Aspects of wave equation migration

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Includes parts joint with Martijn de Hoop, and joint with Bill Symes

Topics

- Wave equation migration as solution of a high-frequency linearized inverse problem

- Angle CIG's in media with multipathing: free of kinematic artifacts

Migration as a linearized inverse problem

Series of papers, e.g. Cohen-Bleistein, Beylkin, Rakesh, Ten Kroode-Smit-Verdel, Nolan-Symes).

Model data by
- Born approximation: write $\frac{1}{c^2(x)} = \frac{1}{c_0^2(x)}(1 + \alpha(x))$
- Ray-theory

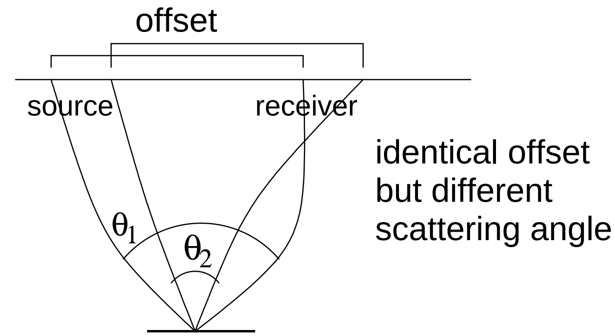
Linearized forward map, given c_0 maps $\alpha \mapsto$ data.

Kirchhoff migration reconstructs most singular part of $\alpha(x)$, if
- Proper weight factors/amplitudes
- Absence of caustics (unstacked), or much less restrictive TIC condition (stacked with "maximal data").

Question: Can you show something similar for wave-equation migration?

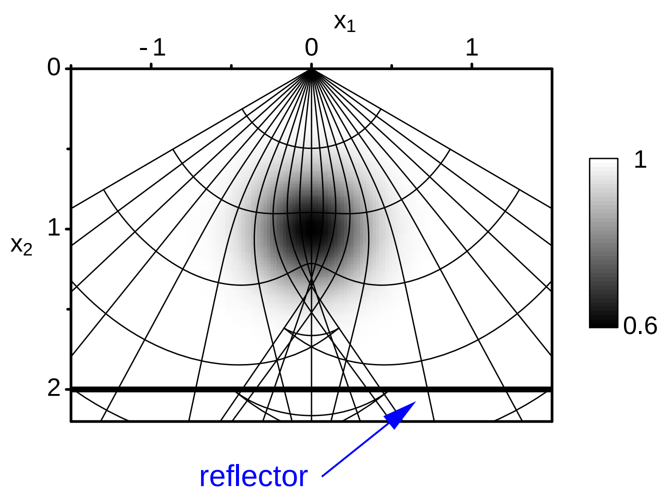
Angle CIG's

- Multiple reflected rays for a single offset
  $\longrightarrow$ images sorted by angle

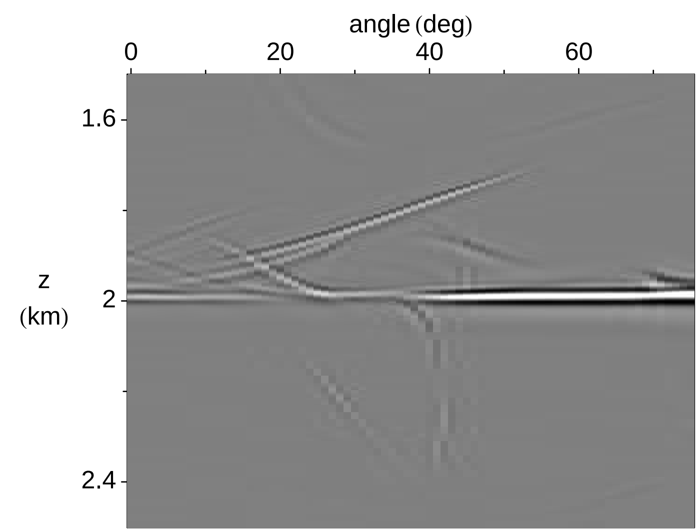


- Artifacts

Example medium, some rays and wave fronts



Kirchhoff angle migration (Xu et al. 2001) for $x_1 = 0$ km



- Artifacts invalidate use of angle gathers for velocity analysis

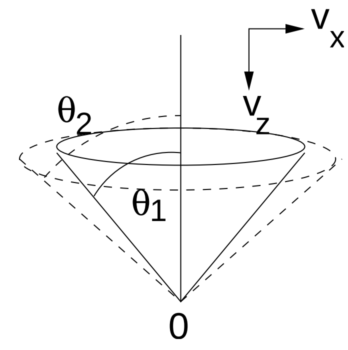
- Artifacts confuse AVO/AVA analysis

Topics

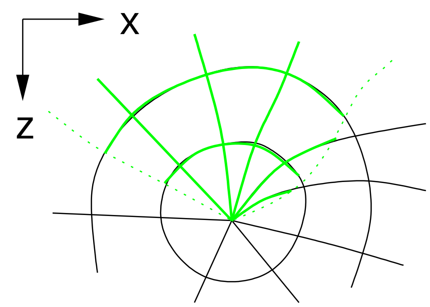
- Wave equation migration as solution of a high-frequency linearized inverse problem
  - Solutions of one-way wave equations $\leftrightarrow$ solutions to full equation
  - Double-square root forward modeling (Stolk & De Hoop, to appear in SIAM J. on Appl. Math)
  - Remarks about imaging

- Angle CIG's in media with multipathing: free of kinematic artifacts
  - Wave equation migration (almost) artifact free (preprint Stolk & De Hoop)
  - Example with both Kirchhoff and wave equation angle gathers

One-way wave equations analysis: summary



Model: Propagation for velocities inside θ_1-cone, strong damping outside θ_2-cone. Subset of wave fronts computed from point source.



x denotes horizontal variable(s)
$u = u(z, x, t)$ a wavefield.
$G_-(z, z_0)$ extrapolation operator.
For $G_-(z, z_0)$ to be unitary for propagating waves, work with a normalized wave field u_-

$$u_-(z, \cdot, \cdot) = G_-(z, z_0) u_-(z_0, \cdot, \cdot),$$
$$u(z, \cdot, \cdot) = Q_-^*(z, x, D_x, D_t) u_-(z, \cdot, \cdot),$$
$$Q = |\partial_t|^{-1/2} \left((c(z, x)^{-2} - \partial_t^{-2} \partial_x^2 \right)^{-1/4}.$$

In this way one obtains correct amplitudes in a smoothly varying medium.

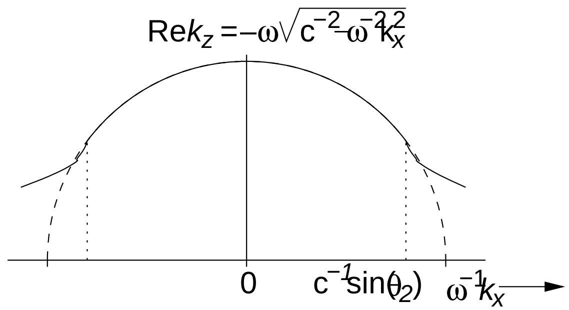
Theoretically we can make this such that

- G_- solves a one way wave equation

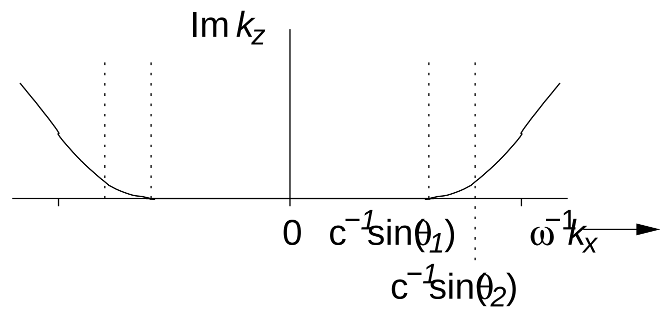
$$\left(\frac{\partial}{\partial z} - iB_-(z,x,-i\partial_x,i\partial_t) - C(z,x,-i\partial_x,i\partial_t)\right) G_-(z,z_0) = 0,$$
$$G_-(z_0,z_0) = \mathrm{Id}\,.$$

  B the square root operator, selfadjoint; C is damping (new).

Square root operator B



Damping operator C



- The Green's function is replaced by

$$G(z_r,x_r,t;z_s,x_s) \approx (Q_-^*\, G_-(z_r,z_s)\, \tfrac{1}{2}\mathcal{H}_t Q_-)\delta_{t=0}\delta_{x=x_s},$$

  with $\mathcal{H}$ the Hilbert transform, on the "upgoing part" of the rays.

- The approximation above is to highest order singularities, modulo a smoother term.

Modeling and upward continuation

Acoustic Born: data modeled by

$$w *_t \delta G, \text{ with } w = w(t) \text{ source wavelet,}$$

$\delta G(z_r, x_r, t; z_s, x_s)$ given by linearization $\frac{1}{c(z,x)^2} \rightarrow \frac{1}{c_0(z,x)^2}(1 + \alpha(z,x))$

$$\delta G(z_r, x_r, t; z_s, x_s) = \int dz \int dx \int dt_0$$

$$G(z_r, x_r, t - t_0; z, x) \frac{(-\alpha(z,x))}{c_0^2(z,x)} \partial_t^2 G(z_s, x_s, t_0; z, x)$$

Introduce Claerbout's "subsurface offset" h

$$\delta G(z_r, x_r, t; z_s, x_s) = \int dz \int dx \int dt_0 \int dh \int dt'$$

$$G(z_r, x_r, t - t_0; z, x+h) \underbrace{\frac{(-\alpha(z,x))}{c_0^2(z,x)} \delta(h)\delta(t')}_{\text{"DSR reflectivity" } r(z, x-h, x+h, t').} \partial_t^2 G(z_s, x_s, t_0 - t'; z, x-h)$$

$x - h =$ "sunken source position"; $x + h =$ "sunken receiver position"

Replace G by extrapolator $G_-(z,z_0)$. Let $G_{-,s}$ act in (x_s,t) variables, $G_{-,r}$ in (x_r,t) variables. We let

$$H(z,z_0) = G_{-,s}(z,z_0)G_{-,r}(z,z_0),$$

upward continuation (kernel contains a time convolution)

Modeling formula

Let Z be some large depth below which $\alpha = 0$. Then

$$\delta G \approx \int_0^Z dz(\partial_t^2)(Q^*_{-,s}Q^*_{-,r})\ H(0,z)\ (Q_{-,s}Q_{-,r})\tfrac{1}{4}\partial_t^2 r(z,y_s,y_r,t)$$

with

$$r(z,y_s,y_r,t) = \frac{\alpha}{c_0^2}(z,\tfrac{y_s+y_r}{2})\,\delta(\tfrac{y_r-y_s}{2})\,\delta(t).$$

DSR equation for $H(z,z_0)$

Using that $B_{-,r}, C_{-,r}$ commute with $G_{-,s}$, we find that

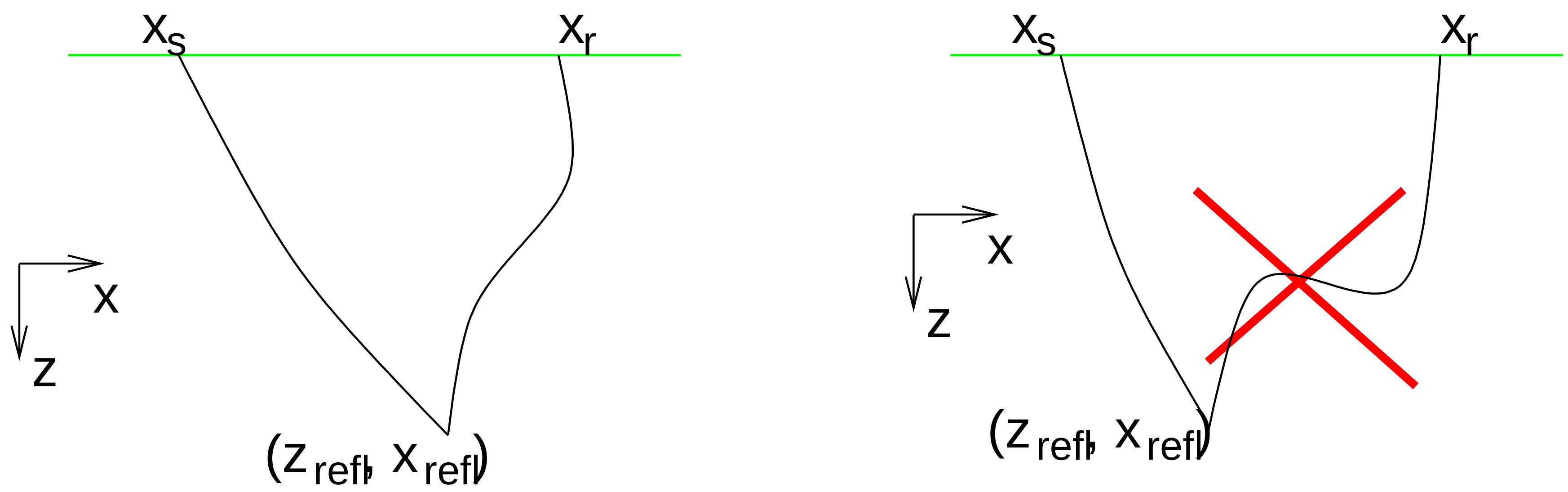
$$(\tfrac{\partial}{\partial z} - iB_{-,s} - iB_{-,r} - C_s - C_r)H(z,z_0) = 0,$$

and

$$H(z_0,z_0) = \mathrm{Id}.$$

DSR assumption

All reflections contributing energy to the data are along a downward traveling incoming ray, and an upward traveling reflected ray.



Adjoints and imaging

Adjoint of upward continuation is downward continuation.

Adjoint of map

$$\frac{\alpha(z,x)}{c_0^2(z,x)} \to r(z,y_s,y_r,t)$$

gives Claerbout's imaging condition $t=0$ and $y_s=y_r$ in downward continuation imaging.

Amplitude factors can also be derived

WE angle transform

Let $R(z, x_m, h, t)$ be downward continued data in midpoint-offset coords.

$$
\begin{aligned}
f_{\text{stack}}(z, x) &= R(z, x, 0, 0) \\
&= \frac{1}{(2\pi)^n} \int\int \widehat{R}(z, x, \theta, \omega)\, d\theta\, d\omega.
\end{aligned}
$$

$p = \frac{\theta}{\omega}$ is a difference of horizontal slownesses. The angle transform is obtained by taking p constant (De Bruin et al. 1990, Prucha et al. 1999)

$$
\begin{aligned}
f_{\text{WE}-\text{angle}}(p, z, x) &= \frac{1}{2\pi} \int \widehat{R}(z, x, p\omega, \omega)\, d\omega \\
&= \frac{1}{(2\pi)^n} \int\int\int e^{i(\theta - p\omega)\cdot h} \widehat{R}(z, x, \theta, \omega)\, dh\, d\theta\, d\omega \\
&= \int R(z, x, h, p \cdot h) dh.
\end{aligned}
$$

Kinematics of downward continuation

High-frequency asymptotic Green's function with multivalued traveltimes

$$G(z_r, x_r, z_s, x_s, t) \approx \sum_{(j)} a^{(j)}(z_r, x_r, z_s, x_s) \mathcal{H}_t^{\sigma^{(j)}} \delta(t - \tau^{(j)}(z_r, x_r, z_s, x_s)).$$

Downward continued data

$$R(z, \frac{y_s + y_r}{2}, \frac{y_r - y_s}{2}, t) = \int\int (...) d(x_s, x_r, t + \tau^{(j)}((0, x_r), (z, y_r)) + \tau^{(k)}((0, x_s), (z, y_s)))\, dx_s\, dx_r.$$

Consider an event at $t = T_{\text{data}}(x_s, x_r)$.

Large contribution to integral: integration traveltime surface tangent to data traveltime surface by stationary phase.

Result of stationary phase

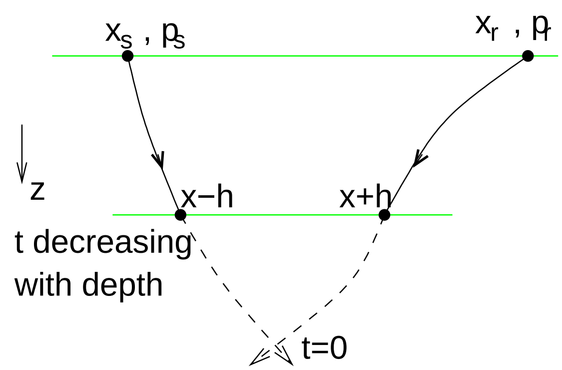
$$t+\tau^{(j)}((0,x_r),(z,y_r))+\tau^{(k)}((0,x_s),(z,y_s)) = T_{\text{data}}(x_s,x_r)$$
$$\frac{\partial}{\partial x_s}(\tau^{(j)}((0,x_r),(z,y_r))+\tau^{(k)}((0,x_s),(z,y_s))) = \frac{\partial}{\partial x_s}T_{\text{data}}(x_s,x_r)$$
$$\frac{\partial}{\partial x_r}(\tau^{(j)}((0,x_r),(z,y_r))+\tau^{(k)}((0,x_s),(z,y_s))) = \frac{\partial}{\partial x_r}T_{\text{data}}(x_s,x_r)$$

$\frac{\partial\tau^{(j)}}{\partial x_s}((z,y_s),(0,x_s))$ and $c(0,x_s)$ fix direction of ray from (z,y_s) to $(0,x_s)$ at $(0,x_s)$, therefore

(z,y_s) must be on the ray determined by $(0,x_s)$ and $p_s = \dfrac{\partial T_{\text{data}}}{\partial x_s}$,

(z,y_r) must be on the ray determined by $(0,x_r)$ and $p_r = \dfrac{\partial T_{\text{data}}}{\partial x_r}$.

Energy in $R(z,x,h,t)$ propagates downward along DSR raypairs ("double rays")



Focusing property: If T_{data} due to reflection at $(z_{\mathsf{refl}}, x_{\mathsf{refl}})$, then for $t=0$, energy is focused at $(z,x,h) = (z_{\mathsf{refl}}, x_{\mathsf{refl}}, 0)$.

Focusing ⇒ angle gathers are artifact free

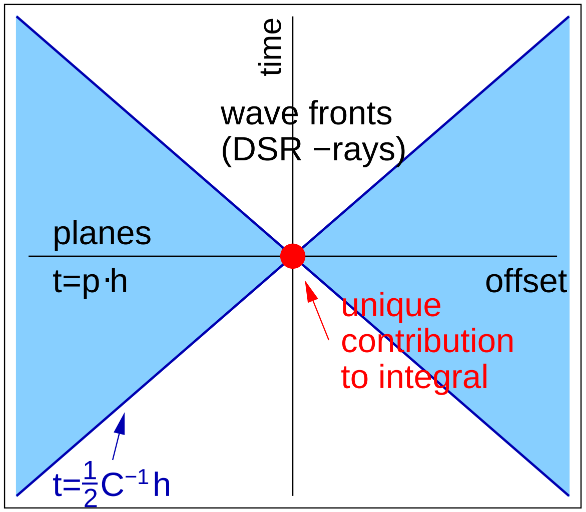
Angle transform

$$f_{\mathsf{WE-angle}}(p,z,x) = \int R(z,x,h,p\cdot h)dh.$$

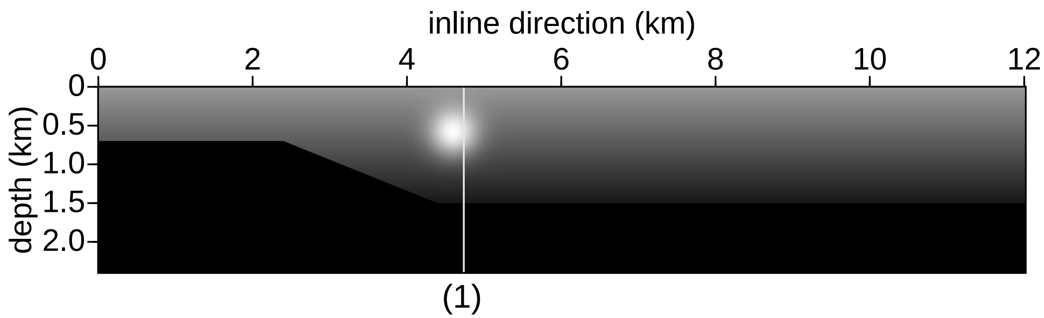
Suppose $c(z,x) < C$ then for focusing rays, energy is present only if

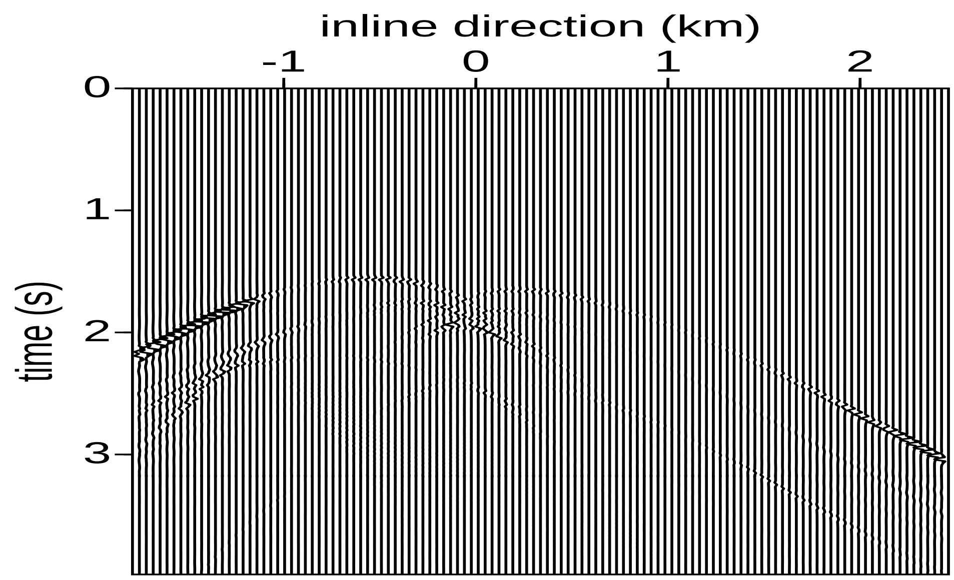
$$h < Ct$$

Now assume $\|p\| < C^{-1}$. Then we have a unique contribution to $f_{\mathsf{WE-angle}}$, with the DSR assumption.

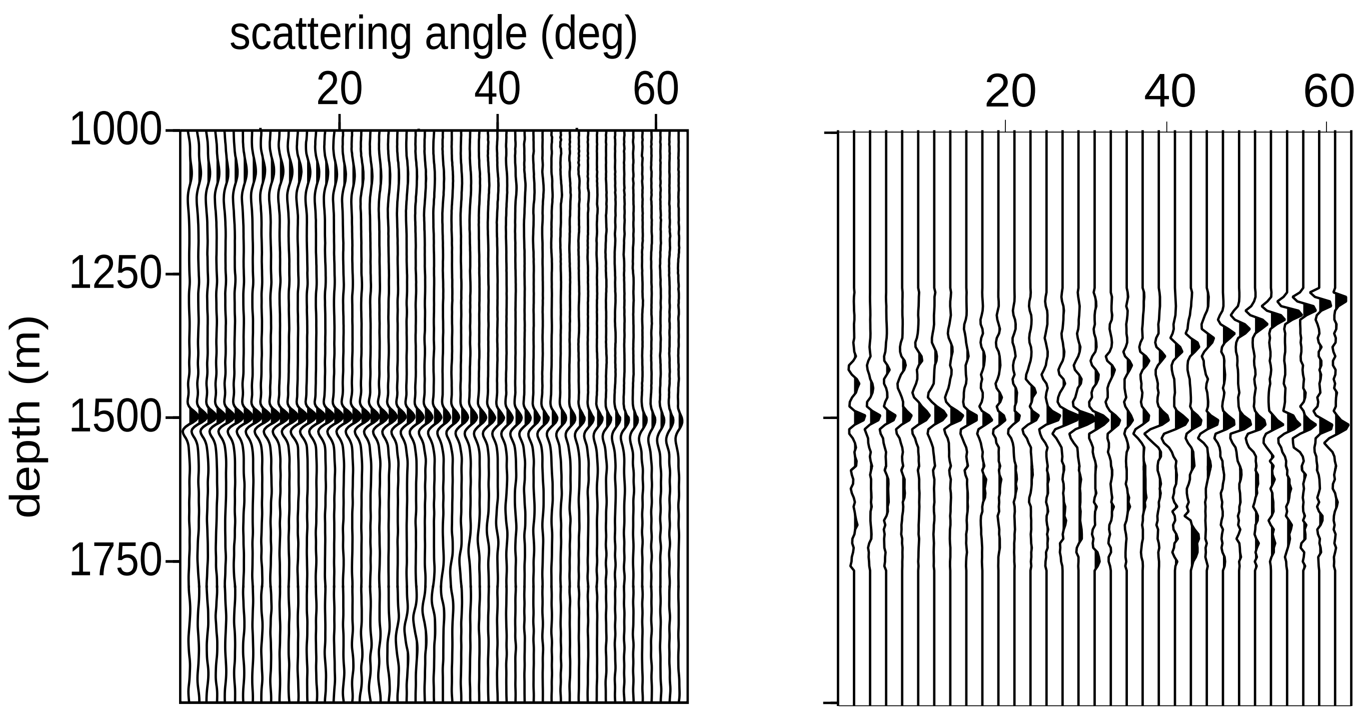


Synthetic example: A simplified model for gas lenses observed in the Valhall field. Top: The model. Bottom: Data, a typical shot.





Angle CIG's at position (1) for synthetic data from the lens using the correct velocity model. Left: Using the wave equation method. Right: Using the Kirchhoff method. Artifacts are present in the Kirchhoff CIG, but absent in the wave equation CIG.



Conclusions

- The DSR modeling formulation provides a way to go from the wave equation to DSR imaging and migration.

  Claerbout's imaging conditions are explained in this way, and amplitude factors can be derived.

- Wave equation angle gathers are artifact free, under the DSR condition. This allows for caustics, unlike with Kirchhoff angle gathers.