Operator Upscaling and Adjoint State Method

Tetyana Vdovina, William Symes

The Rice Inversion Project
Rice University
vdovina@rice.edu

February 20, 2009

Motivation

- **Ultimate Goal:** Use 3D elastic upscaling algorithm as a core in the inverse solver
- **Principal Tool:** Adjoint state method: gradient = forward + adjoint problems
- **Major Question:** How do we upscale the adjoint problem? Is the upscaling algorithm the same as for the forward problem?
  - If it is, can we reuse as much of the existing code as possible?
  - If it is not, what exactly changes and how can we reuse as much of the existing code as possible?

Code Reusability

- Reasons to reuse as much of the code as possible:
  - numerical solution of the elastic wave equation is complicated
  - upscaling of the elastic wave equation is complicated
  - adjoint state method is complicated
- Solution to the code reusability problem: TSOpt

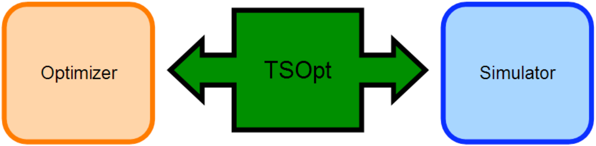


Figure: courtesy of Marco Enriquez

- Need to provide:
  - forward and adjoint update functions
  - simulator/TSOpt and TSOpt/optimizer interfaces

Decisions

3D elastic upscaling vs. 2D acoustic upscaling

Start with acoustics, since elastic upscaling is more complicated

"Optimize then discretize" vs. "Discretize then optimize "

"Optimize then discretize"
- get adjoint for the continuous problem, discretize
- less complicated
- additional discretization error

"Discretize then optimize "
- get adjoint for the discrete problem directly
- more complicated
- no additional errors

Use "discretize then optimize" approach

Forward Acoustic Upscaling Algorithm

- Pressure-acceleration formulation:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + \nabla \cdot (u_x + u_y) = w(t)\delta(x, y),$$
$$u_x = -\frac{\partial p}{\partial x}, \ \ u_y = -\frac{\partial p}{\partial y}$$

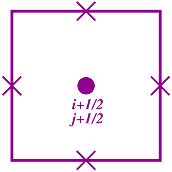
boundary and initial conditions

- Upscale acceleration only:
  - δu_x, δu_y are the subgrid components
  - u_x^c, u_y^c are the coarse components

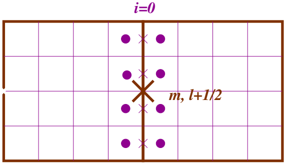
- After discretization mixed finite element method results in to a two-scale difference scheme

Forward Acoustic Upscaling Algorithm (cont.)

Subgrid stencil



Coarse-grid stencil



- $p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = 2p^{n}_{i+\frac{1}{2},j+\frac{1}{2}} - p^{n-1}_{i+\frac{1}{2},j+\frac{1}{2}} + c^2_{i+\frac{1}{2},j+\frac{1}{2}} \Delta t^2 D_x(\delta u_x + u^c_x)^n_{i+\frac{1}{2},j+\frac{1}{2}}$

  $+$ vertical component $+$ source term

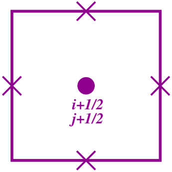
- $(\delta u_x)^{n+1}_{i,j+\frac{1}{2}} = -\dfrac{p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - p^{n+1}_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x},$

- $(u^c_x)^{n+1}_{m,l+\frac{1}{2}} = \sum_j \dfrac{p^{n+1}_{\frac{1}{2},j+\frac{1}{2}} - p^{n+1}_{-\frac{1}{2},j+\frac{1}{2}}}{\Delta x},$

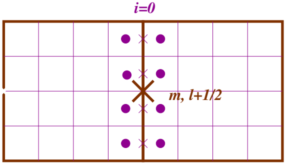
- eqns for vertical component of subgrid and coarse accelerations

Adjoint Acoustic Upscaling Algorithm (cont.)

Subgrid stencil



Coarse-grid stencil



- $p^{n-1}_{i+\frac{1}{2},j+\frac{1}{2}} = 2p^{n}_{i+\frac{1}{2},j+\frac{1}{2}} - p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} + \Delta t^2 D_x(\delta u_x + u_x^c)^n_{i+\frac{1}{2},j+\frac{1}{2}}$
  $+$ vertical component $+$ source term
- $(\delta u_x)^{n-1}_{i,j+\frac{1}{2}} = -\dfrac{c^2_{i+\frac{1}{2},j+\frac{1}{2}} p^{n-1}_{i+\frac{1}{2},j+\frac{1}{2}} - c^2_{i-\frac{1}{2},j+\frac{1}{2}} p^{n-1}_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x},$
- $(u_x^c)^{n-1}_{m,l+\frac{1}{2}} = \sum_j \dfrac{c^2_{\frac{1}{2},j+\frac{1}{2}} p^{n-1}_{\frac{1}{2},j+\frac{1}{2}} - c^2_{-\frac{1}{2},j+\frac{1}{2}} p^{n-1}_{-\frac{1}{2},j+\frac{1}{2}}}{\Delta x},$
- eqns for vertical component of subgrid and coarse accelerations

Implications

- **For acoustic upscaling:**
  - Need to modify update functions for adjoint problem
  - Modifications are minor
- **For elastic upscaling:**
  - Need to modify update functions for adjoint problem
  - Modifications are likely to be more serious than for acoustics

$$\left(\rho\frac{\partial v_1}{\partial t}, w\right) = -\left((\lambda+2\mu)\frac{\partial u_1}{\partial x} + \lambda\frac{\partial u_2}{\partial y} + \lambda\frac{\partial u_3}{\partial z}, \frac{\partial w}{\partial x}\right) + \text{four other inner products} + \text{source term}$$

  - Discretize: $(\lambda+2\mu)\frac{\partial u_1}{\partial x} \approx (\lambda+2\mu)_{i,j,k}\frac{(u_1)_{i+1,j,k} - (u_1)_{i,j,k}}{\Delta x}$
  - Take adjoint: $\frac{(\lambda+2\mu)_{i+1,j,k}(u_1)_{i+1,j,k} - (\lambda+2\mu)_{i,j,k}(u_1)_{i,j,k}}{\Delta x}$

- Reconsider "optimize then discretize" approach?

Current and Future Work

What have we done?

- Derived and implemented linearized and adjoint problems (parallel) for acoustic upscaling problem
- Built an interface between our simulator and TSOpt
- Verified that dot product test works up to machine precision!

What are we doing?

- Building optimizer interface

What is next?

- Do the same for 3D elastic upscaling