Multivalued traveltimes via Liouville equations

Recent developments in level set methods

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Outline

- Overview for multivalued geometrical optics

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- Paraxial formulation
- What’s next

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- High frequency asymptotics for wave equations:

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- H is a homogeneous Hamiltonian of degree one in p.

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- Caustics decomposition (Benamou'99), slowness matching (Symes'96), segment projection (Engquist, etal '02), level sets (Osher, etal'02, Fomel-Sethian'02)

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- Hamiltonian system:

$$\frac{d\tilde{\mathbf{x}}}{dt} = \nabla_{\mathbf{p}} H(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}), \quad \tilde{\mathbf{x}}(0, \mathbf{x}, \mathbf{p}) = \mathbf{x}$$

$$\frac{d\tilde{\mathbf{p}}}{dt} = -\nabla_{\mathbf{x}} H(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}), \quad \tilde{\mathbf{p}}(0, \mathbf{x}, \mathbf{p}) = \mathbf{p}$$

$$\frac{d\tilde{\tau}}{dt} = \tilde{\mathbf{p}} \cdot \nabla_{\tilde{\mathbf{p}}} H(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}), \quad \tilde{\tau}(0, \mathbf{x}, \mathbf{p}) = 0.$$

Liouville equations

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  $$w_t + \nabla_{\mathbf{p}} H \cdot \nabla_{\mathbf{x}} w - \nabla_{\mathbf{x}} H \cdot \nabla_{\mathbf{p}} w = 0$$
- Liouville equation shares the same bicharacteristics as the original nonlinear PDE.

Properties of Liouville equations

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- A linear transport equation for w in phase space.

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- Nonlinear first order eqn: local smooth solutions only.

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$$w_t + c(\mathbf{x})\frac{\mathbf{p}}{|\mathbf{p}|} \cdot \nabla_{\mathbf{x}} w - \nabla_{\mathbf{x}}(c(\mathbf{x})|\mathbf{p}|) \cdot \nabla_{\mathbf{p}} w = 0$$

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- Reduced form (Engquist-Runborg’02):

$$w(t,\mathbf{x},\mathbf{p}) = c(\mathbf{x})\delta(|\mathbf{p}| - \frac{\mathbf{1}}{\mathbf{c}(\mathbf{x})})\mathbf{u}(\mathbf{t},\mathbf{x},\frac{\mathbf{p}}{|\mathbf{p}|})$$

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- Two dimension: **p=**$(r\cos\theta, r\sin\theta)$.

$$\begin{aligned} &u_t + c\cos(\theta)u_{x_1} + c\sin(\theta)u_{x_2} \\ &+ \left(c_{x_1}\sin(\theta) - c_{x_2}\cos(\theta)\right)u_\theta = 0 \end{aligned}$$

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- A similar equation exists in 3-D case.

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- Wavefront for the ray tracing system in reduced phase space $(\mathbf{x}, \mathbf{s})$:

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  - $\mathbf{x} \in \mathcal{R}^3$, $\frac{\mathbf{p}}{|\mathbf{p}|} \in \mathcal{S}^2 \Rightarrow$ Wavefront of dim = 2 and co-dimension = 3; a surface intersected by 3 level sets

Level set motion equation

- Define scalar level set functions: $\phi^k = \phi^k(t, \mathbf{x}, \mathbf{p})$, $k = 1, 2, \cdots, m$ (with m=2-D or 3-D).

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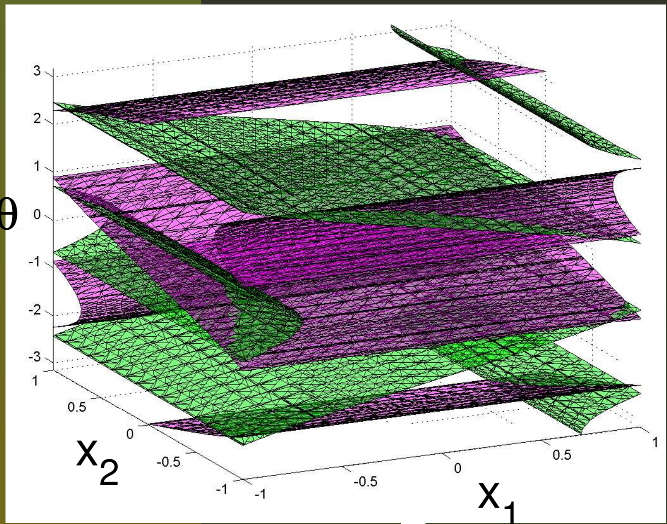
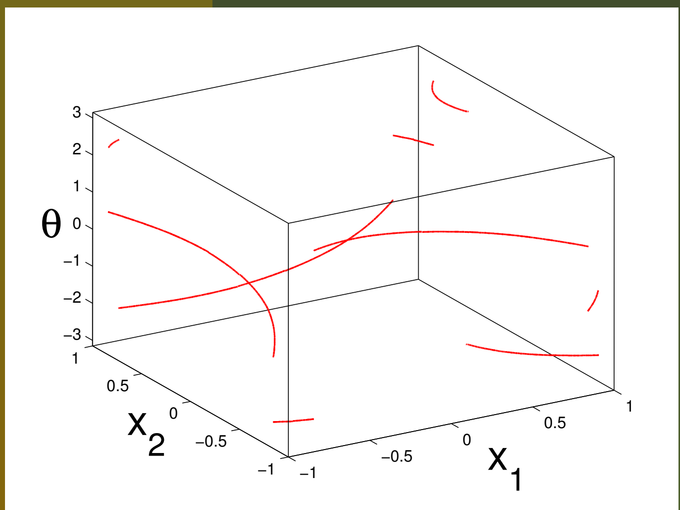
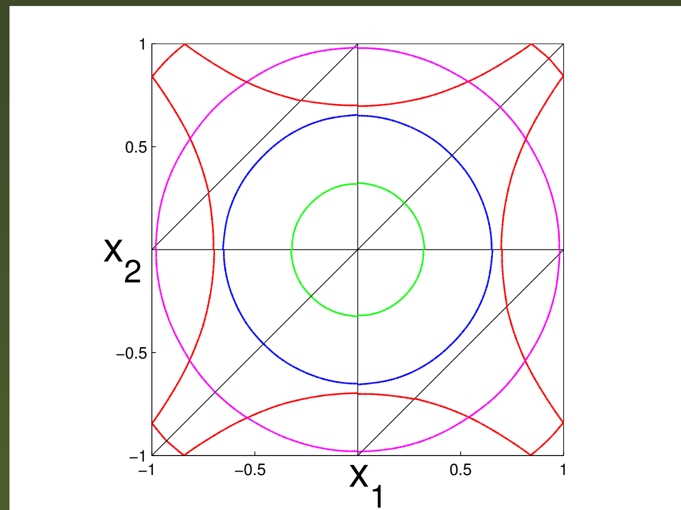
- Initialize those equations with initial wavefronts in phase space

Extract multiple traveltimes





- $\Gamma(t) = \{(\mathbf{x}, \mathbf{p}) : \Phi(t, \mathbf{x}, \mathbf{p}) = 0\}$: wavefront in phase space at time t.



Recent Developments

- Spectral/discontinuous Galerkin (DG) finite-element formulation (Cockburn-Qian-Reitich-Wang'04)

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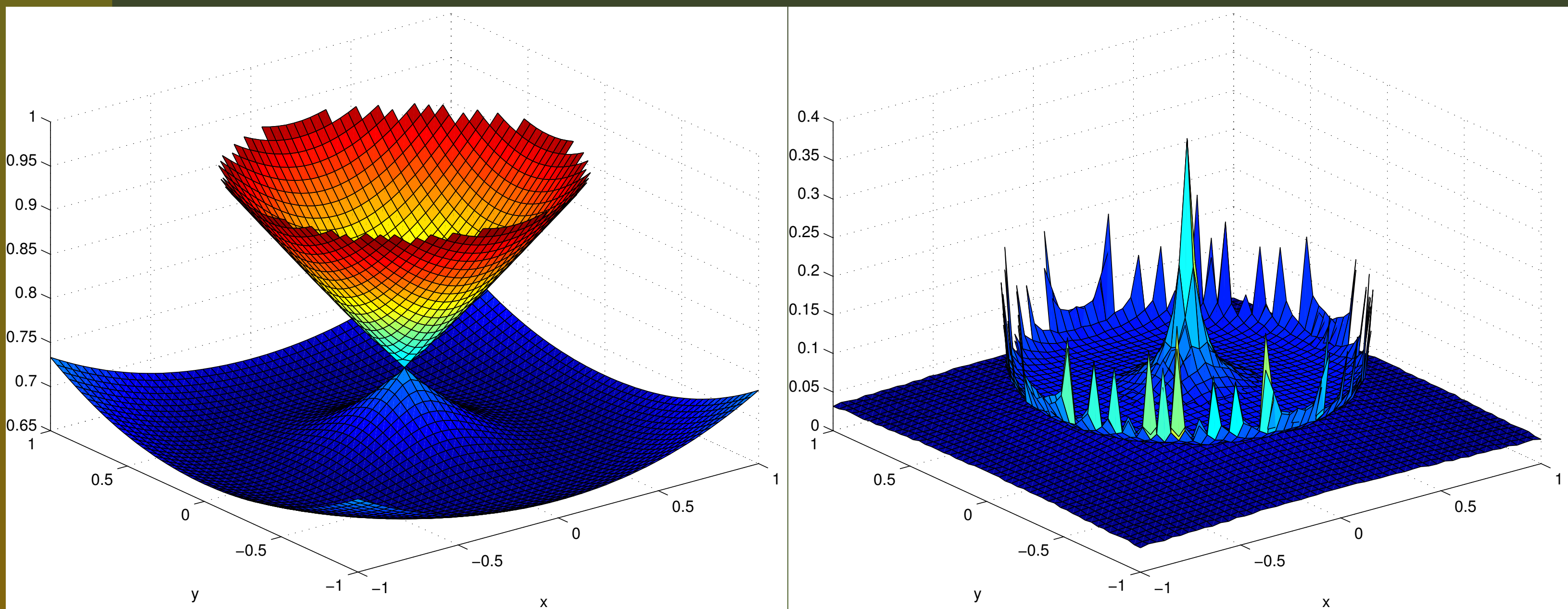
- Paraxial formulation for 3-D geometrical optics (Leung-Qian-Osher'04)



Spectral/DG formulation

- Liouville equation:

$$u_t + c\cos(\theta)u_{x_1} + c\sin(\theta)u_{x_2} + \left(c_{x_1}\sin(\theta) - c_{x_2}\cos(\theta)\right)u_\theta = 0$$

- Pseudo-Spectral formulation:

$$u(x_1, x_2, \theta, t) = \sum_{n=-N}^{N} U_n(x_1, x_2, t)\, e^{in\theta}$$

Spectral/DG formulation: cont.

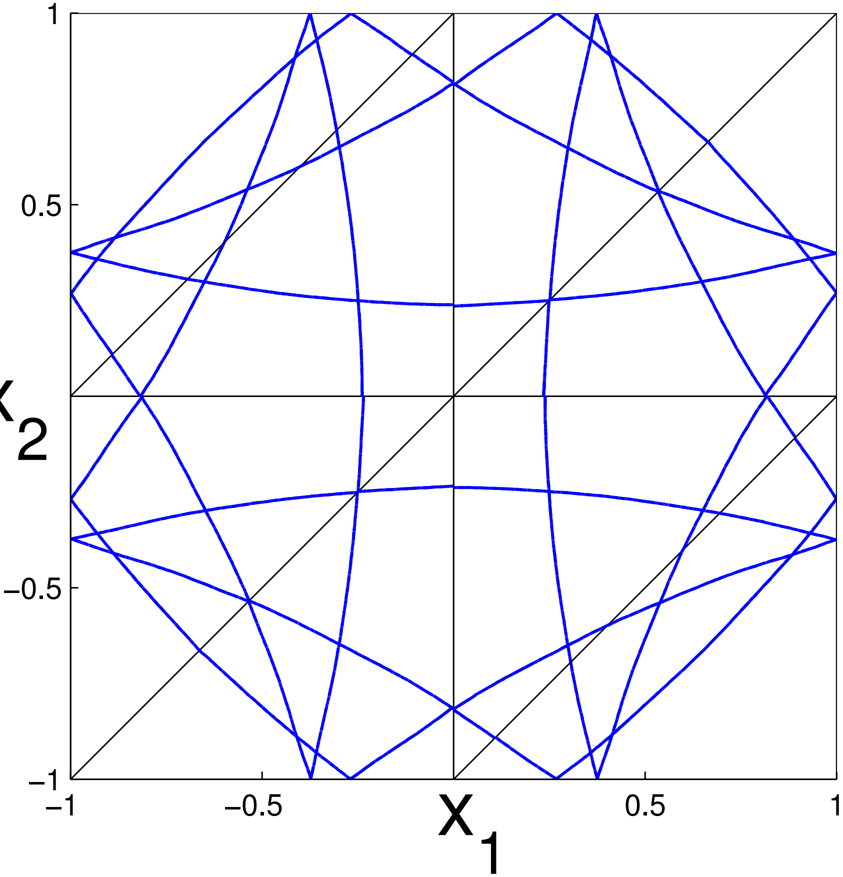
- Strictly symmetric and explicitly diagonalizable hyperbolic system for spectral coefficients:

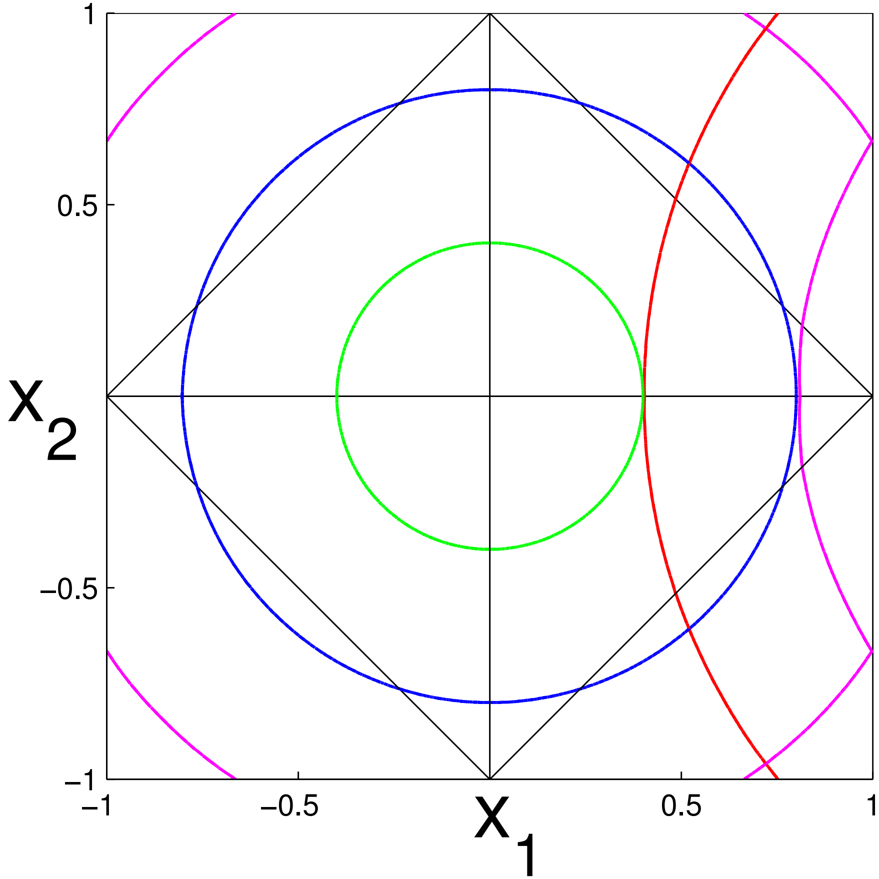
$$\mathbf{U}_t + A_1\mathbf{U}_{x_1} + A_2\mathbf{U}_{x_2} + B\mathbf{U} = 0$$

- Apply discontinuous Galerkin formulation (Cockburn-Shu'01)
- DG: Easily parallelizable and arbitrarily high order accuracy

Spectral/DG formulation: results

- Multiple reflection:





Paraxial Liouville formulation: 3-D

- Paraxial ray tracing system:

$$x_z = \frac{1}{\cos\psi \tan\theta}, \quad y_z = \tan\psi,$$
$$\theta_z = \frac{c_x}{c\cos\psi} - \frac{c_z + c_y \tan\psi}{c\tan\theta},$$
$$\psi_z = \frac{c_z \tan\psi - c_y}{c\sin^2\theta}$$

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- Paraxial Liouville eqns for level sets and traveltimes:

$$\phi_z^m + x_z\phi_x^m + y_z\phi_y^m + \theta_z\phi_\theta^m + \psi_z\phi_\psi^m = 0$$
$$T_z + x_z T_x + y_z T_y + \theta_z T_\theta + \psi_z T_\psi = \frac{1}{c\sin\theta\cos\psi}$$

Paraxial Liouville formulation: cont.

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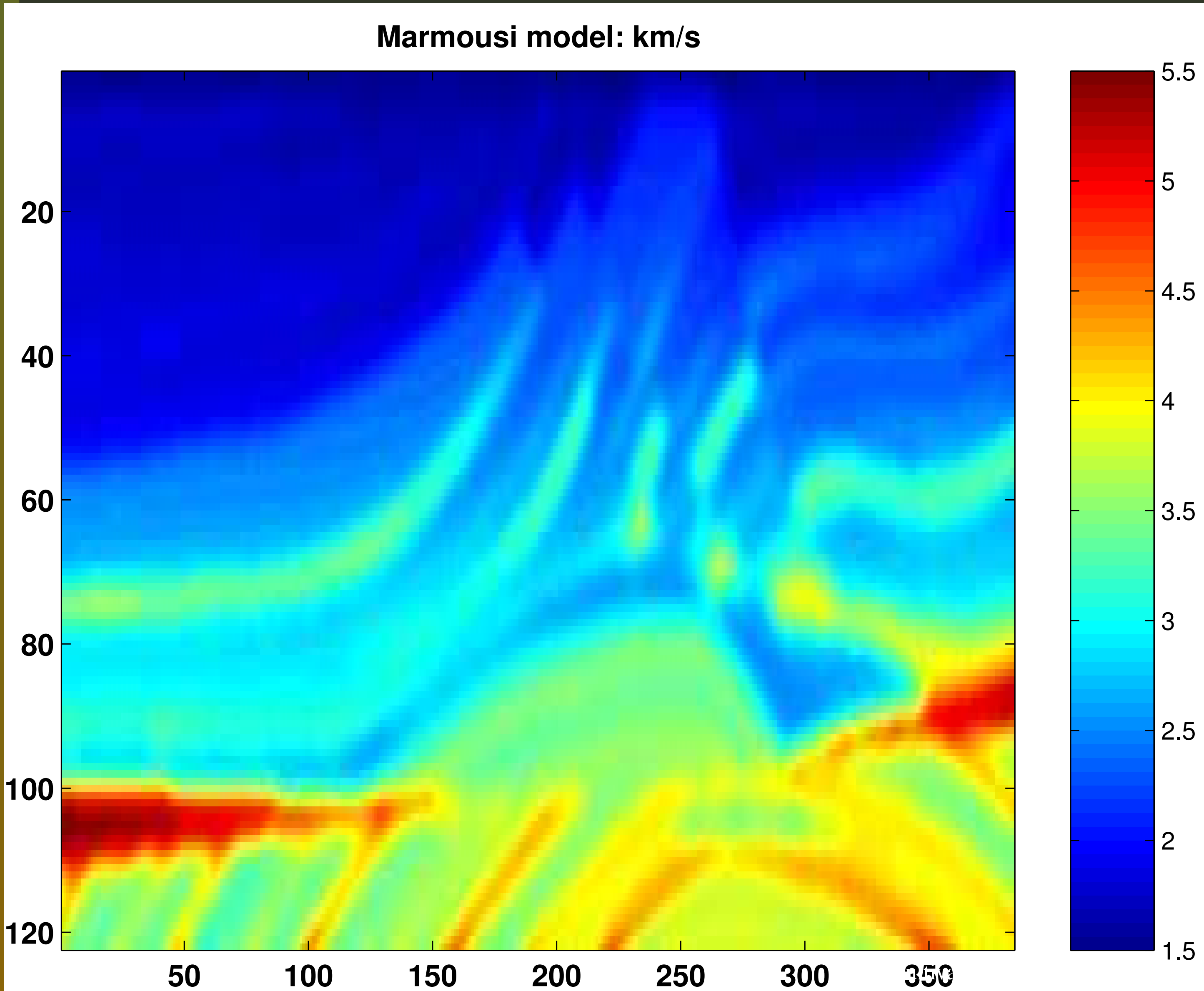
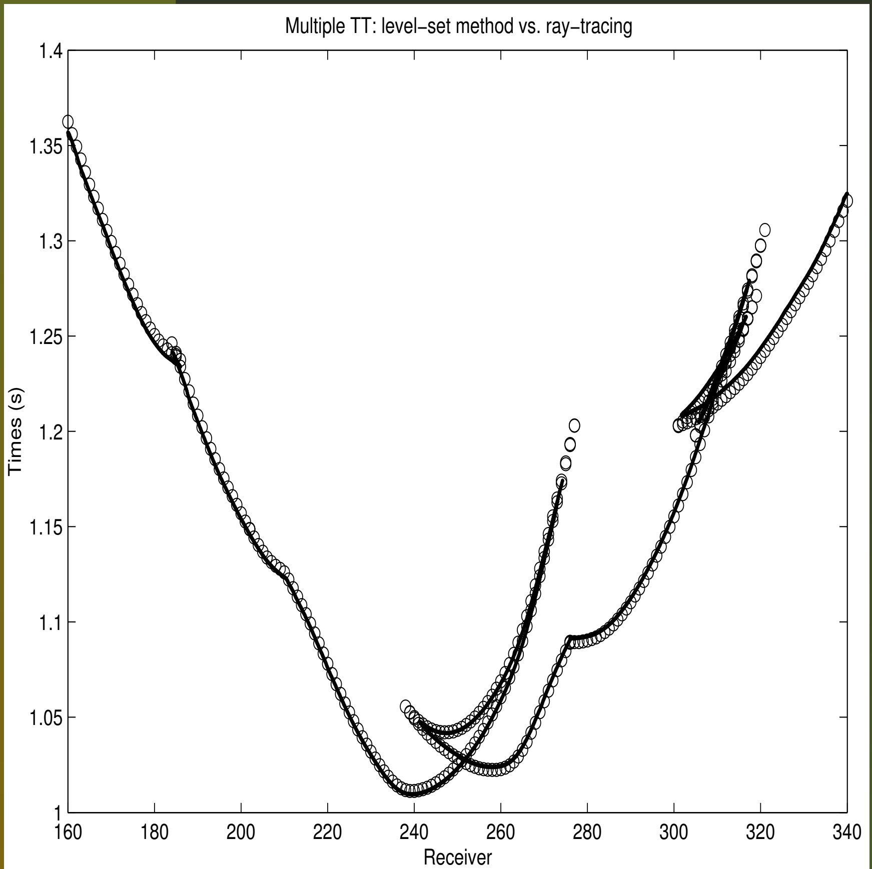
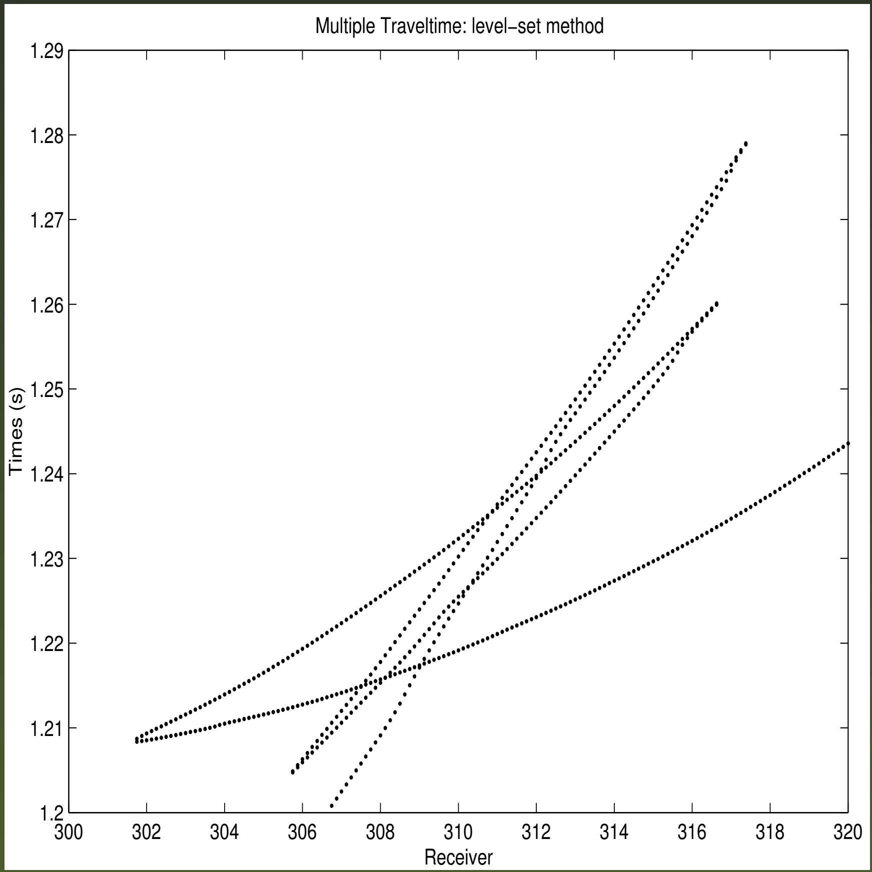
- Amplitude can be computed in the same framework
- Efficient implementation by using Semi-Lagrangian method

Synthetic Marmousi model

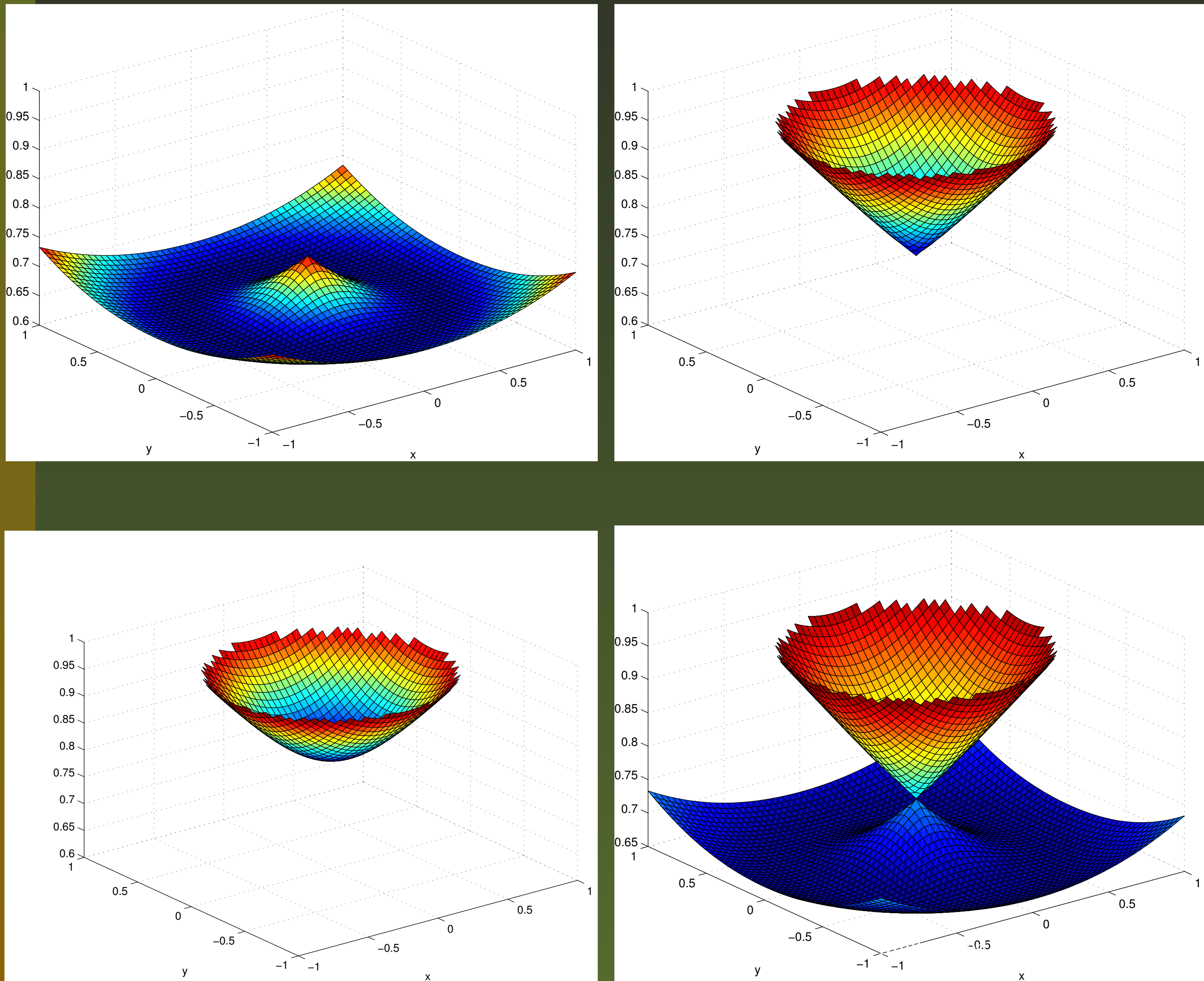


Marmousi: traveltime

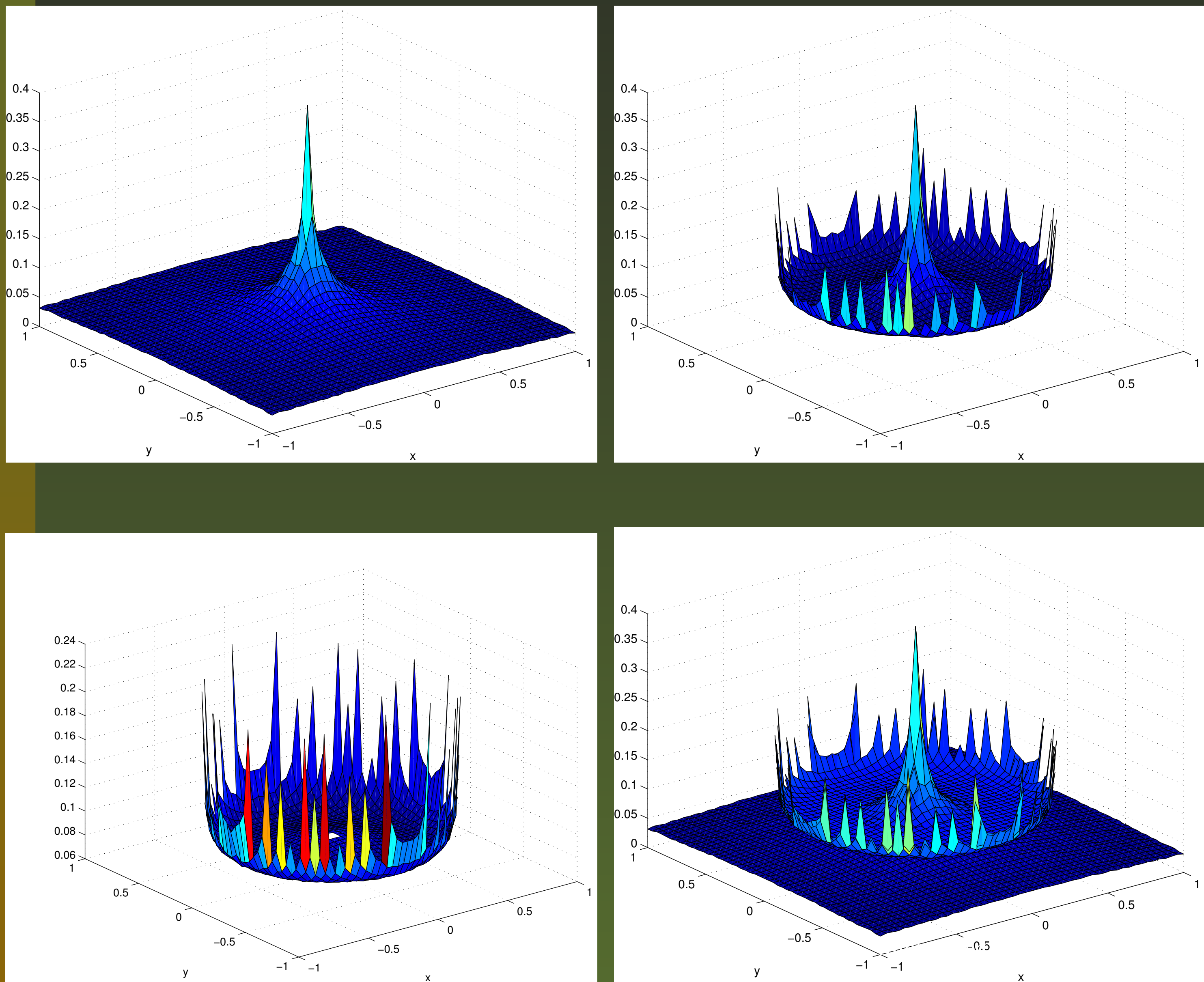




3D Vinje's Gaussian: traveltime



3D Vinje's Gaussian: amplitude



What's next

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- Multivalued high resolution reflection tomography