Effective Media with a Continuum of Scales and Accurate FEM: Viewpoint of Owhadi and Zhang

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Background

Reflection-transmission at interface:

state-of-art is interface-adaptive unstructured mesh FEM - spectral element (Tromp-Komatitch) or DG (Käser-Dumbser, Hesthaven-Warburton - see Xin’s talk later this AM).

Several averaging schemes proposed to rescue regular grid methods, based on effective medium theory (Muir et al. 1992) - hence *scale separation*.

However, typical distributions of elastic parameters in the earth show

- discontinuities, large and small, along interfaces of limited smoothness and spread throughout volumes
- apparent continuum of scales

Major new development: Owhadi’s scale-free effective medium theory (Owhadi-Zhang 2006, 2008).
Review of constant density case (illustration - Igor’s talk):

\[ \frac{1}{\kappa} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = f; \quad p \equiv 0, \quad t \ll 0 \]

\( f \) is “low frequency” (well sampled in time - always the case)

\( \Rightarrow \) pressure time derivatives have “finite energy”, \( 1/\kappa \) bounded \( \Rightarrow \)
pressure Laplacian \( \nabla^2 p \) has “finite energy” \( \Rightarrow \) pressure has two \( L^2 \) derivatives

\( \Rightarrow \) optimal approximation by \( Q^1 \) elements, hence \( O(\Delta t) \)
convergence of pressure derivatives, \( O(\Delta t^2) \) convergence of
pressure itself.
Accuracy and Approximation

General case, discontinuous $\rho$:

$$\frac{1}{\kappa} \frac{\partial^2 p}{\partial t^2} - \nabla \cdot \frac{1}{\rho} \nabla p = f; \quad p \equiv 0, \quad t << 0$$

Displacement, hence acceleration $= \rho^{-1} \nabla p$ continuous

$\Rightarrow$ $p$ has discontinuous derivatives where $\rho$ is discontinuous $\Rightarrow$ no optimal order approximation by $Q^1$ elements $\Rightarrow$ no optimal order convergence (numerical evidence: Tommy’s talk)

Crux of problem: how to create elements with appropriate approximation properties for $p$?
Smoothing via Change of Coordinates

Owhadi’s observation:

Suppose that $F$ is an invertible stationary coordinate transformation, $p(x) = \tilde{p}(F(x))$, that is, $p = \tilde{p} \circ F$. Then

$$\frac{\partial p}{\partial x_i} = \sum_j \frac{\partial F_j}{\partial x_i} \frac{\partial \tilde{p}}{\partial x_j} \circ F$$

so

$$\nabla \cdot \frac{1}{\rho} \nabla p = \sum_j [\nabla \cdot \frac{1}{\rho} \nabla F_j] \frac{\partial \tilde{p}}{\partial x_j} \circ F + \sum_{j,k} \left[ \frac{1}{\rho} \nabla F_j \cdot \nabla F_k \right] \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k} \circ F$$
Smoothing via Change of Coordinates

Set

\[ a_{jk} = \left[ \frac{1}{\rho} \nabla F_j \cdot \nabla F_k \right] \circ F^{-1} \]

and \( \tilde{\kappa} = \kappa \circ F^{-1} \). Then \( \tilde{p} \) solves

\[ \frac{1}{\tilde{\kappa}} \frac{\partial^2 \tilde{p}}{\partial t^2} - \sum_{j,k} a_{jk} \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k} = f \circ F^{-1} \]

provided that the change of coordinates \( F \) is \( \rho \)-harmonic:

\[ \nabla \cdot \frac{1}{\rho} \nabla F_j = 0. \]

On boundary of domain, make \( F \) the identity: \( F_j = x_j \).
Smoothing via Change of Coordinates

What’s really involved:

- the $\rho$-harmonic map $F$ must be a change of coordinates: that is, continuous with a continuous inverse map
- the coefficient matrix $a_{jk}$ must be elliptic, that is,

$$a_{\min} \leq a \leq a_{\max}$$

in the sense of symmetric matrices, for scalars $a_{\max} \geq a_{\min} > 0$ - actually more is necessary - “Cordes type conditions”, see references.

Owhadi-Zhang 2006, 2008: generically OK in 2D, may fail for very high contrast in 3D.
Smoothing via Change of Coordinates

$F$ is coordinate change, $a_{jk}$ satisfies Cordes-type conditions

$\Rightarrow \tilde{p}$ has two $L^2$ derivatives

$\Rightarrow Q^1$ elements $\{\tilde{\phi}_j\}$ optimally approximate $\tilde{p}$

$\Rightarrow$ distorted $Q^1$ elements $\{\phi_j = \tilde{\phi}_j \circ F\}$ optimally approximate $p$

$\Rightarrow$ error in distorted $Q^1$ FE solution of wave equation $= O(\Delta t^2)$ - can also lump mass matrix using distorted elements.

Details - Tommy (next talk).
Perspective

Practical numerical method requires *localization* - construction of *global* $\rho$-harmonic coordinates too expensive.

Construction of distorted elements trivial in 1D for interface problems - what about 2D/3D? Probably needs to be *localized*. How accurately must $F$ be computed?

Our observation: low (typical) density contrast $\Rightarrow$ little difference between ordinary, distorted $Q^1$ FEM.

What about elasticity? Or even acoustics in 1st order (“mixed FEM”) formulation?
Perspective

Critical issue: how do we represent coefficients $\kappa, \rho, ...$?

Main lesson: simple grid sampling not enough - must somehow encode *subgrid* information (cf. also Tanya’s upscaling work)

Proposal: *multiscale representation* (wavelets, curvelets, xxxxlets,...) via *oracle* - able to provide any average required with any precision required.

Meaning for inversion: produces estimates of averages, hence *constraint* on multiscale subgrid structure.