Viscoelastic Inversion and the Range of the Forward Map

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The Viscoelastic Wave Equation

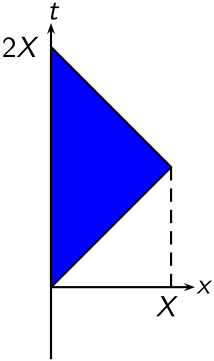
$$
\begin{aligned}
\rho \frac{\partial v_i}{\partial t} &= \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \\
\frac{\partial \sigma_{kl}}{\partial t} &= \sum_{i,j} C_{ijkl} *_t \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),
\end{aligned}
$$

where v is velocity, σ is stress, C is the Hooke tensor, ρ is density, and $*_t$ indicates time convolution.

A One-D Viscoelastic Equation

A simple model for viscoelasticity in one dimension is

$$
\begin{aligned}
0 &= \zeta w_t + p_x \\
0 &= p_t + \zeta w_x + ap \\
&\quad -a\alpha \int_{-\infty}^{t} e^{-\alpha(t-s)} p(x,s)\,ds
\end{aligned}
$$



p is pressure, w is velocity, ζ is impedance, a, and α are attenuation coefficients. x is travel-time.

The Causal Impulse Response

We consider the causal impulse response. If $p(0, t) = \delta(t)$, the impulse response is $w(0, t) = h(t)$. In this context, there are two main problems:

- The forward problem, $\zeta \to h$.
  - What is the range of the forward map?
  - What does this tell us about the attenuation coefficients a and α?
- The inverse problem, $h \to \zeta$.
  - Can we solve the inverse problem explicitly?
  - Can we solve for any of the attenuation coefficients?

The Forward Map

For the forward map, it can be shown that the range of the forward map for the viscoelastic problem is contained within the range of the forward map for the conservative problem.

- Is this containment strict? If yes, then this could give an indication of how to choose the attenuation coefficients.
  - In the continuous case, this is unknown.
  - In a simple discretization of the continuum problem, we can show that the containment is in fact strict.

The Forward Map - Acoustic Transparency

In order to analyze the forward map, we construct the operator $R : L^2[0, 2X] \to L^2[0, 2X]$

$$Rf(t) = \int_0^{2X} [h(t-\tau) + h(\tau - t)] f(\tau)\, d\tau,$$

and consider the modulus of acoustic transparency

$$\epsilon = \inf_{f \neq 0} \frac{\langle f, Rf \rangle}{\langle f, f \rangle}$$

The Forward Map for the Conservative Problem

Theorem (Symes)

For the lossless problem, there is a one-to-one correspondence between $\zeta \in H^1[0, X]$ and h with $\overline{h} \in L^2[0, 2X]$ such that the operator R is positive definite as an operator on $L^2[0, 2X]$.

In other words, for any $\zeta \in H^1[0, X]$, there is a unique impulse response with positive acoustic transparency, and vice versa.

Viscoelastic Transparency

Theorem

In the viscoelastic case, the operator

$$Rf(t) = \int_0^{2X} [h(t-\tau) + h(\tau - t)] f(\tau)\, d\tau$$

mapping $L^2[0, 2X]$ *to itself is positive definite; that is, there exists an* $\epsilon > 0$ *such that* $\langle f, Rf \rangle \geq \epsilon \langle f, f \rangle$ *for all* $f \in L^2[0, 2X]$.

So while we do not have the correspondence that exists in the conservative case, we do have that the range of the viscoelastic forward map is a subset of the conservative forward map.

The Discrete Problem

Turning to the discrete problem, if $p_0^i = \delta_{i0}/2\Delta$ (where Δ is the mesh size in both the x and t directions), then $h_k = w_0^{2k}$ is the discrete impulse response. We construct the discrete version of the R operator by constructing the matrix

$$H = \begin{bmatrix} h_0 & 0 & \dots & & 0 \\ h_1 & h_0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ h_{N-1} & h_{N-2} & \dots & h_0 & 0 \\ h_N & h_{N-1} & \dots & & h_0 \end{bmatrix}$$

The we define $R = H + H^T$.

Discrete Transparency

Theorem

The matrix operator R is positive definite.

Proof: *Inverse Problems*, 22 (6): 1947-1958, 2006.

Corollary

$$\langle f, Rf \rangle \geq \frac{\Delta a_0}{\zeta_0 (1 + 2X\alpha_0)^2} \langle f, f \rangle.$$

This gives a lower bound for the modulus of acoustic transparency.

The geometric optics expansion of the impulse response solution

$$\mathbf{v}(x,t) = \sum_{k=0}^{N} \mathbf{v}^{(k)}(x) f_k(t-\phi(x)) + \mathbf{S}_N(x,t),$$

gives the equation:

$$\eta(x) = \eta(0) + \int_0^x \eta(s) \left[U(s,s^+)\eta(0) e^{\frac{1}{2}\int_0^s a(\xi)d\xi} + \frac{1}{2}a(s) \right] ds,$$

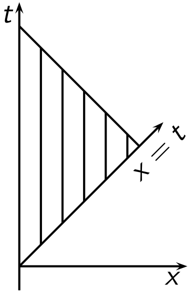
where $U(x,x^+) = U^{(1)} = \lim_{t \to x+} U(x,t)$ (U is the upward travelling wave), and $\eta^2 = \zeta$.

Solving the Inverse Problem

This integral equation for the impedance is a contraction mapping, so it has a unique solution.

We obtain the impedance by performing the following iteration on small slices of the domain:

- Start with a guess for the impedance.
- Use this to solve the differential equation.
- With this solution, find a new impedance using the integral equation.
- With the new impedance, solve the differential equation again.
- Repeat until convergence.



Proof: See *Inverse Problems*, 22 (5): 1869-1882, 2006.