Solving Interface Problems with Finite Elements

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The Rice Inversion Project

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Introduction

Solve the acoustic wave equation (AWE) accurately in $3d$ on regular grids without aligning interfaces.

- Solve for Reflected Waves
- Validate Inversion Algorithms

Overview of Interface Methods

Finite Difference IBM & IIM

- Peskin, 1972
- Leveque and Li, 1994
- Zhang and Leveque, 1997

FD is State-of-the-art
Poor Accuracy without IIM, Symes and Vdovina, 2008
Complicated Implementation
Convergence Theory Messy

Overview of Interface Methods

Immersed Finite Element

- Li, 1998
- Kafafy, 2005

Elliptic and Parabolic Problems
Second Order

Acoustic Wave Equation in 1d

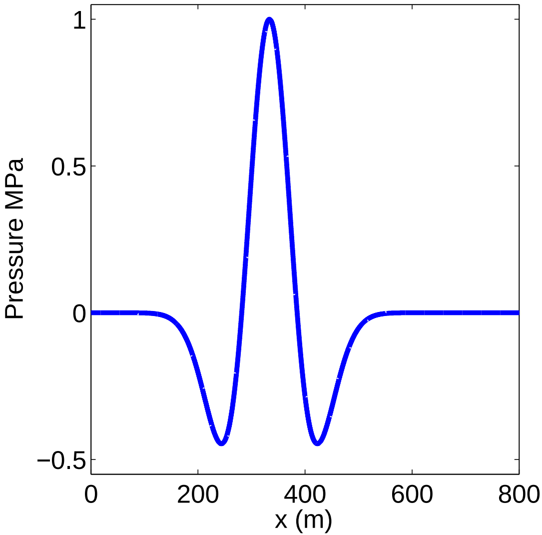
$$\frac{1}{\kappa}\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial u}{\partial x}\right) = 0$$

- Bulk Modulus κ
- Density ρ

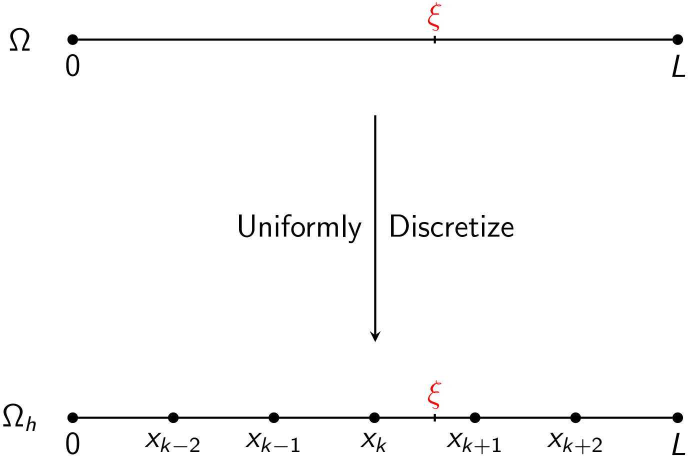
Density and bulk modulus piecewise constant.

- Homogeneous Dirichlet B.C.
- Ricker Wavelet I.C.

Ricker Wavelet

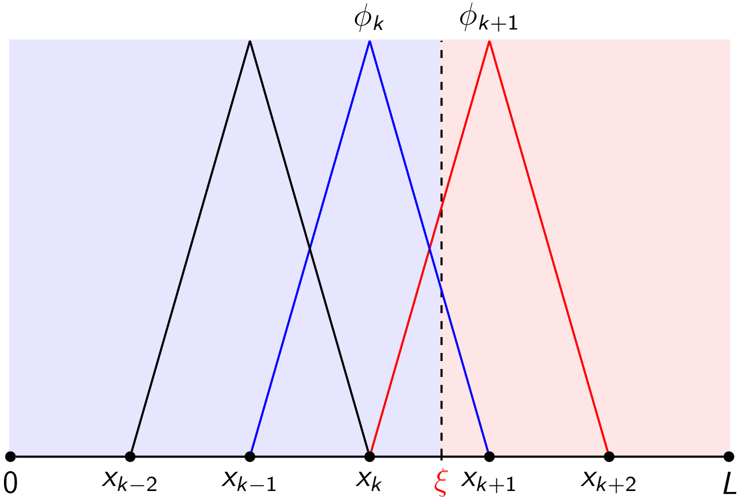


Basic FE Approach



Grid spacing $x_k - x_{k-1} = h$.

Basic FE Approach



Basis property: $\phi_k(x_j) = \delta_{kj}$.

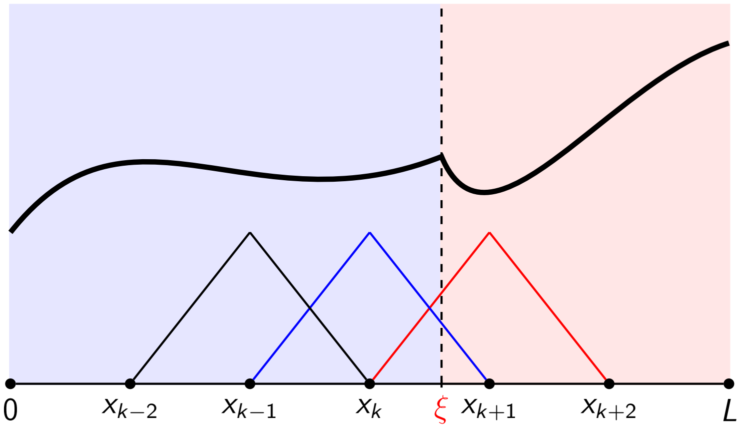
Semi-discrete Equation

$$\mathcal{M}\frac{d^2U}{dt^2} + \mathcal{S}U(t) = 0$$

or

$$\frac{d^2U}{dt^2} = -\mathcal{M}^{-1}\mathcal{S}U(t)$$

Basic FE Approach - Problem



Goals

- Local Modification
- No Grid Tinkering
- Simple Implementation
- Good Convergence Theory

Owhadi and Zhang, 2006 harmonic coordinates.

ρ-Harmonic Coordinates

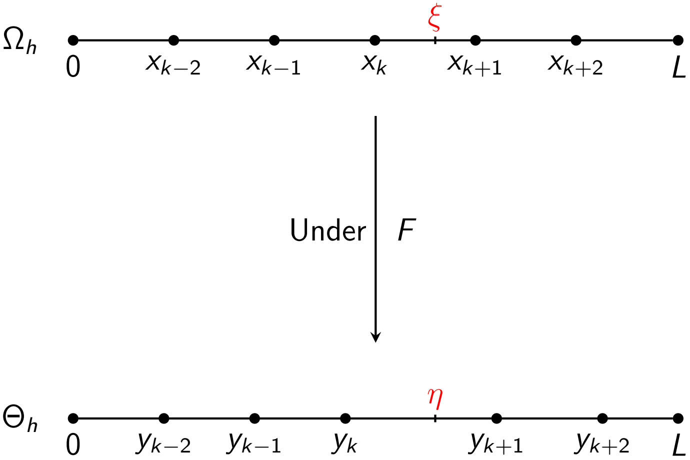
Solve

$$\frac{d}{dx}\left(\frac{1}{\rho}\frac{dF}{dx}\right) = 0,$$

$$F(0) = 0,$$

$$F(L) = L.$$

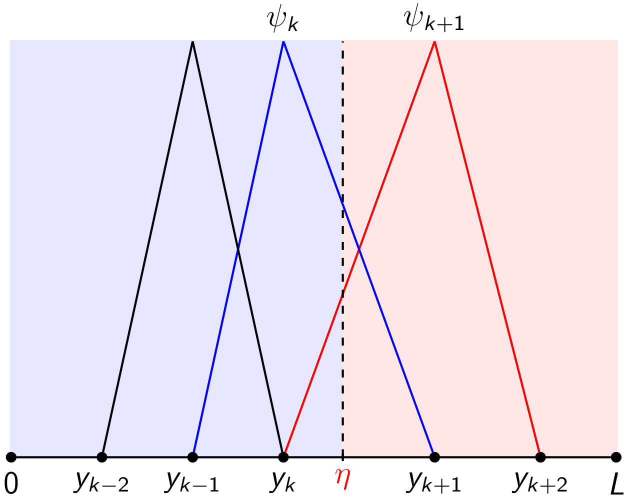
Mapping



Ω_h
ξ
0
x_{k-2}
x_{k-1}
x_k
x_{k+1}
x_{k+2}
L
Under F
Θ_h
η
0
y_{k-2}
y_{k-1}
y_k
y_{k+1}
y_{k+2}
L

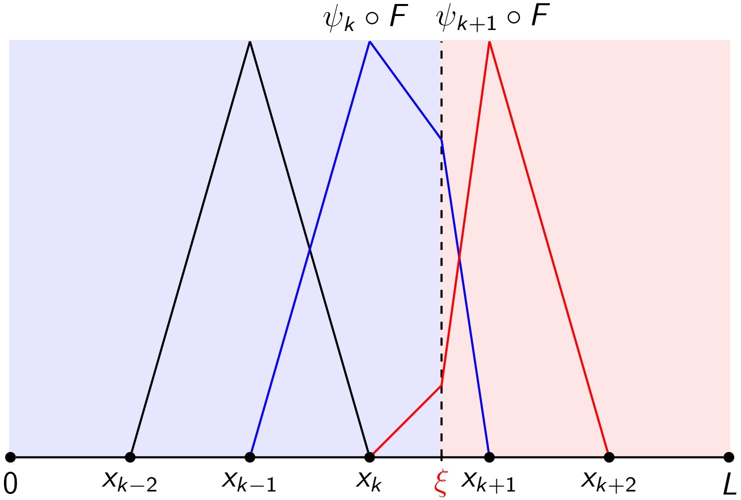
Basis Functions

Construct basis in new grid:



Modified Basis

Map back to Ω_h

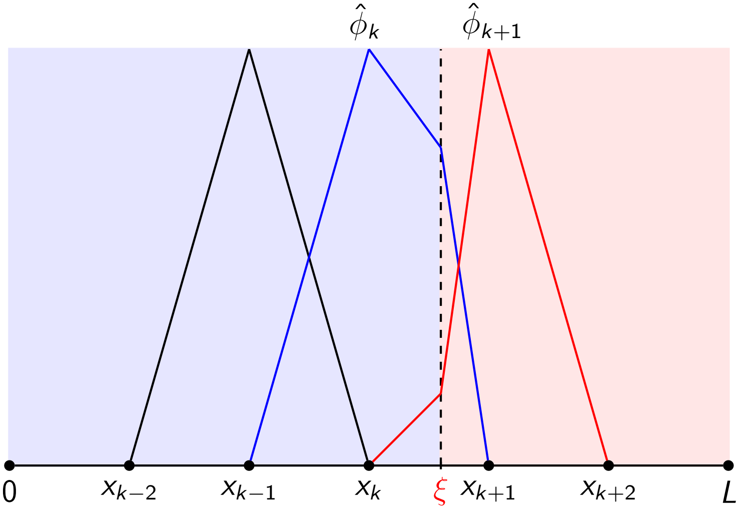


Li, 1998

- PWL basis
- Impose continuity
- Impose zero flux jump condition
- Proved second-order

Li and Ito, 2006

Immersed Finite Element Basis ≡ Owhadi Trick in 1d



Numerical Results

No Interface

	FEM		IFEM	
	e_{L_∞}	rate	e_{L_∞}	rate
h_1	1.13e-02	–	1.13e-02	–
h_2	2.82e-03	2.00	2.82e-03	2.00
h_3	7.09e-04	1.99	7.09e-04	1.99

$$\rho = 2100\,\mathrm{kg/m^3}$$
$$c = 2.3\,\mathrm{m/ms}$$
$$\kappa = 1.1\mathrm{e}4\,\mathrm{MPa}$$

Symes and Vdovina, 2008 Data

	FEM		IFEM	
	e_{L_∞}	rate	e_{L_∞}	rate
h_1	9.11e-03	–	9.15e-03	–
h_2	2.26e-03	2.01	2.29e-03	2.00
h_3	5.70e-04	1.99	5.84e-04	1.97

$$\rho_L = 2100\,\mathrm{kg/m^3} \qquad \rho_R = 2300\,\mathrm{kg/m^3}$$
$$c_L = 2.3\,\mathrm{m/ms} \qquad c_R = 3.0\,\mathrm{m/ms}$$
$$\kappa_L = 1.1\mathrm{e}4\,\mathrm{MPa} \qquad \kappa_R = 2.1\mathrm{e}4\,\mathrm{MPa}$$

High Contrast Density: $\rho_L/\rho_R = 10$

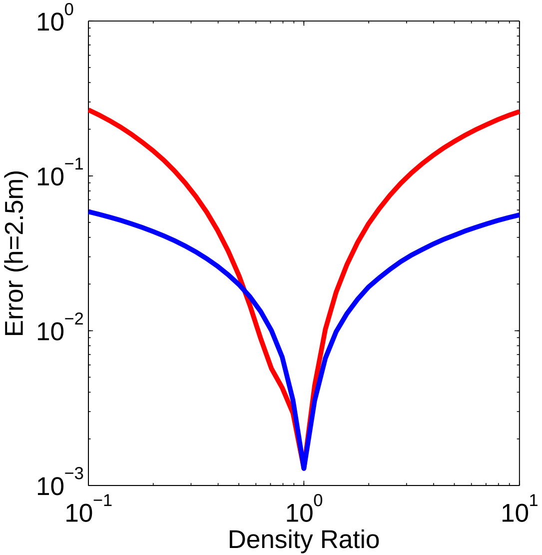
Method	t_2	t_4	t_6	t_8	t_{10}
FEM L_2	1.12	1.12	1.13	1.14	1.14
FEM L_∞	1.14	1.14	1.15	1.16	1.17
IFEM L_2	2.00	2.00	2.00	2.00	2.00
IFEM L_∞	1.99	1.99	1.99	1.99	1.99

$$\rho_L = 2300\,\mathrm{kg/m^3} \qquad \rho_R = 230\,\mathrm{kg/m^3}$$

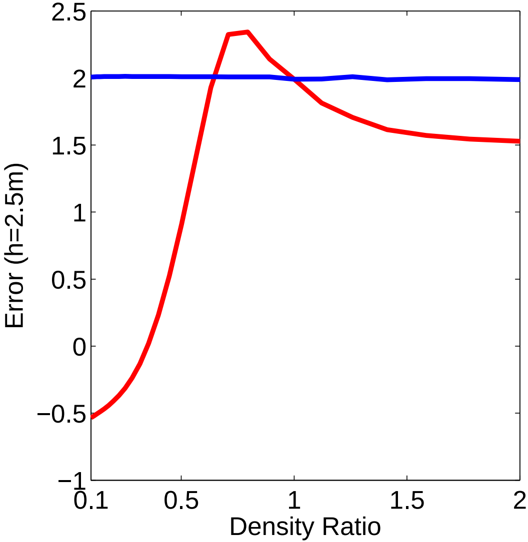
$$c_L = 2.3\,\mathrm{m/ms} \qquad c_R = 3.0\,\mathrm{m/ms}$$

$$\kappa_L = 1.2\text{e}4\,\mathrm{MPa} \qquad \kappa_R = 2.1\text{e}3\,\mathrm{MPa}$$

Error Behavior for Density Ratio, $h = 2.5\,\mathrm{m}$



Error Rate for Density Ratio



Mass Lumping

Mass Lumped Symes and Vdovina, 2008

	FEM		IFEM	
	e_{L_∞}	rate	e_{L_∞}	rate
h_1	8.52e-003	–	8.36e-003	–
h_2	2.16e-003	1.98	2.10e-003	1.99
h_3	5.57e-004	1.96	5.36e-004	1.97

Mass Lumped Large Density Constrast

	FEM		IFEM	
	e_{L_∞}	rate	e_{L_∞}	rate
h_1	6.30e-002	–	3.60e-002	–
h_2	2.18e-002	1.53	8.91e-003	2.01
h_3	2.23e-002	-0.03	2.27e-003	1.97

Summary

- Owhadi Map + FEM in $1d$ *is* IFEM
- FEM accurate for small contrast density
- FEM $\mathcal{O}(h)$ large contrast
- FEM Unstable for Lumping

Future Work

- Extend to Two and Three Dimensions
- Accurate Interface Model
- Harmonic Map Accuracy
- Couple IFEM and FD?
- Mixed FEM?

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