Level set based Eulerian methods for multivalued traveltimes in both isotropic and anisotropic media

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Summary

We consider here a level set based Eulerian method for computing multivalued traveltimes in both isotropic and anisotropic media. A self-intersecting wavefront in two-dimensional (2-D) position space can be lifted up to be a smooth curve in a three-dimensional (3-D) space. This lifted curve in the 3-D space can then be represented as the common zeros of two level set functions. Thus, its evolution follows the evolution of the level set functions under appropriate velocity fields. The projection of the evolved 3-D curve to the original position space yields the possibly self-intersecting wavefront at a desired moment. Numerical experiments, including a sinusoidal waveguide model, the well-known Marmousi model, and a transversely isotropic model, illustrate the effectiveness, accuracy and stability of our new level set based Eulerian method.

Introduction

Seismic inverse scattering theory is based upon geometric optics for modeling high frequency wave propagation (Symes, 1995). Two functions are used to characterize the leading singularity of the wave field, namely traveltime and amplitude (Keller and Lewis, 1995). The traveltime function satisfies the so-called eikonal equation which is a first-order nonlinear partial differential equation. In general, the eikonal equation admits more than one Lipschitz continuous solutions (weak solutions); however, one of the physically relevant solutions, the so-called viscosity solution, is unique, which corresponds to the first-arrival traveltime for the eikonal equation (Crandall and Lions, 1983). Moreover, monotone and higher order accurate Essentially NonOscillatory (ENO) (Osher and Shu, 1991) and Weighted ENO (Jiang and Peng, 2000) finite difference methods applied to the eikonal equation are able to compute this viscosity-solution based first-arrival traveltime accurately and efficiently; see (Osher and Shu, 1991; van Trier and Symes, 1991; Jiang and Peng, 2000; Qian and Symes, 2002a) for references therein.

Nevertheless, the first-arrival wavefront may not carry the most energetic part of the wave-field, and the later-arrival wavefronts may be more interesting and useful for modern high resolution seismic imaging (Liu and Bleistein, 1995). The classic ray tracing method yields both first arrivals and later arrivals, and works for both isotropic and anisotropic media; however, it suffers from some shortcomings; for example, the traveltime is not uniformly distributed to the computational domain and the related interpolation onto a uniform mesh is cumbersome and expensive, although there are some improvements in this regard at the cost of complicated data structures and bookkeeping (Vinje et al., 1993). On the other hand, finite-difference based eikonal solvers for generating first arrivals can be designed to work for both isotropic and anisotropic media with traveltimes distributed uniformly to the computational domain; see (Qian and Symes, 2002b) and references therein. Thus, the challenge is to design finite difference based Eulerian methods for multivalued traveltimes which are able to produce uniformly distributed traveltimes, including both first and later arrivals, and work for both isotropic and anisotropic media.

In principle, there are two kinds of multivaluedness for the traveltime field. One is due to the inhomogeneity of the material parameters, and the other is due to the structure of the material. In isotropic media, the multivaluedness of the traveltime field is only due to the former and the singularity of the wavefront needs some time to develop during the wave propagation; most of the current available multivalued traveltime solvers only deal with this kind of singularity (Engquist et al., 1995; Benamou, 1996; Symes, 1996; Engquist et al., 2001). In anisotropic media, the multivaluedness of the traveltime field may be caused by both the inhomogeneity and the structure of the material; for example, the instantaneous singularity associated with the quasi-transverse (qS) wave in a transversely isotropic (TI) medium is caused by the structure and exists even in a homogeneous TI medium. Recently, Qian et al (Qian et al., 2001) have extended the level set based, phase space ray tracing Eulerian approach for isotropic media, first proposed in Osher et al (Osher et al., 2001), to anisotropic media. The approach was quite successful in computing the multivalued traveltimes in both isotropic and anisotropic media and yielding uniformly distributed traveltime fields. Fomel and Sethian (Fomel and Sethian, 2001) have also proposed a so-called fast phase space method, but the formal computational complexity of their approach is much higher and so far their numerical examples are only for isotropic media.

The level set method was originally designed for problems dealing with codimension one objects, where it has achieved a great amount of success, especially when topological changes occur in the interface (Osher and Sethian, 1988), and thus has found its way into many different applications (Osher and Fedkiw, 2002). Motivated by this, Burchard et al (Burchard et al., 2001) has implemented a vectorial level set approach for capturing codimension two objects in 3-D space; namely, a curve in a 3-D space. The apparent advantage of the vectorial level set formulation is that it provides an Eulerian PDE framework so that a

3-D curve is well represented and resolved in a 3-D space. A self-intersecting wavefront in a 2-D space is a codimension one object there, however, a single level set function in the 2-D space is unable to capture the self-intersection phenomenon. Lifting the self-intersecting wavefront into its smooth curve representation in a 3-D space, though, allows for a different approach. In conjunction with this is the use of the vectorial level set formulation to move the lifted wavefront in the 3-D space according to some velocity field up to the desired time. The 3-D curve can then be projected at that time onto the original 2-D space, thus giving the desired self-intersecting wavefront.

The level set methodology

In the vectorial level set formulation for a curve in a 3-D space, the curve is represented by the intersection between the zero level set surfaces of two level set functions, ϕ and ψ, i.e., where $\phi = \psi = 0$. Under this representation, moving a curve by a certain type of motion is accomplished by evolving the two functions ϕ and ψ in the 3-D space, with the intersection of their zero level sets at the desired time giving the curve of interest.

Let $\Gamma(t) = \{\gamma(s,t) = (x(s,t), y(s,t), z(s,t) : 0 \le s \le 1, t \ge 0\}$ be the curve at time t, and let it be defined by the intersection of the zero level sets of two level set functions,

$$\Gamma(t) = \{\gamma(s,t) : \phi(t, \gamma(s,t)) = \psi(t, \gamma(s,t)) = 0, 0 \le s \le 1\}$$

Differentiating with respect to t the equations $\phi(t, \gamma(s,t)) = 0$ and $\psi(t, \gamma(s,t)) = 0$ yields

$$\nabla\phi(t, \gamma(s,t)) \cdot \gamma_t(s,t) + \phi_t(t, \gamma(s,t)) = 0$$

$$\nabla\psi(t, \gamma(s,t)) \cdot \gamma_t(s,t) + \psi_t(t, \gamma(s,t)) = 0$$

Given a velocity field $W = \gamma_t(s,t) \equiv (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$, the PDE

$$\nabla\phi \cdot \gamma_t + \phi_t = 0$$

thus moves the zero level sets of ϕ according to $\gamma_t = W$. Therefore, the above PDE system moves the curve defined by intersections of the zero level sets of ϕ and ψ according to $\gamma_t = W$. We refer the reader to Burchard et al (2000) for more details and references.

Ray tracing system in reduced phase spaces

The 2-D eikonal equation in an isotropic medium states that

$$|\nabla\tau| = \frac{1}{v}$$

where $v = v(x,z)$ is the velocity. We define the Hamiltonian $H(\mathbf{x}, \mathbf{p}) = \frac{1}{2}(v^2|\mathbf{p}|^2 - 1)$, where $\mathbf{x} = (x,z)$ and $\mathbf{p} = \nabla\tau$. Then by the method of characteristics, we have

$$\frac{d\mathbf{x}}{dt} = v^2\mathbf{p}, \frac{d\mathbf{p}}{dt} = -\frac{1}{v}\nabla_{\mathbf{x}} v$$

Introducing the phase angle θ and parameterizing $\mathbf{p}$ by θ: $p_1 = \frac{\sin\theta}{v}$ and $p_3 = \frac{\cos\theta}{v}$, we have

$$\frac{dx}{dt} = v\sin\theta, \frac{dz}{dt} = v\cos\theta$$

$$\frac{d\theta}{dt} = v_z \sin\theta - v_x \cos\theta$$

Define $W = (\frac{dx}{dt}, \frac{dz}{dt}, \frac{d\theta}{dt})$, giving the desired velocity field for the motion of the level set functions.

Since W is defined by a reduced ray tracing system, the level set motion using the above velocity field as the passive velocity field can be understood as an Eulerian, reduced phase space ray tracing. However, different from traditional ray tracing, the new approach yields wavefronts which are uniformly distributed to the computational domain. Another point worth mentioning is that the eikonal equation is nonlinear while the level set motion equations are linear. The stability condition for finite difference methods applied to the linear advection equation is easily satisfied and guarantee the convergence of the scheme through the Lax Equivalence Theorem; however, the CFL stability condition for the eikonal equation is only a necessary condition for convergence.

Similarly, we can derive the reduced phase space ray tracing system for anisotropic media (Qian et al., 2001). Consider a 2-D anisotropic eikonal equation defined by

$$F(x, z, \tau_x, \tau_z) = F(x, z, p_1, p_3) = 0$$

Parameterize the slowness vector by

$$p_1 = \frac{\cos\theta}{V(x,z,\theta)}, p_3 = \frac{\sin\theta}{V(x,z,\theta)}$$

where θ is the so-called phase angle varying from $-\pi$ to π and V the phase velocity depending on θ. Then the method of characteristics gives us

$$\frac{dx}{dt} = \left(p_1\frac{\partial F}{\partial p_1} + p_3\frac{\partial F}{\partial p_3}\right)^{-1}\frac{\partial F}{\partial p_1}$$

$$\frac{dz}{dt} = \left(p_1\frac{\partial F}{\partial p_1} + p_3\frac{\partial F}{\partial p_3}\right)^{-1}\frac{\partial F}{\partial p_3}$$

$$\frac{d\theta}{dt} = \left(p_1\frac{\partial F}{\partial p_1} + p_3\frac{\partial F}{\partial p_3}\right)^{-1}\left(V\frac{\partial F}{\partial x}\sin\theta - V\frac{\partial F}{\partial z}\cos\theta\right)$$

The above is the velocity field W needed for the level set evolution equations. See Qian et al (2001) for more details.

Implementations

Since the two advection equations in the level set evolution system are decoupled, we only need to consider the approach for solving one of them, with the other following the same strategy. For example, to solve $\phi_t + W \cdot \nabla\phi = 0$

by finite difference methods, we may use high-order finite difference schemes for Hamilton-Jacobi equations, such as third-order WENO Godunov scheme (Jiang and Peng, 2000) for spatial derivatives, and third order Total Variation Diminishing (TVD) Runge-Kutta scheme for time derivatives (Osher and Shu, 1991); see also (Qian and Symes, 2002a) for the application of this scheme to eikonal and advection equations.

For this explicit scheme applied to the linear advection equation, the stability condition states that the time step Δt must be

$$\Delta t \leq C \frac{\Delta x}{\max |W|}$$

where $\Delta x = \Delta z = \Delta\theta$ is the spatial step for a uniform mesh and C is a CFL constant that is less than 1.

In some situations, the appearance of "kinks" in the level set functions may affect the PDE solvers, leading to the use of reinitialization steps to regularize the level set functions and make the zero level sets of ϕ and ψ orthogonal to each other at their points of intersection. See (Osher et al., 2001) for more details.

One last important point is in actually obtaining the location of the intersection of the zero level sets of ϕ and ψ. Each small cube in the 3-D mesh can be broken up into six tetrahedra, inside of which ϕ and ψ can be approximated by hyperplanes. The intersection of the zero level sets of the two hyperplanes can then be computed, giving a small segment of line inside each tetrahedron. The union of all these segments gives an approximation of the curve. See also (Burchard et al., 2001) for more.

Numerical experiments

The first is an anisotropic model, the Greenriver Shale. Figure 1 shows the computed qS wavefronts for this transversely isotropic model; the singularities in the qS wavefronts are due to the non-convexity of the corresponding slowness surface. See (Qian et al., 2001) for more.

The next model is a sinusoidal waveguide model (Symes, 1996), and its velocity is given by

$$v(x, z) = 1 + 0.2 \sin 0.5\pi z \sin 3\pi x$$

The source point is located at $(0.55, 0.05)$ near the center of the top of the domain. Slow regions form lenses and create crossing rays, imperfect foci, and caustics (Symes, 1996). Figure 2 shows the traveltime wavefronts including all arrivals computed by the level-set approach.

Figure 3 shows the traveltime wavefronts for the well-known Marmousi model including all arrivals by the level set approach, where the source is located at $(6, 2.808)$ km. The multivalued traveltime in the lower part of the model challenges modern seismic imaging methods (Liu and Bleistein, 1995).

Conclusions

We have summarized the vectorial level set based Eulerian approach for performing phase space ray tracing and verified that the approach is able to capture both developing singularities and instantaneous singularities appearing in wave propagation; especially, the approach is applicable to both isotropic and anisotropic eikonal equations.

Acknowledgements

L.-T. Cheng is supported by NSF Grant #0112413. S. Osher is supported by AFOSR Grant #F49620-01-1-0189. J. Qian thanks Profs. B. Cockburn, F. Santosa and W. Symes for their interests in this research.

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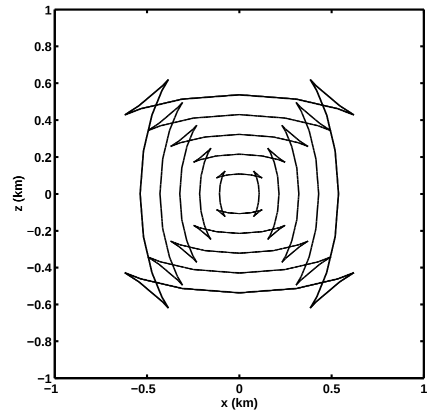

Fig. 1: The traveltime wavefronts for qS waves in Greenriver shale.

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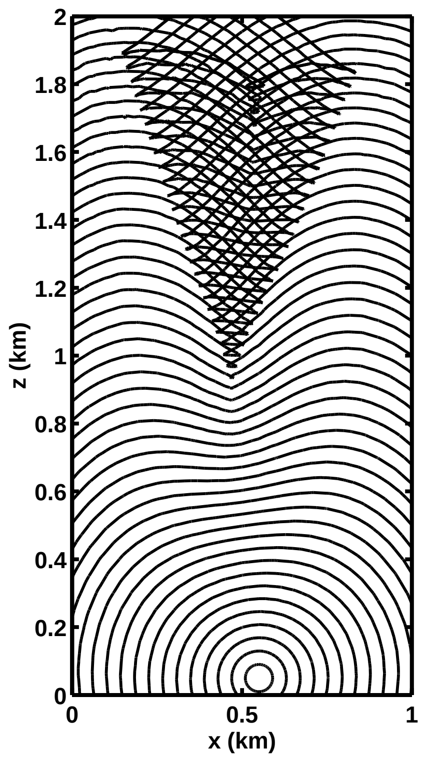

Fig. 2: The traveltime wavefronts for a sinusoidal model: all arrivals

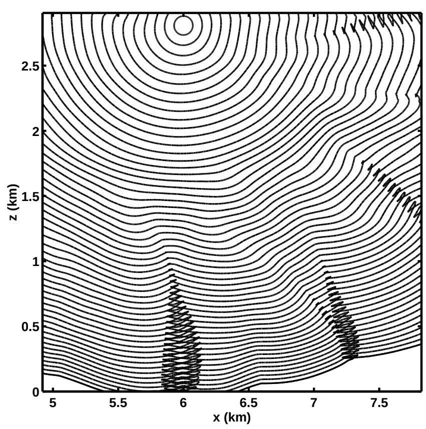

Fig. 3: The traveltime wavefronts for the Marmousi model: a windowed portion of the original model; the energy is carried upon later arrivals.