Nonlinear Generalization of Claerbout's Extended Scattering Model

William W. Symes *

ABSTRACT

The Claerbout extension of linearized seismic modeling provides a framework for velocity analysis and imaging. It extends the reflectivity volume by introducing additional degrees of freedom, in subsurface offset. The generalization to nonlinear scattering makes the velocity field into an operator, the kernel variables of which play the role of the "sunken" source and receiver in the linear theory. Formal linearization of this nonlinear extended scattering model about the multiplication operator by a positive velocity field recovers Claerbout's linear extended model. The generalization to (anisotropic) linear elasticity is straightforward.

Extended Scattering

Let $\mathcal{V}$ denote the bounded measurable functions on $\mathbf{R}^3$ whose reciprocals are also bounded. The acoustic (constant density) scattering operator $\mathcal{F} : \mathcal{V} \rightarrow L^2(Y)$ is defined by

$$\mathcal{F}[v] = u(\cdot,\cdot,\cdot;v)|_Y$$

where the acoustic potential field $u(\mathbf{x},t,\mathbf{x}_s;v)$ satisfies

$$\left(v^{-2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u(\cdot,\cdot,\mathbf{x}_s;v) = w(t)\delta(\mathbf{x}-\mathbf{x}_s) \tag{1}$$

with appropriate initial and boundary conditions, and $Y = \{(\mathbf{x}_r,t,\mathbf{x}_s)\}$ is the acquisition manifold.

Note that the field u only has properties which allow identification of $L^2(Y)$ as the range under special circumstances, eg. when v is smooth near source and receiver points and the source wavelet w is square-integrable.

Denote by $\bar{\mathcal{V}}$ the bounded positive selfadjoint operators on $L^2(\mathbf{R}^3)$. Given $\bar{v} \in \bar{\mathcal{V}}$, the generalized acoustic potential field $\bar{u}(\mathbf{x},t,\mathbf{x}_s;\bar{v})$ satisfies

$$\left(\bar{v}^{-2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \bar{u}(\cdot,\cdot,\mathbf{x}_s;\bar{v}) = w(t)\delta(\mathbf{x}-\mathbf{x}_s) \tag{2}$$

*The Rice Inversion Project, Department of Computational and Applied Mathematics, Rice University, Houston TX 77251-1892 USA, email `symes@caam.rice.edu`

The extended forward modeling operator is

$$\bar{\mathcal{F}}[\bar{v}] = u(\cdot,\cdot,\cdot;\bar{v})|_Y$$

Define the extension map $\chi : \mathcal{V} \rightarrow \bar{\mathcal{V}}$ by

$$\chi[v]f(\mathbf{x}) = v(\mathbf{x})f(\mathbf{x}),\ \ f \in L^2(\mathbf{R}^3)$$

χ is continuous, eg. with the L^∞ norm in domain and the operator norm in the range.

Note that $\bar{\mathcal{F}}\chi = \mathcal{F}$, so the foregoing actually does define an extension of $\mathcal{F}$.

Linearization

Formally expand $\bar{\mathcal{F}}$ about $\chi[v_0], v_0 \in \mathcal{V}$:

$$\bar{\mathcal{F}}[\chi[v_0] + \delta\bar{v}] \simeq \mathcal{F}[v_0] + D\bar{\mathcal{F}}[\chi[v_0]][\delta\bar{v}]$$

The formal perturbation operator $D\bar{\mathcal{F}}$ is given by

$$D\bar{\mathcal{F}}[\chi[v_0]][\delta\bar{v}] = \delta\bar{u}|_Y,$$

where

$$\left(v_0^{-2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta\bar{u}(\cdot,\cdot,\mathbf{x}_s;v_0,\delta\bar{v}) = 2v_0^{-1}\delta\bar{v}\left[v_0^{-2}\frac{\partial^2 u_0}{\partial t^2}(\cdot,\cdot,\mathbf{x}_s;v_0)\right] \tag{3}$$

and u_0 solves (1) with $v = v_0$. Introduce the Schwarz kernel R of the operator $2v_0^{-1}\delta\bar{v}v_0^{-2}$:

$$(2v_0^{-1}\delta\bar{v}v_0^{-2})f(\bar{\mathbf{x}}_r) = \int dy\, R(\bar{\mathbf{x}}_r,\bar{\mathbf{x}}_s)f(\bar{\mathbf{x}}_s)$$

and the causal Green's function (retarded fundamental solution) G_0 of the operator on the left hand side of (3). Then we can write

$$D\bar{\mathcal{F}}[\chi[v_0]][\delta\bar{v}](\mathbf{x}_r,t,\mathbf{x}_s) = \int\int\int d\bar{\mathbf{x}}_r\, d\bar{\mathbf{x}}_s\, dt'\, G_0(\mathbf{x}_r,t-t',\bar{\mathbf{x}}_r)R(\bar{\mathbf{x}}_r,\bar{\mathbf{x}}_s)G_0(\bar{\mathbf{x}}_s,t',\mathbf{x}_s) \tag{4}$$

which is immediately recognizable as the Claerbout extension of the linearized acoustic scattering operator, with extended reflectivity volume $R(\bar{\mathbf{x}}_r,\bar{\mathbf{x}}_s)$, see eg. (Biondi et al., 2003). The reason for the peculiar notation is now also clear: $\bar{\mathbf{x}}_s$ and $\bar{\mathbf{x}}_r$ are the "sunken" source and receiver coordinates of Claerbout's conception.

"Physical" reflectivity corresponds to $\delta\bar{v}$ in the range of χ, which is equivalent to R taking the form

$$R(\bar{\mathbf{x}}_r,\bar{\mathbf{x}}_s) = r(\bar{\mathbf{x}}_s)\delta(\bar{\mathbf{x}}_r - \bar{\mathbf{x}}_s)$$

in which $r = 2v_0^{-3}\delta v$ and the kernel of $\delta\bar{v}$ is $\delta v(\bar{\mathbf{x}}_s)\delta(\bar{\mathbf{x}}_r - \bar{\mathbf{x}}_s)$. For such "physical" R, (4) defines the usual linearized or "Born" scattering operator with reflectivity volume r.

For a more complete account of the relation between this extension of Born scattering and Claerbout's survey-sinking shot-geophone migration concept (Claerbout, 1985), see (Symes, 2002; Stolk and De Hoop, 2001).

As explained (Stolk and De Hoop, 2001), for instance, the "DSR" assumption, that rays carrying significant energy do not turn horizontal, naturally invites partial imposition of the "physical" constraint, by requiring that

$$R(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) = \tilde{R}(\bar{\mathbf{x}}'_r, \bar{\mathbf{x}}'_s)\delta(\bar{z}_r - \bar{z}_s) \tag{5}$$

in which the prime denotes the horizontal coordinate subvector. This constraint also makes sense in the nonlinear setting, in which it becomes

$$(\bar{v}f)(\cdot, z) = \tilde{v}(f(\cdot, z)). \tag{6}$$

Here $\tilde{v}$ is a bounded measurable function of z with values in positive bounded selfadjoint operators on $L^2(\mathbf{R}^2)$. Clearly (5) is the linearization of (6).

We will refer to both (5) and (6) as the DSR constraint.

The Layered Case

Suppose for simplicity that sources and receivers are assumed to occupy the same depth plane, i.e. $z_r = z_s$, and write $\mathbf{x}_r = (x_r, y_r, z_r)$ etc. Define the medium model $\bar{v} \in \bar{\mathcal{V}}$ to be *layered* iff the seismogram $\bar{\mathcal{F}}[\bar{v}]$, and more generally the acoustic potential field $\bar{u}$, depends only on the horizontal components of offset $x_r - x_s, y_r - y_s$, respectively $x - x_s, y - y_s$. For physical media ($\bar{v} = \chi[v]$) this implies that $v = v(z)$. In general, it is easy to see that $\bar{v}$ is layered if and only if

$$\bar{v}f(\bar{\mathbf{x}}_r) = \int\int\int d\bar{\mathbf{x}}_s C_{\bar{v}}(\bar{\mathbf{x}}'_r - \bar{\mathbf{x}}'_s, \bar{z}_r, \bar{z}_s) f(\bar{\mathbf{x}}_s)$$

for some suitable distribution $C_{\bar{v}}$, i.e. $\bar{v}$ acts as a convolution in the horizontal variables. Formally, the Fourier transform of $C_{\bar{v}}$ in the horizontal variables should be positive, in the sense of yielding a positive operator when evaluated on a positive test function.

With the imposition of the "DSR" constraint, the kernel may be expressed as

$$C_{\bar{v}}(\bar{\mathbf{x}}', \bar{z}_r, \bar{z}_s) = c(\bar{\mathbf{x}}', \bar{z}_s)\delta(\bar{z}_r - \bar{z}_s)$$

Thus the "sunken source and receiver" points $\bar{\mathbf{x}}_s$, and $\bar{\mathbf{x}}_r$ reside at the same depth level.

The action of $\bar{v}$ (hence of $\bar{\mathcal{F}}$) in this case is computable inexpensively via the Fourier transform, which also decomposes the evolution problem (2) into a suite of 1D problems:

$$\left(\frac{1}{\hat{c}^2(k_x, k_y, z)}\frac{\partial^2}{\partial t^2} + k_x^2 + k_y^2 - \frac{\partial^2}{\partial z^2}\right)\hat{u}(k_x, k_y, z, t) = w(t)\delta(z - z_s) \tag{7}$$

Differential Semblance

A natural annihilator of the range of the extension map χ is given in terms of the distribution kernel $C_{\bar{v}}$ of $\bar{v}$:

$$\bar{v} \in \mathcal{R}(\chi) \Leftrightarrow (\bar{\mathbf{x}}_r - \bar{\mathbf{x}}_s)C_{\bar{v}}(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) = 0$$

Define an operator W on $\mathcal{L}(L^2(\mathbf{R}^3))$ by multiplying the kernel by the offset function:

$$C_{W\bar{v}} = \phi(\bar{\mathbf{x}}_r - \bar{\mathbf{x}}_s)(\bar{\mathbf{x}}_r - \bar{\mathbf{x}}_s)C_{\bar{v}}$$

in which $\phi \in C_0^\infty(\mathbf{R}^3)$ is identically $= 1$ near the origin. Inclusion of such a cutoff is necessary to render $W\bar{v}$ bounded, and also corresponds to practical necessity - computational implementations can use only a finite range of offsets. The operator W, so defined, is continuous in the operator norm on $\mathcal{L}(L^2(\mathbf{R}^3))$ and vanishes on the range of χ.

As explained in for example (Symes, 2004), such an annihilator leads to a reformulation of the seismic inverse problem. An optimization problem of differential semblance type expressing this equivalence is

$$\min_{\bar{v}} \|W\bar{v}\|_H \text{ subj } \bar{\mathcal{F}}[\bar{v}] = d \tag{8}$$

The norm $\|\cdot\|_H$ should be a Hilbert norm on some subspace of $\mathcal{L}(L^2(\mathbf{R}^3))$; it is not clear at this point what are the reasonable choices.

Approximation of (8) about a multiplication operator yields the differential semblance problem studied in (Shen et al., 2003).

For the layered case, Fourier transform leads to an annihilator incorporating a factor of ∇_{k_x,k_y}, i.e. truly a differential semblance operator. The problem analogous to (8) is similar to the plane wave differential semblance problem studied in (Symes, 1991).

Elasticity

The nonlinear generalization of the Claerbout extension is easy to formulate for any variant of linear elasticity, and indeed for a wide class of linear hyperbolic systems. It is required only that the equations of motion be written in the form

$$\left(A\frac{\partial}{\partial t} + \mathbf{D}\cdot\nabla\right)U = F$$

in which A is a positive definite bounded measurable symmetric matrix-valued function of position, $\mathbf{D}$ is a 3-vector of constant matrices, and U and F represent the system state and energy source respectively. For elasticity, U is a vector consisting of stress and velocity components, and A is a block matrix whose blocks include ρI (ρ being the material density) and the inverse Hooke tensor.

The Claerbout extension for such a system simply consists in replacing A by a positive definite self-adjoint operator. Formally, the rest of the framework developed above carries over without alteration. Of course, the acoustic scattering problem and its Claerbout extension can be formulated in this way as well.

Acknowledgements

This work was supported in part by National Science Foundation, and by the sponsors of The Rice Inversion Project (TRIP).

REFERENCES

Biondi, B., Tisserant, T., and Symes, W. (2003). Wavefield-continuation angle-domain common-image gathers for migration velocity analysis. In *Expanded Abstracts, Society of Exploration Geophysicists, 73rd Annual International Meeting*. SEG.

Claerbout, J. (1985). *Imaging the Earth's Interior*. Blackwell Scientific Publishers, Oxford.

Shen, P., Symes, W., and Stolk, C. (2003). Differential semblance velocity analysis by wave-equation migration. In *Expanded Abstracts, Society of Exploration Geophysicists, 73rd Annual International Meeting*. SEG.

Stolk, C. C. and De Hoop, M. V. (2001). Seismic inverse scattering in the 'wave-equation' approach. Preprint 2001-047, The Mathematical Sciences Research Institute, http://msri.org/publications/preprints/2001.html.

Symes, W. (1991). Layered velocity inversion: a model problem from reflection seismology. *SIAM J. Math. Anal.*, 22:680–716.

Symes, W. W. (2002). The Mathematics of Seismic Inverse Scattering: lecture notes for the C. I. M. E. summer school, Martina Franca, Italy. Technical report, The Rice Inversion Project, Rice University, Houston, Texas, USA.

Symes, W. W. (2004). Extensions and Seismic Inverse Scattering: Lecture at Opening Conference of IPRPI, April 2004. Technical report, The Rice Inversion Project, Rice University, Houston, Texas, USA.