Solving Interface Problems with Finite Elements

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The Rice Inversion Project

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Introduction

Solve the acoustic wave equation (AWE) accurately in 3d on regular grids without aligning interfaces.

▶ Solve for Reflected Waves
▶ Validate Inversion Algorithms
Overview of Interface Methods

Finite Difference IBM & IIM

- Peskin, 1972
- Leveque and Li, 1994
- Zhang and Leveque, 1997

FD is State-of-the-art
Poor Accuracy without IIM, Symes and Vdovina, 2008
Complicated Implementation
Convergence Theory Messy
Overview of Interface Methods

Immersed Finite Element

- Li, 1998
- Kafafy, 2005

Elliptic and Parabolic Problems
Second Order
Acoustic Wave Equation in 1d

\[
\frac{1}{\kappa} \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left( \frac{1}{\rho} \frac{\partial u}{\partial x} \right) = 0
\]

- Bulk Modulus \( \kappa \)
- Density \( \rho \)

Density and bulk modulus piecewise constant.
- Homogeneous Dirichlet B.C.
- Ricker Wavelet I.C.
Ricker Wavelet
Basic FE Approach

Grid spacing $x_k - x_{k-1} = h$. 

Uniformly Discretize

Uniformly

Grid spacing $x_k - x_{k-1} = h$. 

Basic FE Approach

Grid spacing $x_k - x_{k-1} = h$. 

Uniformly Discretize

Uniformly
Basic FE Approach

Basis property: $\phi_k(x_j) = \delta_{kj}$. 
Semi-discrete Equation

\[ \mathcal{M} \frac{d^2 U}{dt^2} + SU(t) = 0 \]

or

\[ \frac{d^2 U}{dt^2} = -\mathcal{M}^{-1}SU(t) \]
Basic FE Approach - Problem

\[ x_k - 2 \leq x \leq x_k - 1 \leq x_k \leq x_k + 1 \leq x_k + 2 \leq L \]

Diagram with points labeled: 0, \( x_{k-2} \), \( x_{k-1} \), \( x_k \), \( \xi \), \( x_{k+1} \), \( x_{k+2} \), and \( L \).
Goals

- Local Modification
- No Grid Tinkering
- Simple Implementation
- Good Convergence Theory

Owhadi and Zhang, 2006 harmonic coordinates.
$\rho$-Harmonic Coordinates

Solve

$$\frac{d}{dx} \left( \frac{1}{\rho} \frac{dF}{dx} \right) = 0,$$

$$F(0) = 0,$$

$$F(L) = L.$$
Mapping

$\Omega_h$

$0 \quad x_{k-2} \quad x_{k-1} \quad x_k \quad x_{k+1} \quad x_{k+2} \quad L$

$\Theta_h$

$0 \quad y_{k-2} \quad y_{k-1} \quad y_k \quad y_{k+1} \quad y_{k+2} \quad L$

Under $F$
Basis Functions

Construct basis in new grid:

\[ \eta_{y_k - 2} y_k - 2 y_k - 1 y_k + 1 y_k + 2 \]

\[ \psi_k \quad \psi_{k+1} \]

0 \quad y_{k-2} \quad y_{k-1} \quad y_k \quad \eta \quad y_{k+1} \quad y_{k+2} \quad L
Modified Basis

Map back to $\Omega_h$

\[ \psi_k \circ F \quad \psi_{k+1} \circ F \]
Li, 1998

- PWL basis
- Impose continuity
- Impose zero flux jump condition
- Proved second-order
Li and Ito, 2006

Immersed Finite Element Basis ≡ Owhadi Trick in 1d
Numerical Results
No Interface

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>IFEM</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$e_{L_{\infty}}$</td>
<td>rate</td>
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<tr>
<td>$h_1$</td>
<td>1.13e-02</td>
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$\rho = 2100 \text{ kg/m}^3$

$c = 2.3 \text{ m/ms}$

$\kappa = 1.1e4 \text{ MPa}$
## Symes and Vdovina, 2008 Data

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\[
\begin{align*}
\rho_L &= 2100 \text{ kg/m}^3 \\
c_L &= 2.3 \text{ m/ms} \\
\kappa_L &= 1.1e4 \text{ MPa} \\
\rho_R &= 2300 \text{ kg/m}^3 \\
c_R &= 3.0 \text{ m/ms} \\
\kappa_R &= 2.1e4 \text{ MPa}
\end{align*}
\]
High Contrast Density: $\rho_L / \rho_R = 10$

<table>
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<tr>
<th>Method</th>
<th>$t_2$</th>
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<th>$t_6$</th>
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<td>IFEM $L_\infty$</td>
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<td>1.99</td>
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<td>1.99</td>
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</tr>
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</table>

$\rho_L = 2300 \text{ kg/m}^3$ \hspace{2cm} $\rho_R = 230 \text{ kg/m}^3$

$c_L = 2.3 \text{ m/ms}$ \hspace{2cm} $c_R = 3.0 \text{ m/ms}$

$\kappa_L = 1.2e4 \text{ MPa}$ \hspace{2cm} $\kappa_R = 2.1e3 \text{ MPa}$
Error Behavior for Density Ratio, $h = 2.5 \text{ m}$
Error Rate for Density Ratio

![Graph showing the error rate for density ratio with a peak at a density ratio of 1.5.](image)
Mass Lumping
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## Mass Lumped Large Density Contrast

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<tr>
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Summary

- Owhadi Map + FEM in 1d is IFEM
- FEM accurate for small contrast density
- FEM $O(h)$ large contrast
- FEM Unstable for Lumping
Future Work

- Extend to Two and Three Dimensions
- Accurate Interface Model
- Harmonic Map Accuracy
- Couple IFEM and FD?
- Mixed FEM?


