricker's wavelet simulation on $[0, 1800\ m] \times [-15, 15\ m]$ with interface at 900 m,



Highlights of this project

traces of the true and numerical solutions at 500 m, $h = 10$ m,
fifth-order DG scheme



red line: true solution blue $+$: numerical solution

- first peak $\rightarrow$ trace of the transmitted wave
- second peak $\rightarrow$ trace of the reflected wave

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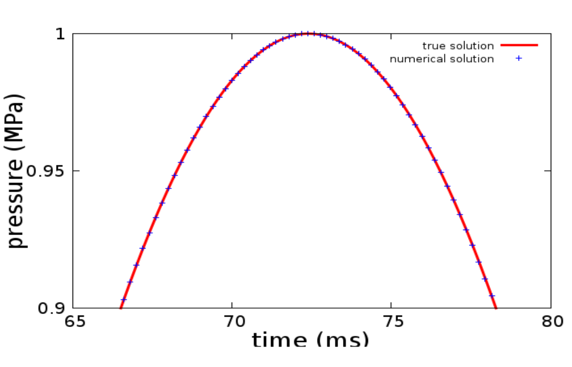

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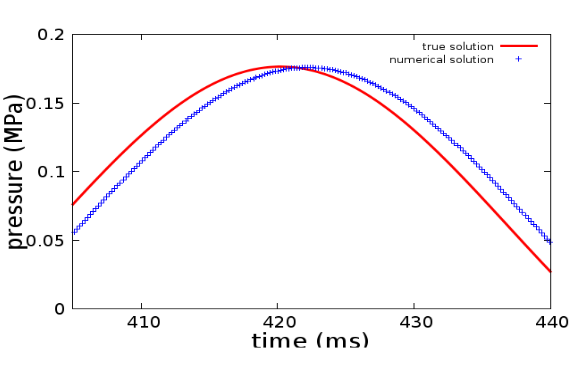

- first peak $\longrightarrow$ trace of the transmitted wave
- second peak $\longrightarrow$ trace of the reflected wave

Highlights of this project

I will propose two techniques to reduce this error:

- mesh alignment for unstructured mesh
- local mesh refinement

Outline

Outline

Acoustic Wave Equation in Heterogeneous Material

2-D case:

$$\begin{aligned}
\rho(x,y)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} &= 0 \\
\rho(x,y)\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} &= 0 \\
\frac{1}{\kappa(x,y)}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= S(x,y,t)
\end{aligned}$$

subject to initial conditions and boundary conditions

- p: acoustic pressure
- (u, v): particle velocity
- S: a source term
- ρ: density; κ: bulk modulus
- t: time variable
- $(x, y) \in \Omega$: spatial variables

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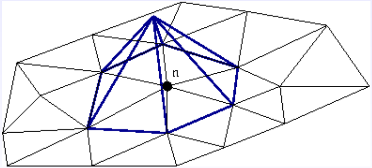
Numerical Method

- FD (iwave by Igor Terentyev, 2008)
  - $\surd$ easy to implement
  - $\surd$ high-order scheme
  - $\surd$ explicit semi-discrete form
  - $\times$ complex geometry
- finite element method - spectral element method (Tromp-Komatitsch)
  - $\times$ easy to implement
  - $\surd$ high-order scheme
  - $(\surd)$ explicit semi-discrete form
    - usually FEM $\rightsquigarrow$ linear system in each time step
    - mass lumping $\rightsquigarrow$ explicit semi-discrete form (Igor's talk)
  - $\surd$ complex geometry

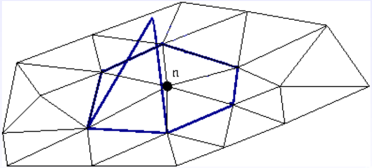
Numerical Method

- discontinuous Galerkin method (DG) (Hesthaven-Warburton)
  - × easy to implement
  - √ high-order scheme
  - √ explicit semi-discrete form

    the support of each basis function over only one element and several basis functions defined on one element
    $\rightsquigarrow$ block diagonal mass matrix
  - √ complex geometry



support of a linear FEM basis



support of a linear DG basis

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Comparison between FD and DG

accuracy comparison:

- different order schemes
- different grid size

efficiency comparison:

- mesh generation and mesh structure
- computation load for updating in each time step
- constraints on the time step size
- memory consumption, memory access pattern and frequency
- parallelism (MPI, CUDA)
- ...

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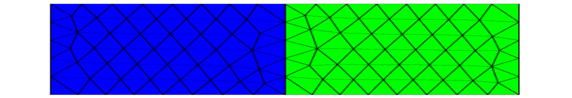
Outline

Plane Wave

- computation domain:



- blue and green stand for two material: $(\rho_L\ c_L)$,$(\rho_R\ c_R)$
- true solution,

$$
\begin{aligned}
x < 0 &\quad : \\
p(x,y,t) &= f(t - \frac{x}{c_L}) - \frac{\rho_L c_L - \rho_R c_R}{\rho_L c_L + \rho_R c_R} f(t + \frac{x}{c_L}) \\
u(x,y,t) &= \frac{1}{\rho_L c_L}(f(t - \frac{x}{c_L}) + \frac{\rho_L c_L - \rho_R c_R}{\rho_L c_L + \rho_R c_R} f(t + \frac{x}{c_L})) \\
x \geq 0 &\quad : \\
p(x,y,t) &= \frac{2\rho_R c_R}{\rho_L c_L + \rho_R c_R} f(t - \frac{x}{c_R}) \\
u(x,y,t) &= \frac{2}{\rho_L c_L + \rho_R c_R} f(t - \frac{x}{c_R})
\end{aligned}
$$

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Plane Wave: Sine Wave

dimensionless example

- computation domain: $[-3,3] \times [-1,1]$
- $$\rho_L = 1.0 \ \rho_R = 0.5$$
  $$\kappa_L = 1.0 \ \kappa_R = 2$$
  $$time = 2$$
- sine wave: $f = \sin(2\pi x)$
- source term $S = 0$;
- initial conditions: true solution at $time = 0$
- boundary conditions:
  - upper and lower → reflection boundary condition
  - left → inflow boundary condition
  - right → outflow boundary condition

Plane Wave: Sine Wave

convergence test

h	N	$\|p_h - p\|_\infty$	$\|u_h - u\|_\infty$	$\|v_h - v\|_\infty$	R
0.2	1	0.2865	0.3232	0.1123	1.84
0.1	1	0.0799	0.1009	0.0303	1.98
0.05	1	0.0203	0.0265	0.0078	-
0.2	2	0.0402	0.0628	0.0204	2.61
0.1	2	0.0066	0.0094	0.0030	2.91
0.05	2	8.76e-4	0.0012	3.95e-4	-

h: grid size

N: polynomial order in DG

$N = 1$: piecewise linear basis function $\Rightarrow$ second order scheme

$N = 2$: piecewise quadratic basis function $\Rightarrow$ third order scheme

$$R = \frac{\log \|p_H - p\| - \log \|p_h - p\|}{\log H - \log h}$$

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Plane Wave: Ricker's Wavelet

dimensional example

- computation domain: $[0, 1800\ m] \times [-15\ m, 15\ m]$
-

$$\rho_L = 2100\ kg/m^3\ c_L = 2.3\ m/ms$$
$$\rho_R = 2300\ kg/m^3\ c_R = 3.0\ m/ms$$
$$time = 600ms$$

- f is a Ricker's wavelet with central frequency $f_0 = 10\ Hz$:

$$f(t) = (1 - 2(\pi f_0(t - t_0))^2)e^{-(\pi f_0(t-t_0))^2},$$

- source term $S = 0$;
- initial conditions: true solution at $time = 0$
- boundary conditions:
  - upper and lower $\rightarrow$ reflection boundary condition
  - left $\rightarrow$ inflow boundary condition
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Plane Wave: Ricker's Wavelet

convergence test

h	N	$\|p_h - p\|_\infty$	$\|u_h - u\|_\infty$	$\|v_h - v\|_\infty$	R
10	1	0.0125	0.0154	0.0045	2.51
5	1	0.0022	0.0037	0.0012	1.86
2.5	1	6.04e-4	1.00e-3	3.14e-4	-
10	2	9.81e-4	0.0014	3.17e-4	2.96
5	2	1.26e-4	1.85e-4	4.14e-5	2.96
2.5	2	1.62e-5	2.34e-5	5.23e-6	-

h: grid size

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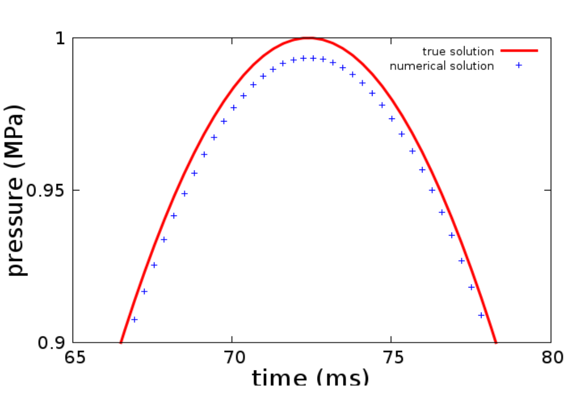
$$R = \frac{\log \|p_H - p\| - \log \|p_h - p\|}{\log H - \log h}$$

Plane Wave: Ricker's Wavelet

traces of the true and numerical solutions at 500 m, $h = 10$ m
$N = 1$ (second-order scheme)



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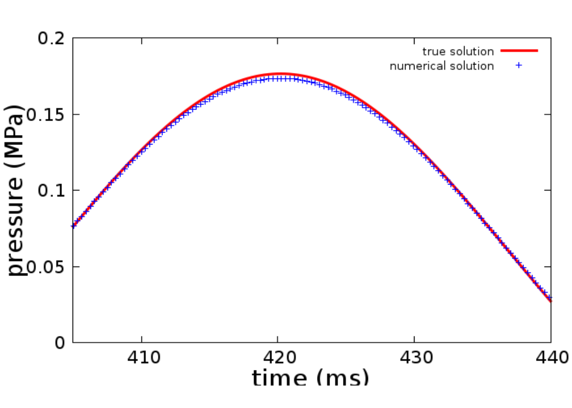
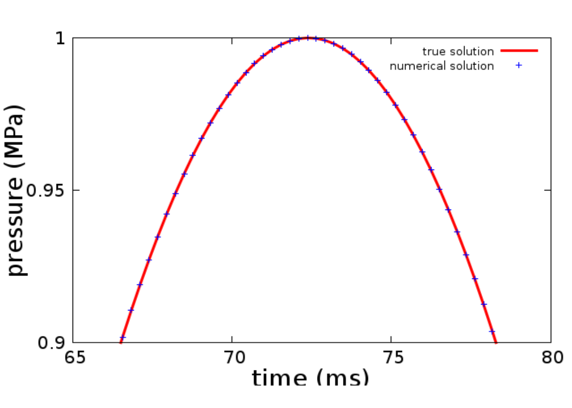


Plane Wave: Ricker's Wavelet

traces of the true and numerical solutions at 500 m, $h = 10$ m
$N = 2$ (third-order scheme)



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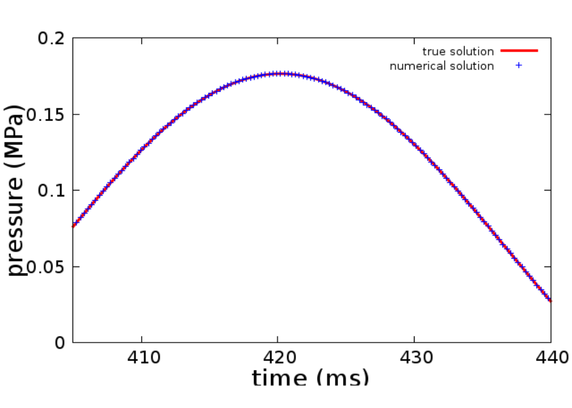
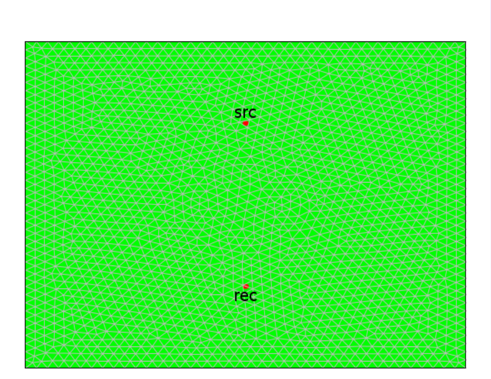


Point Source Wave + Free Surface Boundary

- computation domain: $[-0.5, 0.5] \times [-0.5, 0.5]$
- zero initial conditions; $\rho = 1.0$, $\kappa = 1.0$
- point source at $x_s = (0, 1/4)$, $f_0 = 10$, $t_0 = 1.2/f_0$

$$S(x,t) = (t - t_0)e^{-(\pi f_0 (t-t_0))^2}\delta(x - x_s)$$



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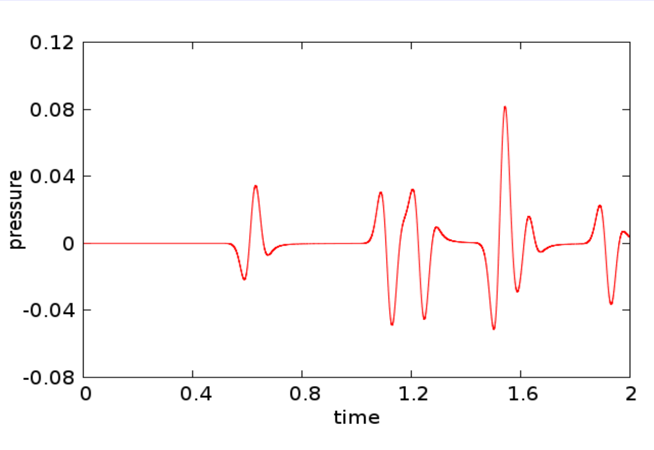
$$S(x,t) = (t - t_0)e^{-(\pi f_0 (t - t_0))^2}\delta(x - x_s)$$

Point Source Wave + Free Surface Boundary

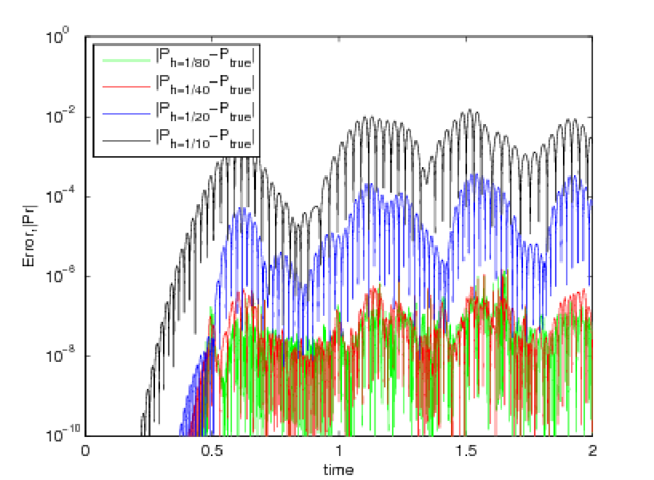
trace of analytic solution at $(0, -0.25)^1$



[1] generated by FORTRAN code acfree.f by Thomas Hagstrom, provided by Dr. Warburton

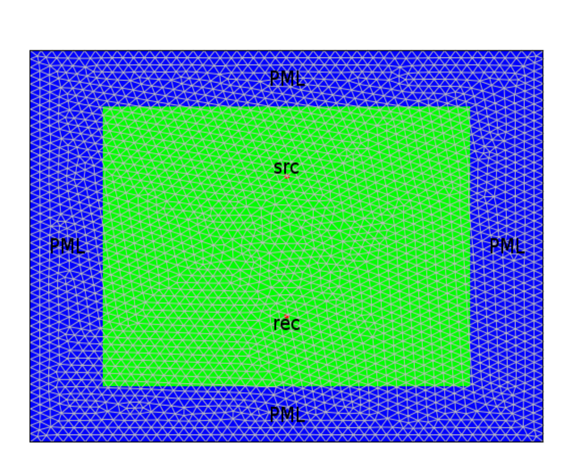
Point Source Wave + Free Surface Boundary

trace error of analytic solution and numerical solution, $N = 5$ (sixth-order scheme)



Point Source Wave + PML Boundary

- computation domain: $[-0.7, 0.7] \times [-0.7, 0.7]$
  non-PML domain: $[-0.5, 0.5] \times [-0.5, 0.5]$



Point Source Wave + PML Boundary

- computation domain: $[-0.7, 0.7] \times [-0.7, 0.7]$
  non-PML domain: $[-0.5, 0.5] \times [-0.5, 0.5]$

Outline

Unstructured Mesh Techniques

numerical error associated with FD of wave propagation in discontinuous media (Symes and Vdovina, 2008):

- higher order component corresponding to the truncation error
- a first-order error due to the misalignment between numerical grids and material interfaces

numerical examples show the numerical error of DG has the same components

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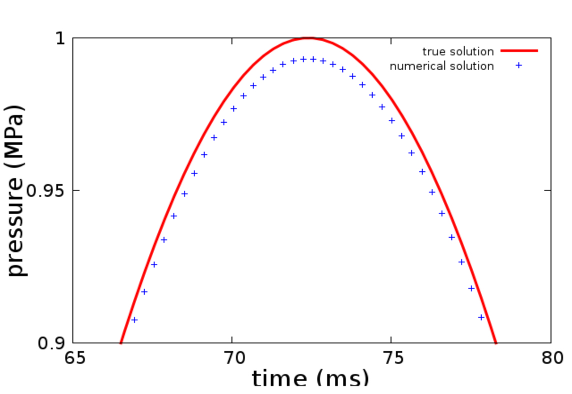
Plane Wave: Ricker's Wavelet

- the setting is the same
- mesh grid does not align with material interface

traces of the true and numerical solutions at 500 m, $h = 10$ m

$N = 1$ (second-order scheme)



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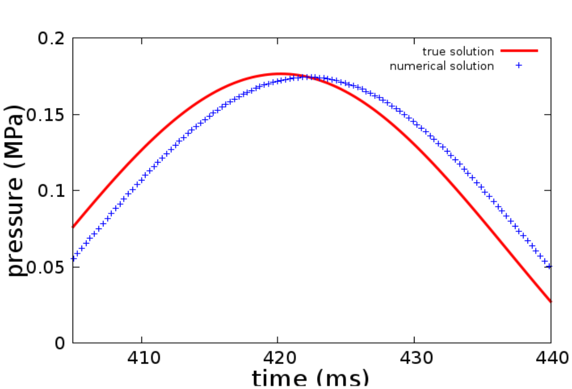

Plane Wave: Ricker's Wavelet

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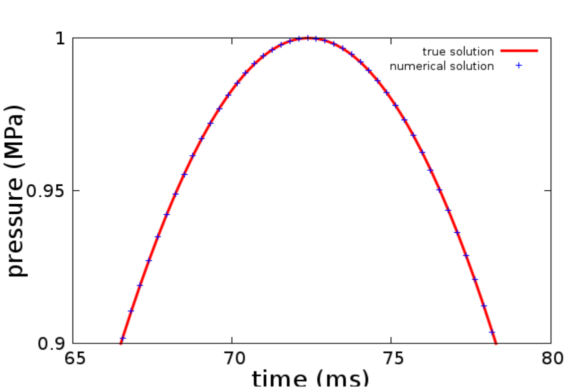
traces of the true and numerical solutions at 500 m, $h = 10$ m

$N = 2$ (third-order scheme)



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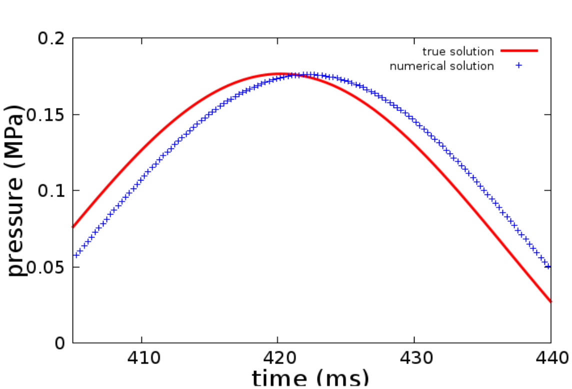

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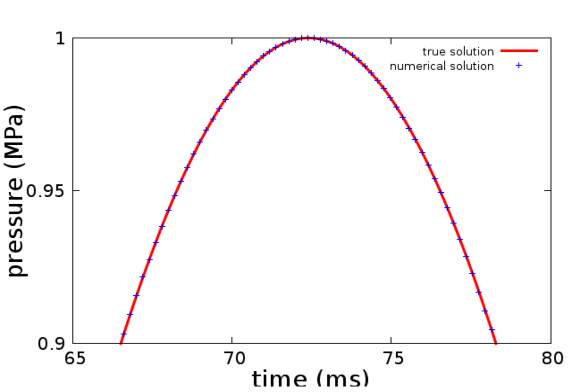
traces of the true and numerical solutions at 500 m, $h = 10$ m

$N = 4$ (fifth-order scheme)



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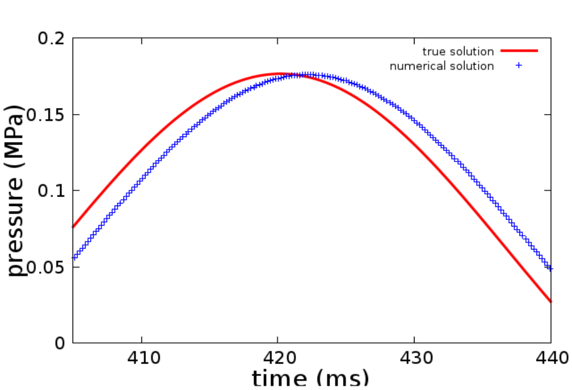
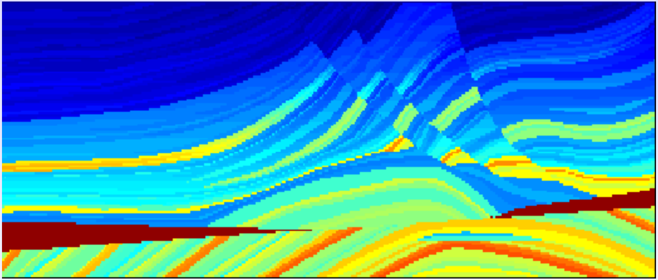


Unstructured Mesh Techniques

two techniques to reduce the error caused by mesh misalignment

- curved element



Marmousi model

difficulties: complex structure, highly curved interface

Unstructured Mesh Techniques

- local mesh refinement near the interface (SPICE project)

refinement procedure

1. initial mesh $\{\mathcal{T}_k\}_k$, $\Omega = \bigcup_k \mathcal{T}_k$
2. compute material contrast indicator $\mathcal{I}_k$ on $\mathcal{T}_k$
3. if $\mathcal{I}_k >$ threshold, refine $\mathcal{T}_k$
4. assemble the new mesh $\{\tilde{\mathcal{T}}_{\tilde{k}}\}_{\tilde{k}}$

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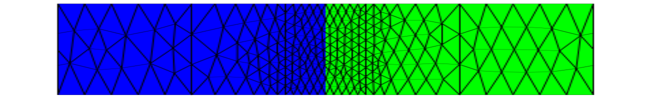
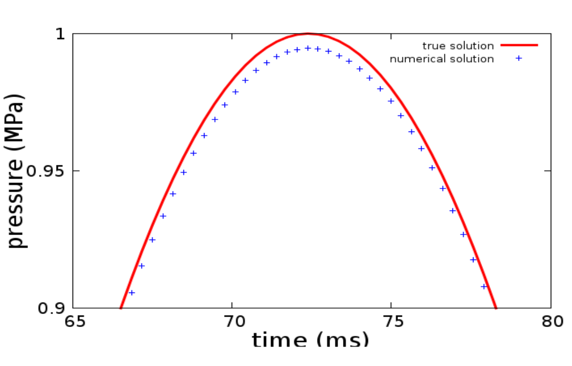
Plane Wave: Ricker's Wavelet

- using the refined mesh grid



traces of the true and numerical solutions at 500 m, $h = 10$ $N = 1$ (second-order scheme)



Plane Wave: Ricker's Wavelet

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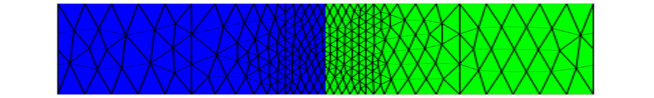

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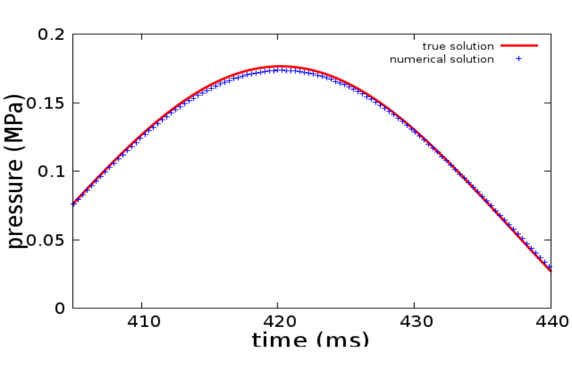

Plane Wave: Ricker's Wavelet

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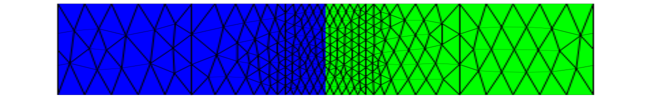
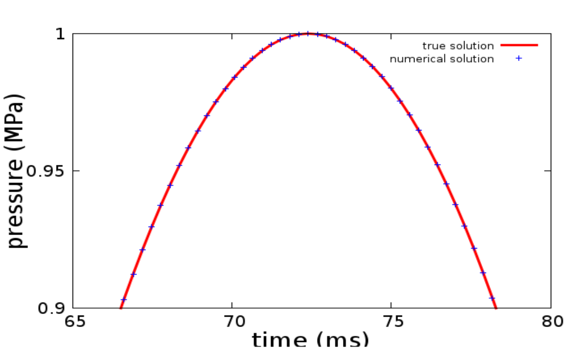


traces of the true and numerical solutions at 500 m, $h = 10$ $N = 2$ (third-order scheme)



Plane Wave: Ricker's Wavelet

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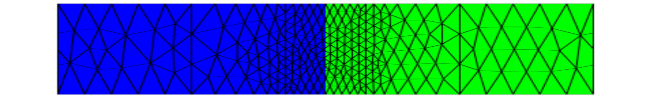

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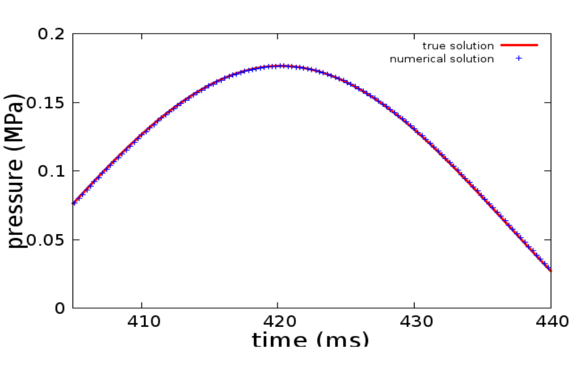

summary:

- discontinuous Galerkin method for solving acoustic wave equation
- convergence tests: plane wave and point source wave
- mesh techniques for reducing numerical error

future work:

- a framework for comparison between FD and DG
- local mesh refinement
- 3-D examples

Thank You!

Thanks

- Prof. Tim Warburton (CAAM, Rice Univ.)