Eulerian Gaussian beams for high frequency wave propagation

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Outline

- Geometrical optics and Gaussian beams
- Lagrangian Gaussian beams: basics
- Eulerian Gaussian beams: global Cartesian coordinates
- Numerical results
- Conclusions and future work

Geometrical optics and Gaussian beams

- Traditional geometrical optics yields unbounded amplitude at caustics.
- A Gaussian beam around a central ray always has regular behavior at caustics and interference of multiple arrivals is achieved by summing up a bundle of Gaussian beams (Cerveny'82, White'87, etc).
- Traditional GBs are based on Lagrangian ray tracing.
- Combining the GB ansatz (Ralston'83, Tanushev-Qian-Ralston'06) with the paraxial Liouville formulation (Qian-Leung'04,'06) $\Rightarrow$ Eulerian GB summation method (Leung-Qian-Burridge'06).

Lagrangian Gaussian beams

Eikonal and transport equations

- Wave equation for $U(x, z, \omega)$,

$$\nabla^2 U(x, z, \omega) + \frac{\omega^2}{v^2(z, x)} U(x, z, \omega) = -\delta(x - x_s)\delta(z - z_s),$$

$$\Omega = \{(x, z) : x_{\min} \le x \le x_{\max}, 0 \le z \le z_{\max}\},$$

where ω frequency, $v(z, x)$ velocity, and (z_s, x_s) a source point.

- GO ansatz:

$$\begin{aligned} \left(\frac{\partial \tau}{\partial x}\right)^2 + \left(\frac{\partial \tau}{\partial z}\right)^2 &= \frac{1}{v^2(x, z)}, \\ \nabla \tau \cdot \nabla A + \frac{1}{2} A \nabla^2 \tau &= 0. \end{aligned}$$

Paraxial Eikonal equations

- (Gray-May'95, Qian-Symes'02)

$$\frac{\partial \tau}{\partial z} - \sqrt{\frac{1}{v^2} - \left(\frac{\partial \tau}{\partial x}\right)^2} = 0, \quad z \geq 0, \quad x_{\min} \leq x \leq x_{\mathrm{ma}}$$

$$\tau(0,x) = \tau_0(x), \quad \mathrm{Im}\,\tau_0 \geq 0, \quad \nabla\tau|_{z=0} = \xi(x),$$

where $\tau_0(x)$ and $\xi(x)$ are given complex smooth functions satisfying the compatibility conditions.

Eikonal eqns: initial condition

- At $(z_s, x_s) = (0, x_s)$, specify initial conditions,

$$\tau_0(x_s) = 0, \quad \xi_1(x_s;\theta_s) = \frac{\sin\theta_s}{v(0,x_s)}, \qquad |\theta_s| \le \theta_{\max} < \frac{\pi}{2},$$

where

$$(x_s, \theta_s) \in \Omega_p = \{(x,\theta) : x_{\min} \le x \le x_{\max}, |\theta| \le \theta_{\max}\}.$$

- Construct a τ in a neighborhood of the source:

$$\begin{aligned} \tau_0(x;x_s) &= \tau_0(x_s) + \xi_1(x_s;\theta_s)\cdot(x - x_s) \\ &\quad + i\frac{\epsilon}{2}(x - x_s)^2 \cos^2\theta_s. \end{aligned}$$

Gaussian beam theory (1)

- Let the central ray of a beam be given by $x = X(z)$, travel time by $\tau = T(z)$, and the Hamiltonian $H(z, X, p) = -\sqrt{\frac{1}{v^2(z,X)} - p^2}$, where $p(z) = \tau_x(z, X(z))$.
- Ray tracing system:

$$\dot{X}(z) = H_p = \frac{p}{\sqrt{\frac{1}{v^2} - p^2}}, X|_{z=0} = x_s;$$

$$\dot{p}(z) = -H_X = \frac{-v_X}{v^3\sqrt{\frac{1}{v^2} - p^2}}, p|_{z=0} = \xi_1(x_s; \theta_s);$$

$$\dot{T}(z) = \frac{1}{v^2\sqrt{\frac{1}{v^2} - p^2}}, T|_{z=0} = \tau_0(x_s).$$

Gaussian beam theory (2)

- Dynamic ray tracing (DRT) system, where $B(z; x_s, \theta_s) = \frac{\partial p(z; x_s, \theta_s)}{\partial \alpha}$ and $C(z; x_s, \theta_s) = \frac{\partial X(z; x_s, \theta_s)}{\partial \alpha}$, and $\epsilon > 0$,

$$\begin{aligned} \dot{B}(z) &= -H_{X,p}B - H_{X,X}C, \quad B(z)|_{z=0} = i\epsilon \cos^2\theta_s, \\ \dot{C}(z) &= H_{p,p}B + H_{p,X}C, \quad C(z)|_{z=0} = 1. \end{aligned}$$

Traveltime near a central ray

- Gaussian beam theory implies that $\mathrm{Im}(BC^{-1})$ remains positive if it is positive initially, i.e. if ϵ is positive. (Leung-Qian-Burridge'06).
- By $\tau_x = p$ and $\tau_{xx} = \delta p/\delta x = (\partial p/\partial \alpha)/(\partial x/\partial \alpha) = B/C$, in the neighborhood of X,

$$\begin{aligned}\tau(z, x; x_s, \theta_s) &= T(z; x_s, \theta_s) + p(z) \cdot (x - X(z)) \\ &\quad + \frac{1}{2}(x - X(z))^2 B(z) C^{-1}(z),\end{aligned}$$

- Let the angle the central ray of a beam makes with the z-direction at z be the arrival angle $\Theta(z; x_s, \theta_s)$, and let $p(z) = \frac{\sin \Theta(z)}{v(z, X(z))}$.

Lagrangian systems

- The ray tracing system, the DRT system, and amplitude nonzero everywhere (L-Q-B'06)

$$\frac{dX}{dz}(z) = \tan\Theta, X(0) = x_s,$$

$$\frac{d\Theta}{dz}(z) = \frac{1}{v}(v_z \tan\Theta - v_x), \Theta(0) = \theta_s,$$

$$\frac{dT}{dz}(z) = \frac{1}{v(z, X(z; x_s, \theta_s)) \cos\Theta(z; x_s, \theta_s)}, T|_{z=0} = 0,$$

$$\dot{B}(z) = -H_{X,p}B - H_{X,X}C, B(z)|_{z=0} = i\epsilon\cos^2\theta_s,$$

$$\dot{C}(z) = H_{p,p}B + H_{p,X}C, C(z)|_{z=0} = 1;$$

$$A(z; x_s, \theta_s) = \frac{\sqrt{C(0)v(z, X(z))\cos\theta_s}}{\sqrt{v(z_s, x_s)C(z; x_s, \theta_s)\cos\Theta(z)}}.$$

Lagrangian GB summation

- The wavefield due to one Gaussian beam parameterized with initial take-off angle θ_s is

$$\Psi(z,x;x_s,\theta_s) = \psi_0 A(z;x_s,\theta_s)\exp[i\omega\tau(z,x;x_s,\theta_s)]\,,$$

where $p(z) = \frac{\sin\Theta(z;x_s,\theta_s)}{v(z,X(z;x_s,\theta_s))}$ and

$$\begin{aligned}\tau(z,x;x_s,\theta_s) &= T(z;x_s,\theta_s) + p(z)\cdot(x - X(z)) + \\ &\frac{1}{2}(x - X(z))^2 B(z)C^{-1}(z).\end{aligned}$$

- The wavefield generated by a point source at x_s,

$$U(z,x;x_s) = \int_{-\pi/2}^{\pi/2}\Psi(z,x;x_s,\theta_s)d\theta_s.$$

Eulerian Gaussian beams

Paraxial Liouville equations

- (Qian-Leung'04,06). Introduce a function,

$$\phi = \phi(z, x, \theta) : [0, z_{\max}] \times \Omega_p \longrightarrow [x_{\min}, x_{\max}],$$

such that, for any $x_s \in [x_{\min}, x_{\max}]$ and $z \in [0, z_{\max}]$,

$$\Gamma(z; x_s) = \{(X(z), \Theta(z)) : \phi(z, X(z), \Theta(z)) = x_s\}$$

gives the location of the reduced bicharacteristic strip $(X(z), \Theta(z))$ emanating from the source x_s with takeoff angles $-\theta_{\max} \le \theta_s \le \theta_{\max}$.

- Differentiate with respect to z to obtain

$$\phi_z + u\phi_x + w\phi_\theta = 0\,,$$
$$\phi(0, x, \theta) = x, \quad (x, \theta) \in \Omega_p.$$

Level sets (1)

- For a fixed $x_s \in [x_{\min}, x_{\max}]$, the location where $\phi(0, x, \theta) = x_s$ holds is

  $$\Gamma(0; x_s) = \{(x, \theta) : x = x_s, -\theta_{\max} \le \theta \le \theta_{\max}\},$$

  which states that the initial takeoff angle varies from $-\theta_{\max}$ to $\theta_{\max}$ at the source location x_s.
- Evolving level set equations will transport source locations (“tag”) to any z according to the vector fields u and w.
- Given $z \in [0, z_{\max}]$ and $x_s \in [x_{\min}, x_{\max}]$, the set $\Gamma(z; x_s)$ is a curve in Ω_p, which defines an implicit function between X and Θ.

Level sets (2)

- When $z = 0$, $\Gamma(0; x_s)$ is a vertical line in Ω_p, indicating that the rays with takeoff angles from $-\theta_{\max}$ to $\theta_{\max}$ emanate from the source location x_s.
- When $z \neq 0$, $\Gamma(z; x_s)$ being a curve indicates that for some $X = x^*$ there are more than one $\Theta = \theta_a^*$ such that $\phi(z, x^*, \theta_a^*) = x_s$, implying that more than one rays emanating from the source x_s reach the physical location (z, x^*) with different arrival angles θ_a^*.

Multivaluedness: illustration

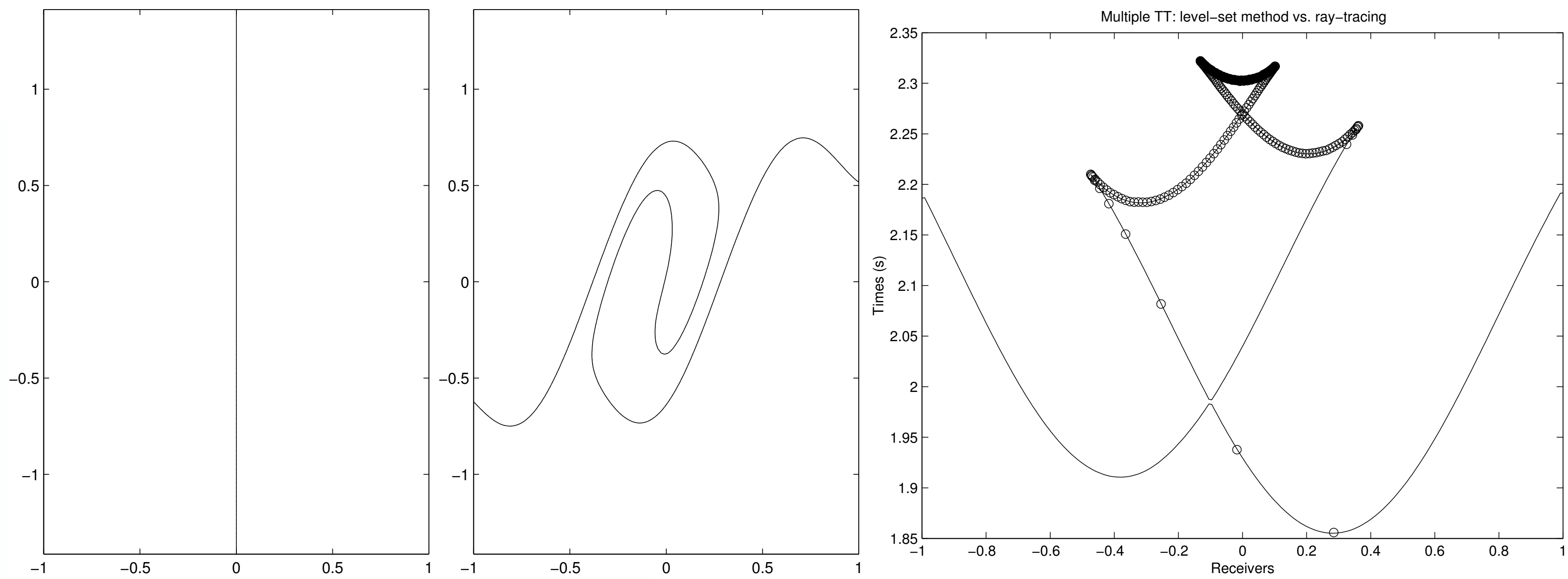

Figure 1: Multivaluedness.

Takeoff angles, traveltimes

- To sum Gaussian beams, parametrize $\Gamma(z;x_s)$ with takeoff angles: transport the initial takeoff angle, ψ,

$$\psi_z + u\psi_x + w\psi_\theta = 0\,,$$
$$\psi(0,x,\theta) = \theta, \quad (x,\theta) \in \Omega_p.$$

- For each point $(x^*,\theta^*) \in \Gamma(z;x_s)$, the unique takeoff angle is $\psi(z,x^*,\theta^*)$.
- Map $\Gamma(z;x_s)$ into $\psi(z,\Gamma(z;x_s)) \subset [-\theta_{\max},\theta_{\max}]$; mapping from $\Gamma(z;x_s)$ to $\psi(z,\Gamma(z;x_s))$ is 1-1.
- The traveltime for those multiple rays:

$$T_z + uT_x + wT_\theta = \frac{1}{v\cos\theta}, \quad T(0,x,\theta) = 0\,.$$

Liouville for B and C

- B and C satisfy

$$
\begin{aligned}
&B_z + uB_x + wB_\theta = -H_{x,p}B - H_{x,x}C, \\
&B(x,\theta,z=z_s) = i\epsilon\cos^2\theta\,, \\
&C_z + uC_x + wC_\theta = H_{p,p}B + H_{x,p}C, \\
&C(x,\theta,z=z_s) = 1\,.
\end{aligned}
$$

- The Eulerian amplitude is

$$
A(z,x,\theta) = \frac{\sqrt{C(0)v(z,x)\cos\psi(z,x,\theta)}}{\sqrt{v(z_s,x_s)C(z,x,\theta)\cos\theta}}.
$$

Eulerian GB summation (1)

- Gaussian beam summation formula in phase space,

$$U(z,x;x_s) = \int_{I(z;x_s)} \frac{A(z,x',\theta')}{4\pi} \times \exp\left[i\omega\tau(z,\ x;\ x',\ \theta') + \frac{i\pi}{2}\right]\ d\theta_s,$$

where $(x',\theta') \in \Gamma(z;x_s)$, $\psi = \psi(z,x',\theta')$,

$$\begin{aligned} I &= \psi(z,\Gamma(z;x_s)) \\ &= \{\theta_s : \theta_s = \psi(z,x',\theta') \text{ for } (x',\theta') \in \Gamma(z;x_s)\} \\ &\subset [-\theta_{\max},\theta_{\max}]. \end{aligned}$$

Eulerian GB summation (2)

- Traveltime

$$\tau(z,x;x',\theta') = T(z,x',\theta') + \frac{(x-x')\sin\theta'}{v(z,\ x')} + \frac{1}{2}(x-x')^2 BC^{-1}(z,x',\theta')$$

- $I = I(z;x_s)$ is an interval because $\Gamma(z;x_s)$ is a continuous curve and the takeoff angle parametrizes this curve continuously.
- Efficient numerical procedures (Leung-Qian-Burridge'06).

Waveguide model

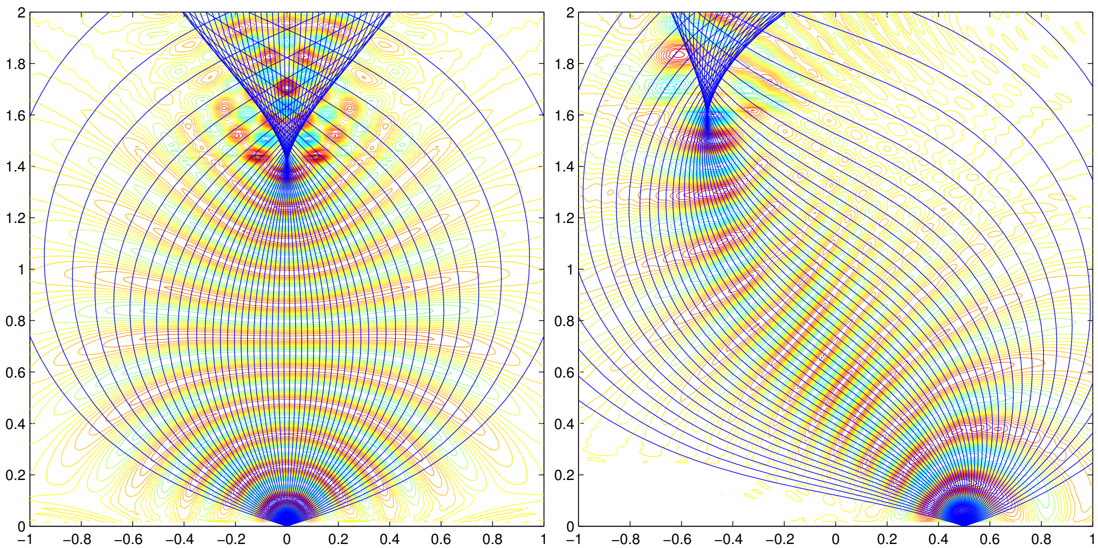

Figure 2: $\omega = 8\pi$. $x_s = 0$ and $x_s = 0.5$.

Sinusoidal model

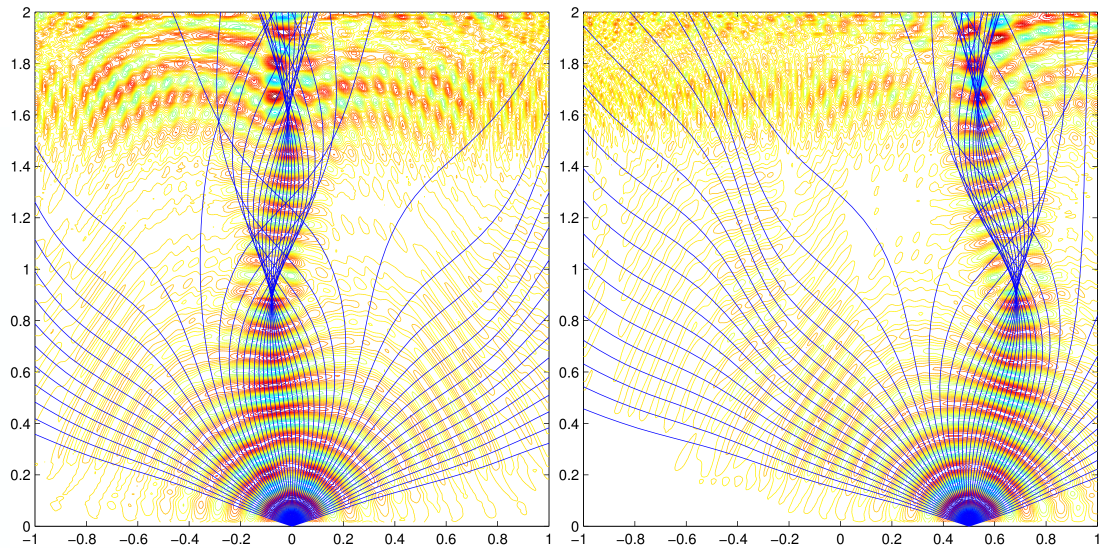

Figure 3: $\omega = 16\pi$. $x_s = 0$ and $x_s = 0.5$.

Conclusion and future work

- Developed a Eulerian Gaussian beam method for high frequency waves.
- Future work consists of
  - 3-D implementation ...
  - incorporating this into seismic migration ...
  - open to suggestions ...