Multi-pole Representation of Small Acoustic Sources*

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Dedicated to Professor Yih-Hsing Pao on the occasion of his 70th birthday

Abstract

Band-limited acoustic sources of small spatial extent produce radiation patterns well approximated by those of multi-poles. Assumed limits on dimensionless size ka and source efficiency lead to explicit rules for the necessary length of the multi-pole series.

Introduction

Acoustic sources represent physical mechanisms which introduce energy into the radiating acoustic field, usually *via* a right hand side in the acoustic wave equation. Piezoelectric transducers and marine air-gun arrays are important examples of such mechanisms. The physical extent of these systems may be small in relation to the wavelengths of the generated radiation. For example, typical air-gun arrays are a few tens of meters in extent at most, and generate sound fields with a useful upper band-limit of perhaps 100 Hz, corresponding to a wavelength of roughly 15 meters in sea-water. If K denotes the effective upper spatial frequency band-limit of the wave field, and a the radius of a sphere enclosing the source equipment, then the dimensionless measure of size Ka is in the range 1 – 3 for these examples.

The asymptotic form of acoustic radiation far from the source is termed the *far-field pattern* (see eg. [2]; a careful definition appears below). Simple examples show that sources of small extent may radiate arbitrarily complex far-field patterns, but

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the same examples suggest that these sources may be very inefficient, in the sense that the power radiated is small, relative to the "energy" input to the source (see eg. [3], pp 150-153).

The issue of efficiency vs. complexity bears on the representation of acoustic sources. A small source might well be approximated for purposes of analysis and modeling by one of point support. Sources of point support are necessarily linear combinations of the Dirac delta and its derivative, i.e. a so-called *multi-pole series* (see for example [6], pp. 169-171). Such simple parameterizations are attractive for modeling purposes, and essential to investigations of source calibration as an inverse (parameter-fitting) problem [5]. The critical question which must be resolved in order to make effective use of multi-pole series is: how many terms in the series should be retained?

Our purpose in this paper is to formulate and justify a precise answer to this question: given an assumed source efficiency, dimensionless size Ka, and an error tolerance for fitting far-field patterns, we give an explicit prescription for determining the number of multi-pole terms required. For small Ka up to around 5, and for reasonable efficiencies and tolerances on the order of 1%, at most tenth order multi-poles are required, and usually lower order than that.

Our result hinges on a very general definition of input power, which may or may not be appropriate for particular source mechanisms. It is motivated by the example of the isotropic point radiator model, which we regard as in some sense representing an optimally efficient source. Otherwise its definition depends only on general properties of the acoustic wave equation. We believe that the modifications of this input power model necessary to accommodate the energy characteristics of particular systems will not alter our essential conclusions in any drastic way. For example, similar results should hold for small antennae; so far as we are aware, the enormous literature on antenna design does not contain such an analysis, relying only on size constraints rather than on particular configurations of antenna elements.

The paper begins with a review of the linear acoustic laws of motion and energy, and a discussion of the isotropic radiator example culminating in our definition of input power. Subsequent sections analyze the far-field of small sources, the relation between the mean square of the far field pattern and the output power, and the approximation properties of the far field for small distributed sources. From these results we extract a rule for truncating the spherical harmonic expansion of the far-field pattern under limits on efficiency and Ka, from which follow immediately the desired results on truncating multi-pole series.

A model of input and output power

The acoustic wave equation relates pressure $p(\mathbf{x},t)$, velocity $\mathbf{v}(\mathbf{x},t)$, material density $\rho(\mathbf{x})$, Lamé's modulus $\lambda(\mathbf{x})$, and an external body force density $\mathbf{f}(\mathbf{x},t)$

$$\rho\frac{\partial \mathbf{v}}{\partial t} = \nabla p + \mathbf{f},\ \frac{\partial p}{\partial t} = \lambda \nabla \cdot \mathbf{v}.$$

This paper assumes that:

- the parameters ρ and λ are independent of spatial location $\mathbf{x}$;
- for a real $a > 0$ and $T > 0$, $\mathbf{f}(\mathbf{x},t)$ vanishes outside of the ball B_a of radius a, i.e. for $|\mathbf{x}| > a$, or $t < 0$ or $t > T$;
- $p, \mathbf{v} \equiv 0$ for $t \ll 0$.

Under these conditions, specification of a reasonably regular external force density $\mathbf{f}$ uniquely determines the acoustic response, i.e. the fields $p, \mathbf{v}$. Moreover p is the unique solution of the scalar wave equation

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \nabla \cdot \mathbf{f},$$

where

$$c = \sqrt{\frac{\lambda}{\rho}},$$

is the sound speed.

The total acoustic energy density of the field is

$$E = \frac{1}{2\lambda}p^2 + \frac{\rho}{2}\mathbf{v}\cdot\mathbf{v}.$$

The wave equation implies that the rate of change of acoustic energy outside any domain Ω containing the ball of radius a is identical to an integral over the boundary $\partial\Omega$

$$\frac{d}{dt}\int_{\mathbb{R}^3\setminus\Omega} d\mathbf{x}\, E = -\int_{\partial\Omega} dS\ p\,\mathbf{v}\cdot\mathbf{n},$$

in which $\mathbf{n}$ denotes the *outward* normal to Ω and dS is the surface area element on $\partial\Omega$. The integrand $p\mathbf{v}\cdot\mathbf{n}$ is the *output power density*, and can be expressed entirely in terms of pressure using the first half of the wave equation (Newton's law)

$$\mathbf{v}\cdot\mathbf{n} = \frac{1}{\rho}\int_{-\infty}^{t} dt'\,\nabla p\cdot\mathbf{n}.$$

Thus the total power output by the source, which is equal to the total energy in the field at any sufficiently large time, is

$$P_{\mathrm{out}} = -\frac{1}{\rho}\int_{-\infty}^{\infty} dt \int_{\partial\Omega} dS\ p \int_{-\infty}^{t} dt'\,\nabla p\cdot\mathbf{n}.$$

Of course this expression is independent of Ω so long as $B_a \subset \Omega$.

The main hypothesis of this paper, on which all else depends, is that the total energy of the field is *finite*

$$\int_{-\infty}^{\infty} dt \int_{I\!R^3 \backslash B_a} dx\, E(\mathbf{x}, t) < \infty.$$

The *acoustic field potential* $u(\mathbf{x}, t)$ provides a convenient representation of *causal* finite energy fields ($p, \mathbf{v} \equiv 0$ for $t \ll 0$)

$$u(\mathbf{x}, t) = \int_{-\infty}^{t} dt'\, p(\mathbf{x}, t'); \quad p = \frac{\partial u}{\partial t}, \ \mathbf{v} = \frac{1}{\rho} \nabla u.$$

In terms of u, the total energy and power output through S_a are respectively

$$E = \int_{-\infty}^{\infty} dt \int_{I\!R^3 \backslash B_a} dx \left\{ \frac{1}{2\lambda} \left| \frac{\partial u}{\partial t} \right|^2 + \frac{1}{2\rho} \nabla u \cdot \nabla u \right\},$$

$$P_{\mathrm{out}} = -\frac{1}{\rho} \int_{-\infty}^{\infty} dt \int_{S_a} dS\, \frac{\partial u}{\partial t} \nabla u \cdot \mathbf{n}, \tag{1}$$

and of course $E = P_{\mathrm{out}}$. It also follows that both $\partial u / \partial t$ and ∇u are square integrable.

One immediate consequence of the finite energy assumption is a representation theorem for sources. Denote by β a function of a real variable satisfying

$$\beta(r) = 1 \ \text{ if } \ r > 1, \quad \beta(r) = 0 \ \text{ if } \ r < 1,$$

and set $\beta_a(\mathbf{x}) = \beta(|\mathbf{x}|/a)$. Then

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \beta_a u = 2 \nabla u \cdot \nabla \beta_a + u \nabla^2 \beta_a.$$

Thus

> *The potential of a finite energy causal acoustic field, unforced outside the ball of radius a, is identical outside the ball of radius $2a$ to the solution of a wave equation of the form*
>
> $$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) u = f, \tag{2}$$
>
> *in which $f(t, \mathbf{x})$ is square integrable and zero outside the ball of radius $2a$.*

We will call f a *square integrable representation* of the external force density. Clearly square integrable representations are nonunique; the various possibilities differ by *nonradiating* source functions.

The special case of a *point dilatational source* will be important: such a source located at the origin is parametrically

$$\mathbf{f}(\mathbf{x},t) = w(t)\nabla\delta(\mathbf{x}).$$

The corresponding acoustic potential $u(t,\mathbf{x})$ satisfies

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u(\mathbf{x},t) = \left(\int_{-\infty}^{t} dt'\, w(t')\right) \nabla^2\delta(\mathbf{x}); \quad u(\mathbf{x}, t \ll 0) \equiv 0,$$

and is related to the solution $u_0(\mathbf{x},t)$ of

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u_0(\mathbf{x},t) = \left(\int_{-\infty}^{t} dt'\, w(t')\right) \delta(\mathbf{x}); \quad u_0(\mathbf{x}, t \ll 0) \equiv 0,$$

by

$$u(\mathbf{x},t) = \nabla^2 u_0(\mathbf{x},t) = \left(\int_{-\infty}^{t} dt'\, w(t')\right) \delta(\mathbf{x}) - \frac{1}{c^2}\frac{\partial^2 u_0}{\partial t^2}(\mathbf{x},t).$$

So the *radiating part* of the point dilatational field, i.e. the part which is nonzero outside of B_a, is

$$u(\mathbf{x},t) = \frac{\frac{dw}{dt}\left(t-\frac{r}{c}\right)}{4\pi r}, \ r = |\mathbf{x}| > a,$$

and its power output is

$$\begin{aligned} P_{\text{out}} &= -\frac{1}{\rho}\int_{-\infty}^{\infty} dt \int_{S_a} dS\, \frac{\frac{d^2w}{dt^2}\left(t-\frac{a}{c}\right)}{4\pi a} \int_{-\infty}^{t} dt'\, \frac{\partial}{\partial r}\left(\frac{\frac{d^2w}{dt^2}\left(t'-\frac{r}{c}\right)}{4\pi r}\right)_{r=a} \\ &= \frac{1}{4\pi\rho c}\int_{-\infty}^{\infty} dt\, \left(\frac{d^2w}{dt^2}\right)^2(t). \end{aligned}$$

On the other hand, the "β" construction above shows that u is the solution of a wave equation with square integrable right hand side

$$f(\mathbf{x},t) = -\left(2\nabla\beta_a(\mathbf{x})\cdot\nabla\left(\frac{\frac{dw}{dt}\left(t-\frac{r}{c}\right)}{4\pi r}\right) + \nabla^2\beta_a(\mathbf{x})\left(\frac{\frac{dw}{dt}\left(t-\frac{r}{c}\right)}{4\pi r}\right)\right).$$

That is, even though a finite energy point source force density is *not* itself square integrable, its radiated field is *identical* to that produced by a square integrable density.

Suppose that the source time function $w(t)$ is *transient*, i.e. $w(t) = 0, t < 0$ or $t > T$. Then there exist constants c_* and c^* depending on a and T but not otherwise on the solution, for which

$$c_*\int dt \int dx\, |f|^2 \le P_{\text{out}} \le c^* \int dt \int dx\, |f|^2.$$

That is, for a point dilatational source, the power output is within a factor of the mean square of a square integrable representation.

Without an analysis of a specific acoustic energy transfer mechanism, an explicit account of source input power is impossible. Nonetheless, a few constraints suggest a crude model. Linear combinations of point dilatational sources provide a reasonable low-order approximation of marine seismic air-gun arrays, for example. The activation energy or power input for such a point source array is the sum of the power inputs $P_{\rm in}$ of the components. If the *efficiencies*, ie. $P_{\rm out}/P_{\rm in}$, of all component sources are the same, then each input power is proportional to the mean square of the corresponding square integrable representation. If the various point sources in an array are far apart compared to the time duration T and the radius a used in the "β" construction, then the square integrable representations so constructed are orthogonal, and the mean square of the sum of the square integrable representations is the sum of the mean squares, which is proportional to the total input power. With this special case as motivation, we take for the remainder of this paper as the crude model of input power

$$P_{\rm in} = K \int dt \int dx |f(\mathbf{x},t)|^2,$$

The constant K should be chosen so that the right hand side has dimensions of energy. Tracking the units through, you see that the quantity

$$\frac{ca}{4\rho} \int dt \int dx |f(\mathbf{x},t)|^2$$

has units of energy, so K should be chosen as

$$K = K_0 \frac{ca}{4\rho}$$

(the factor of 4 will turn out to be convenient), and the dimensionless constant K_0 adjusted so that the maximal achievable efficiency is whatever it should be.

In the sequel we will need a corresponding expression for $P_{\rm in}$ in terms of the Fourier transform in time $\hat{f}$, expressed as a function of spatial angular frequency $k = \omega/c$, ω being temporal angular frequency. From the Plancherel Theorem and the relation between k and ω, it follows that

$$P_{\rm in} = K_0 \frac{c^2 a}{4\rho} \int dk \int dx |\hat{f}(\mathbf{x},k)|^2, \qquad (3)$$

A standard manipulation ("energy estimate") using integration by parts and the Cauchy-Schwarz inequality shows that $P_{\rm in}$, so defined, always bounds $P_{\rm out}$, i.e. the efficiency of any source is at most unity, for proper choice of K_0. However the efficiency is usually much smaller. In the succeeding sections we will show explicitly how to compute the output power, and how much smaller than input power it can be.

Far-field map for sources of small support

The Fourier transform of the potential $u(\mathbf{x}, t)$ is

$$\hat{u}(\mathbf{x}, k) = \int_{-\infty}^{\infty} dt\, u(\mathbf{x}, t) e^{i\omega t},$$

where $k = \omega/c$. This implies that $\hat{u}$ satisfies Helmholtz's equation with a frequency domain distributed source $\hat{f}(\mathbf{x}, k)$, which is the Fourier transform of the time domain distributed source $f(\mathbf{x}, t)$ in (2). The causal Green's function provides an integral representation of the solution

$$\hat{u}(\mathbf{x}, k) = \int_{B_a} d\mathbf{y} \hat{f}(\mathbf{y}, k) \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}.$$

Following [2], p. 20, write

$$\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} = \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} \left\{ e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}} + O\left(\frac{1}{|\mathbf{x}|}\right) \right\}$$

where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$. Combining the last two equations, we obtain

$$\hat{u}(\mathbf{x}, k) = \frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|} \int_{B_a} d\mathbf{y} \hat{f}(\mathbf{y}, k) e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}} + O\left(\frac{1}{|\mathbf{x}|^2}\right). \tag{4}$$

The spherical harmonic expansion of the source is

$$\hat{f}(r, \varphi, \theta, k) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} F_n^j(r, k) Y_n^j(\varphi, \theta). \tag{5}$$

in which φ, θ, and r are the usual spherical coordinates. The spherical harmonics Y_n^j are normalized following [1]

$$Y_n^m(\varphi, \theta) = \sqrt{\frac{(2n+1)}{4\pi} \frac{(n-|m|)!}{(n+|m|)!}}\, P_n^{|m|}(\cos\varphi) e^{im\theta},$$

and form an orthonormal system

$$\int_{-\pi/2}^{\pi/2} d\varphi \sin\varphi \int_0^{2\pi} d\theta\, \overline{Y_n^j(\varphi, \theta)} Y_n^{j'}(\varphi, \theta) = \delta_{jj'}\delta_{nn'}.$$

Inserting (5) in (4) shows that for large R,

$$\begin{aligned}
\hat{u}(R, \varphi, \theta, k) &= \frac{e^{ikR}}{4\pi R} \int_0^a r^2 dr \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \int_{-\pi/2}^{\pi/2} \sin\varphi d\varphi \int_0^{2\pi} d\theta\, F_n^j(r, k) Y_n^j(\varphi, \theta) e^{-ikr\hat{x}\cdot\hat{y}} \\
&\quad + O\left(\frac{1}{R^2}\right) \\
&= \frac{e^{ikR}}{R} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \int_0^a r^2 dr\, F_n^j(r, k) \frac{1}{i^n} j_n(kr) Y_n^j(\varphi, \theta) + O\left(\frac{1}{R^2}\right).
\end{aligned}$$

Thus the *far-field pattern* ([2], p. 20) is

$$\hat{u}_\infty(\varphi,\theta,k) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n}\int_0^a r^2 dr\, F_n^j(r,k)\frac{1}{i^n} j_n(kr) Y_n^j(\varphi,\theta). \tag{6}$$

For small kr,

$$j_n(kr) = \frac{(kr)^n}{1\cdot 3\cdots(2n+1)} + O\left((kr)^{n+2}\right).$$

([2], p. 28) from which it follows that the higher spherical harmonic coefficients of the far-field pattern are much smaller than those of the source.

The relation between output power and the size of the far field

It will also be useful to view the field $\hat{u}$ as the outgoing solution of the (homogeneous) Helmholtz equation outside the sphere of radius a with Dirichlet data

$$\hat{u}(a,\varphi,\theta,k) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} C_n^j(k) Y_n^j(\varphi,\theta). \tag{7}$$

Because u is real, $\overline{\hat{u}}(\cdots,k) = \hat{u}(\cdots,-k)$, and $|\hat{u}|^2$ is even in k. Therefore $|C_n^j(k)|^2$ is also even. Following [4], the outgoing solution to the Helmholtz equation in $r > a$ with this Dirichlet data is

$$\hat{u}(r,\varphi,\theta,k) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} C_n^j(k)\frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} Y_n^j(\varphi,\theta).$$

and the corresponding Neumann data on $r = a$ is,

$$\frac{\partial u}{\partial r}(a,\varphi,\theta,k) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} C_n^j(k)\frac{k h_n^{(1)\prime}(ka)}{h_n^{(1)}(ka)} Y_n^j(\varphi,\theta). \tag{8}$$

Plancherel's Theorem applied to (1) gives an expression for the output power as an integral over frequency k. Truncating this to the passband $-K \le k \le K$, obtain the *band-limited output power* radiated from this sphere as

$$\overline{P_{\text{out}}} = -\frac{c^2}{\rho}\int_{-K}^{K} ik\,dk \int_{S_a} dS\ \overline{\hat{u}}\ \frac{\partial \hat{u}}{\partial r}. \tag{9}$$

The factor c^2 is from the change of variable from temporal frequency to wavenumber. Together with (8) and (7) this implies that

$$\overline{P_{\text{out}}} = -\frac{c^2}{\rho}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\int_{-K}^{K} k^2 dk \frac{i h_n^{(1)\prime}(ka)}{h_n^{(1)}(ka)} a^2 |C_n^j(k)|^2.$$

In the appendix, we will show that

$$\frac{h_n^{(1)\prime}(z)}{h_n^{(1)}(z)} = -\frac{n+1}{z} + \cdots \text{odd powers of } z \cdots + i\beta_n z^{2n} + \cdots, \tag{10}$$

where

$$\beta_n = \frac{n!(n-1)!2^{2n-1}}{(2n)!(2n-1)!}.$$

That is, the first even power of z in the Laurent expansion of the nth order Hankel logarithmic derivative is of the order $2n$, and its coefficient is $i\beta_n$. Since $|C_n^j(k)|^2$ is even and the passband is symmetric about zero frequency, only even powers of k survive the integration and so that

$$\overline{P_{\text{out}}} = \frac{c^2}{\rho} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \int_{-K}^{K} dk \beta_n (ka)^{2n+2} |C_n^j(k)|^2 + O((Ka)^{2n+4}). \tag{11}$$

Colton and Kress [2] give the far-field pattern in terms of the Dirichlet data (7) as

$$\hat{u}_\infty(\varphi, \theta, k) = \frac{1}{k} \sum_{n=0}^{\infty} \frac{1}{i^{n+1} h_n^{(1)}(ka)} \sum_{j=-n}^{n} C_n^j(k) Y_n^j(\varphi, \theta).$$

Recalling that u is the acoustic potential, an appropriate measure of size of the band-limited *pressure* far-field pattern is the mean square of $k\hat{u}$

$$\begin{aligned} \|k\hat{u}_\infty\|^2 &= \int_{-K}^{K} dk\, k^2 \int_{-\pi/2}^{\pi/2} d\varphi \sin\varphi \int_0^{2\pi} d\theta\, |\hat{u}_\infty(\varphi, \theta, k)|^2 \\ &= \int_{-K}^{K} dk \sum_{n=0}^{\infty} \frac{1}{\left|h_n^{(1)}(ka)\right|^2} \sum_{j=-n}^{n} |C_n^j(k)|^2. \end{aligned}$$

For small ka, the asymptotics of $h^{(1)}(z)$ imply that

$$\|k\hat{u}_\infty\|^2 \simeq \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \int_{-K}^{K} dk \, \frac{(ka)^{2n+2}}{[1 \cdot 3 \cdot 5 \cdots (2n-1)]^2} |C_n^j(k)|^2. \tag{12}$$

Since

$$\frac{1}{[1 \cdot 3 \cdot 5 \cdots (2n-1)]^2} = 4\beta_n,$$

we conclude that the band-limited mean square of the pressure far-field pattern is $4\rho/c^2$ times the band-limited output power, to leading order in ka.

Approximation properties of small distributed sources

The foregoing results provide an estimate for the error caused by truncation of the spherical harmonic expansion of the far-field pattern

$$\hat{u}_\infty(\varphi, \theta, k) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} U_n^j(k) Y_n^j(\varphi, \theta).$$

Equation (6) relates the spherical harmonic coefficients of the forcing term to those of the far-field pattern

$$U_n^j(k) = \int_0^a r^2 dr\, F_n^j(r,k) \frac{j_n(kr)}{i^n}.$$

Denote by $\hat{u}_\infty^{(N)}$ the N-term truncation of the spherical harmonic expansion of $\hat{u}_\infty$. Then

$$\begin{aligned}
\|k(\hat{u}_\infty - \hat{u}_\infty^{(N)})\|^2 &= \sum_{n=N+1}^{\infty} \sum_{j=-n}^{n} \int_{-K}^{K} dk\, k^2 |U_n^j(k)|^2 \\
&= \sum_{n=N+1}^{\infty} \sum_{j=-n}^{n} \int_{-K}^{K} dk\, k^2 \left| \int_0^a r^2 dr\, F_n^j(r,k) \frac{j_n(kr)}{i^n} \right|^2 .
\end{aligned}$$

Estimate the right hand side using the Cauchy-Schwarz inequality

$$\begin{aligned}
& \left| \int_0^a r^2 dr\, F_n^j(r,k) \frac{j_n(kr)}{i^n} \right|^2 \\
\leq\ & \int_0^a r^2 dr \left| F_n^j(r,k) \right|^2 \int_0^a r^2 dr \left| j_n(kr) \right|^2 .
\end{aligned}$$

Set

$$\begin{aligned}
d_n^2 &= \frac{1}{a} \max_{|k|<K} k^2 \int_0^a r^2 dr\, |j_n(kr)|^2 \\
&\simeq \frac{(Ka)^{2n+2}}{\gamma_{n+1}\gamma_n}
\end{aligned}$$

to leading order in Ka, where $\gamma_n = 1 \cdot 3 \cdots (2n+1)$. Evidently $d_n \to 0$ very rapidly with increasing n. Then

$$\|k(\hat{u}_\infty - \hat{u}_\infty^{(N)})\|^2 \leq d_N^2 a \int_{-K}^{K} dk \sum_{n=0}^{\infty} \sum_{j=-n}^{n} |F_n^j(r,k)|^2. \tag{13}$$

The *relative* error in the far-field pattern due to order N truncation

$$\begin{aligned}
\frac{\|k(\hat{u}_\infty - \hat{u}_\infty^{(N)})\|^2}{\|k\hat{u}_\infty\|^2} &\leq \frac{d_N^2 a \int_{-K}^{K} dk\, \sum_{n=0}^{\infty} \sum_{j=-n}^{n} |F_n^j(r,k)|^2}{\|k\hat{u}_\infty\|^2} \\
&= \frac{d_N^2 \frac{c^2 a}{4\rho} \int_{-K}^{K} dk\, \sum_{n=0}^{\infty} \sum_{j=-n}^{n} |F_n^j(r,k)|^2}{\frac{c^2}{4\rho}\|k\hat{u}_\infty\|^2} \\
&= d_N^2 \frac{\overline{P_{\text{in}}}}{\overline{P_{\text{out}}}}.
\end{aligned} \tag{14}$$

using the definition (3) of input power P_{in}, with obvious band limitation, and the result of the previous section relating far-field mean square to output power.

N	$Ka = 1$	$Ka = 2$	$Ka = 5$
0	5.7735e−01	1.1547e+00	2.8868e+00
1	1.4907e−01	5.9628e−01	3.7268e+00
2	2.5198e−02	2.0158e−01	3.1497e+00
3	3.1746e−03	5.0794e−02	1.9841e+00
4	3.1906e−04	1.0210e−02	9.9706e−01
5	2.6681e−05	1.7076e−03	4.1689e−01
6	1.9107e−06	2.4457e−04	1.4927e−01
7	1.1965e−07	3.0631e−05	4.6739e−02
8	6.6576e−09	3.4087e−06	1.3003e−02
9	3.3329e−10	3.4129e−07	3.2548e−03
10	1.5165e−11	3.1059e−08	7.4050e−04

Table 1: Tabulated values of d_N against N for various values of Ka.

Truncation of multi-pole expansions

The end result (14) of the last section expresses a relation between the *relative* error e_N in the far-field pattern due to order N truncation, and the source *efficiency* ϵ, i.e. the ratio of output to input (band-limited) power

$$e_N \leq \frac{d_N}{\sqrt{\epsilon}} \tag{15}$$

in which d_N is given by (13). Evidently d_N decreases very rapidly with N; just how rapidly is the content of Table 1, which displays values of d_N for a few small values of Ka and N.

The import of the estimate (15) is that for moderate values of efficiency, say above 1%, very few terms of the spherical harmonic expansion are required to approximate the far-field pattern of a source only a few wavelengths across to moderate accuracy, say 1%. In fact for $Ka \simeq 5$ only 11 terms are required, for a smaller source with $Ka = 1$ only 5 terms. In effect, the radiation patterns of such sources are very limited; achievement of complex radiation patterns using small sources requires enormous energy input relative to output, i.e. very low efficiency.

Another consequence of (15) is a rule for determining the length of an equivalent multi-pole source representation. Recall that multi-pole sources have point support, so are necessarily linear combinations of derivatives of the Dirac delta function in space, i.e. the result of applying to the delta function a linear partial differential operator, the coefficients of which are functions of t. In fact, a subset of such operators suffice to generate all radiation patterns with finite spherical harmonic series, namely those linear partial differential operators expressible in terms of spherical harmonics. Denote by $\mathcal{Y}_n^j$ the homogeneous harmonic polynomial of degree n satisfying

$$\mathcal{Y}_n^j(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) = Y_n^j(\varphi, \theta).$$

The series
$$M^{(N)}(\mathbf{x},t) = \sum_{n=0}^{N} \sum_{j=-n}^{n} m_n^j(t)\mathcal{Y}_n^j\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\delta(\mathbf{x}).$$
defines a (singular) acoustic source, a *harmonic multi-pole source.* Denote by $u^{(N)}$ the causal solution of the wave equation
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)u^{(N)} = M^{(N)}.$$
The superposition principle implies that
$$\begin{aligned} u^{(N)}(\mathbf{x},t) &= \sum_{n=0}^{N} \sum_{j=-n}^{n} \mathcal{Y}_n^j\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{m_n^j\left(t-\frac{r}{c}\right)}{4\pi r}\right) \\ &= \sum_{n=0}^{N} \sum_{j=-n}^{n} \frac{1}{4\pi c^n r}\frac{d^n m_n^j}{dt^n}\left(t-\frac{r}{c}\right)Y_n^j(\varphi,\theta) + O\left(\frac{1}{r^2}\right). \end{aligned}$$
Fourier transformation and extraction of the leading order term gives the far-field pattern
$$u_\infty^{(N)}(\varphi,\theta,k) = \sum_{n=0}^{N} \sum_{j=-n}^{n} (ik)^n \hat{m}_n^j(k) Y_n^j(\varphi,\theta),$$
i.e. precisely the form of the truncated far-field pattern in the last section, with the identification
$$U_n^j(k) = (ik)^n \hat{m}_n^j(k).$$
That is, the harmonic multi-pole sources generate all truncated far-field patterns.

Discussion

We have argued in the preceding sections that efficiency and (small) dimensionless size (Ka) strongly limit the variety of radiation patterns output by an acoustic source, and justify its approximation by a finite multipole expansion. We have given explicit rules for selecting the length of the expansion in terms of assumed efficiency, Ka, and far field pattern approximation error tolerance.

The reader should remain aware of the caveat mentioned in our definition of input power: we have avoided modeling any particular source mechanism. Our definition of input power is therefore based only on general properties of the acoustic wave equation and the energy analysis of a single example, the isotropic point radiator. Since our notion of efficiency hinges on the this definition, it has the strength and weakness of generality: it applies to all sources, and to none in particular. Therefore we hope in the future to specialize our analysis to some specific classes of sources, such as piezoelectric transducers and marine seismic airgun arrays, and to make use of more detailed modeling of energy budgets for such systems to give a more precise accounting of efficiency. We conjecture however that the general form of our result will persist: that is, that rather short multipole series representations of these sources will be adequate so long as their spatial extent encompasses only a small number of wavelengths.

Appendix

We provide a derivation of the formula in (10). We start with the equation for the derivatives of spherical Hankel functions ([4], p.680)

$$\frac{h_n^{(1)\prime}(z)}{h_n^{(1)}(z)} + \frac{n+1}{z} = \frac{h_{n-1}^{(1)}(z)}{h_n^{(1)}(z)}.$$

Recall from definition that $h_n^{(1)}(z) = j_n^{(1)}(z) + iy_n^{(1)}(z)$, and that the spherical Bessel functions can be written as [1]

$$j_n^{(1)}(z) = A_n z^n S_n(z), \quad A_n = \frac{n!2^n}{(2n+1)!}, \quad S_n(z) = \sum_{j=0}^{\infty} a_j^{(n)} z^{2j},$$

$$y_n^{(1)}(z) = B_n z^{-n-1} T_n(z), \quad B_n = -\frac{(2n)!}{2^n n!}, \quad T_n(z) = \sum_{j=0}^{\infty} b_j^{(n)} z^{2j}.$$

The coefficients $a_j^{(n)}$ and $b_j^{(n)}$ can be found in [1].

Now, we use the above expressions to write

$$\begin{aligned} \frac{h_{n-1}^{(1)}(z)}{h_n^{(1)}(z)} &= \frac{A_{n-1}S_{n-1}(z)\, z^{n-1} + iB_{n-1}T_{n-1}(z)\, z^{-n}}{A_n S_n(z)\, z^n + iB_n T_n(z)\, z^{-n-1}} \\ &= \frac{A_{n-1}S_{n-1}(z)\, z^{n-1} + iB_{n-1}T_{n-1}(z)\, z^{-n}}{iB_n T_n(z)\, z^{-n-1}} \left[1 - i\frac{A_n S_n(z)\, z^n}{B_n T_n(z)\, z^{-n-1}}\right]^{-1}. \end{aligned}$$

Rearranging and using small z, we can rewrite the equation as

$$\frac{h_{n-1}^{(1)}(z)}{h_n^{(1)}(z)} = \left[\frac{B_{n-1}T_{n-1}(z)}{B_n T_n(z)} z - i\frac{A_{n-1}S_{n-1}(z)}{B_n T_n(z)} z^{2n}\right]\left[1 + i\frac{A_n S_n(z)}{B_n T_n(z)} z^{2n+1} + \cdots\right].$$

Observe that $T_n(z)$ and $S_n(z)$ are series in even powers of z. Therefore, the ratio $T_{n-1}(z)/T_n(z)$ for small z is again a series in even powers of z. Similarly, the ratios $S_{n-1}(z)/T_n(z)$ and $S_n(z)/T_n(z)$ are also series in even powers of z for small z. The first term in $S_n(z)$ and $T_n(z)$, $a_0^{(n)}$ and $b_0^{(n)}$, are both unity. Therefore, we can write, for small z,

$$\begin{aligned} \frac{h_{n-1}^{(1)}(z)}{h_n^{(1)}(z)} &= \Big[\frac{B_{n-1}}{B_n} z(1 + \text{even powers of } z) \\ &\quad - i\frac{A_{n-1}}{B_n} z^{2n}(1 + \text{even powers of } z)\Big] \\ &\quad \times \left[1 + i\frac{A_n}{B_n} z^{2n+1}(1 + \text{even powers of } z)\right]. \end{aligned}$$

Therefore, to leading z^{2n}, we have

$$\frac{h_{n-1}^{(1)}(z)}{h_n^{(1)}(z)} = \text{odd powers of } z + \cdots - i\frac{A_{n-1}}{B_n} z^{2n} + \cdots.$$

Substituting the formula for A_{n-1} and B_n leads to (10).

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