Source Inversion: Not a Miracle

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ABSTRACT

An accurate representation of the source is a requirement for the detailed imaging of a medium. A simultaneous source/reflectivity inversion provides the means to obtain the desired detail. We analyze perturbations in the seismogram to show why it is possible to invert for both the source and reflectivity. We then describe and implement an alternating source/reflectivity inversion scheme in the offset-time domain.

I. Introduction: Why invert for the Source?

Recent research suggests that the source estimate can affect the earth parameter estimates output from seismic inversion (Ziolkowski, 1991; Minkoff and Symes, 1997; Frazer and Sun, 1998). This is due to the fact that model seismograms (generated with model sources) are compared to real seismograms. If the model seismograms are incorrect (due to incorrect sources), the earth parameters may be incorrect as well. All aspects of the source (wave form, strength, and radiation pattern) may influence the success of inversion. The sources used in actual data collection are anisotropic and have a notable radiation pattern (Ziolkowski, 1991; Minkoff and Symes, 1997).

Given that proper inclusion of the source into an imaging algorithm is a worthy goal, another question immediately arises: Is it possible to invert for both the source and earth parameters? In other words, do the source and reflectivity affect the seismogram in unique ways: when including source inversion within a reflectivity inversion are we really obtaining the source and reflectivity or simply over fitting the data: can the source and reflectivity destructively combine to hide wiggles in the seismograms?

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We can partly answer this question via an examination of the convolutional model of the seismogram in an acoustic layered medium

$$S = f *_t G, \tag{1}$$

where f is the source, $*_t$ is a convolution in time, and G is the Green's function. The Green's function is linear in earth parameter fluctuations, so we can rewrite this equation as

$$S = f *_t r \circ \zeta, \tag{2}$$

where $\circ$ is a functional composition, ζ is a depth function and

$$r = \frac{\delta v}{v} \tag{3}$$

is the reflectivity. This equation shows the scale ambiguity inherent in the seismogram. Doubling the strength of the source and halving the strength of the reflectivity generates no net change in the seismogram. The scale ambiguity is a degree of freedom which cannot be directly treated. Other constraints allow for a lessening of its ambiguity, but that is not the subject of this paper. For the sake of argument, we acknowledge that the scale ambiguity exists and constrain the size of the source as follows.

$$\|f(t)\|_2 = \int_{-\infty}^{\infty} f^2(t)dt = 1. \tag{4}$$

This allows us to proceed in answering the question of uniqueness. Specifically, do the source and reflectivity have different effects on the seismogram?

This question was answered to varying degrees by previous authors. Specifically, a quasi-impulsive, isotropic point source can be uniquely recovered from reflection seismograms using linear perturbation approximations (Bube et al., 1988). In an acoustic constant density, constant background velocity medium, perturbations in the source uniquely determine perturbations in plane wave seismograms (Lewis, 1989). In addition, if the material velocity does not change too rapidly the source can be uniquely determined within a given pass band (Symes and Minkoff, 1995). Minkoff and Symes also showed that as the slowness aperture is widened the extent to which the reflectivity and source can be separately determined improves.

The proofs of the above statements are all fairly technical. In section II. we present a simpler (yet mathematically rigorous) account of Lewis' treatment of the source/reflectivity inversion (SRI) problem. The worked discussed in the preceding paragraph dealt with the $p - \tau$ domain. Harlan (1989) asked analogous questions for the offset domain. Harlan used an algorithm similar to that used by Lewis and Minkoff and Symes.

In section III we formulate an algorithm for the SRI problem in the offset domain similar to that of (Harlan, 1989). We note that the algorithms used in prior work tended to be inefficient. The algorithm is implemented using the object-oriented structure of the Hilbert Class Libraries (HCL). It splits the non-linear (bi-linear) simultaneous SRI into an alternating (linear in the source and linear in the reflectivity) SRI. What makes our implementation new is the use of the adjoint map in the offset-time domain. Since the SRI problem is formulated within the HCL structure we can test the effectiveness of other algorithms in comparison to the alternation code. In section IV. we give some preliminary numerical results.

II. Why can we Invert for the Source?

The fundamental question in inversion is one of existence and uniqueness. In other words, given some seismograms containing information about a medium (and the source of waves within the medium), is it possible to isolate the information pertaining only to the medium (or wave source). In one sense the answer to this question is no. No, because of a scale ambiguity in the seismogram. If one chooses to ignore the scale ambiguity by constraining the size of the source, the question of uniqueness still remains. Looking at wiggles in a seismogram, can we discern exactly what part of the wiggles are due to the source and what part of the wiggles are due to the reflectivity? For the case of a single seismogram the answer to this question is a definitive no, one cannot tell the difference. With seismograms recorded at different locations (offsets), however, is there a possibility of discrimination?

Consider an acoustic medium with constant velocity v and a continuous reflectivity $r(z) \neq 0$ with finite length. If we act upon the medium with a continuous wave source $f(\tau) \neq 0$ with finite duration, the travel time $\tau(z,p)$ as a function of depth z and slowness p is given by

$$\tau(z,p) = \int_0^z \sqrt{\frac{1}{v^2} - p^2} = z\sqrt{\frac{1}{v^2} - p^2}. \tag{5}$$

By performing a change of variables we can generate a depth function $\zeta(\tau,p)$ which has the following form:

$$\zeta(\tau,p) = \frac{\tau}{\sqrt{\frac{1}{v^2} - p^2}} = \frac{v\tau}{\sqrt{1 - v^2p^2}}. \tag{6}$$

This depth function allows us to write the convolutional model for a plane wave reflection seismogram $S(\tau,p)$ as a convolution in time of the source wavelet $f(\tau)$ and the reflectivity $r(z)$.

$$S(\tau,p) = f(\tau) *_t r(z) \circ \zeta(\tau,p) \tag{7}$$

A first order linear perturbation in the seismogram $\delta S(\tau,p)$ then involves a perturbation in the source $\delta f(\tau)$ and a perturbation in the reflectivity $\delta r(z)$ as follows:

$$\delta S(\tau,p) = \delta f(\tau) *_t r(z) \circ \zeta(\tau,p) + f(\tau) *_t \delta r(z) \circ \zeta(\tau,p). \tag{8}$$

Since the velocity v is held constant we can absorb the depth function into the reflectivity and rewrite the perturbed seismogram as:

$$\delta S(\tau,p) = \delta f(\tau) *_t r\left(\frac{v\tau}{\sqrt{1-v^2p^2}}\right) + f(\tau) *_t \delta r\left(\frac{v\tau}{\sqrt{1-v^2p^2}}\right). \tag{9}$$

This is the formulation we need to answer the question of uniqueness. After constraining the size of the source, do perturbations in the seismogram uniquely determine perturbations in the source and the reflectivity? Mathematically, this a question of linear independence, and can be phrased as: if the perturbation to the seismogram is equal to zero,

$$\delta S(\tau,p) = \delta f(\tau) *_t r\left(\frac{v\tau}{\sqrt{1-v^2p^2}}\right) + f(\tau) *_t \delta r\left(\frac{v\tau}{\sqrt{1-v^2p^2}}\right) = 0 \tag{10}$$

does it have to be true that the perturbations in the reflectivity and the source are also zero,

$$\Rightarrow \delta f(\tau) \equiv 0 \equiv \delta r(z)??? \tag{11}$$

The answer to this question is yes. To prove that this is true we will first show that if $\delta S(t) = 0$ then it must be true that $\delta f(t) \equiv 0$. To do this we begin with a quick look at the effects constraining the source has on the first order linear perturbation. Recall the constraint placed on the source (4). This constraint must hold for the first order linear perturbation.

$$1 = \int_{-\infty}^{\infty} f^2(t)dt = \int_{-\infty}^{\infty} (f(t)+\delta f(t))^2 dt. \tag{12}$$

By expanding the integrand this becomes

$$1 = \int_{-\infty}^{\infty} f^2(t)dt + \int_{-\infty}^{\infty} 2f(t)\delta f(t)dt + \int_{-\infty}^{\infty} \delta f(t)^2 dt. \tag{13}$$

Since we have fixed the size of the source, the first term on the RHS is 1. The third term on the RHS is second order and is ignored. This implies that

$$\int_{-\infty}^{\infty} f(t)\delta f(t)dt = 0. \tag{14}$$

In other words, the first order perturbation to the source is perpendicular to the source.

With the first order linear perturbation defined we can proceed with the analysis to show that it must be zero. By performing a Fourier transform, (9) can be written as

$$\widehat{\delta f}(\omega)\hat{r}\left(\frac{\sqrt{1-v^2p^2}}{v}\omega\right) + \hat{f}(\omega)\widehat{\delta r}\left(\frac{\sqrt{1-v^2p^2}}{v}\omega\right) = 0. \tag{15}$$

We then annotate the spatial frequency ξ as follows:

$$\xi = \frac{\omega}{v}\sqrt{1 - v^2 p^2}. \tag{16}$$

Since $f(t) \neq 0$ and $r(z) \neq 0$ there must exist some ζ, ω, ξ such that $\widehat{f}(\omega) \neq 0$ and $\widehat{r}(\xi) \neq 0$. Some algebra within the pass band will then change (10) to

$$\frac{\widehat{\delta f}}{\widehat{f}}\left(\frac{v\xi}{\sqrt{1 - v^2 p^2}}\right) = -\frac{\widehat{\delta r}}{\widehat{r}}(\xi). \tag{17}$$

In this equation we can, by the continuity of $f(t)$ and $r(z)$, fix ξ (keep the RHS constant) and vary p (change the LHS). This implies that for some open interval of p

$$\frac{\widehat{\delta f}}{\widehat{f}}\left(\frac{v\xi}{\sqrt{1 - v^2 p^2}}\right) = constant_1. \tag{18}$$

If we further restrict $p \in (p_{min}, p_{max})$ then by the continuity of ξ, $\omega \in (\omega_{min}, \omega_{max})$. This implies

$$\frac{\widehat{\delta f}}{\widehat{f}}(\omega) = constant_1, \forall\omega \in (\omega_{min}, \omega_{max}). \tag{19}$$

Now, since $f(t)$ has a finite duration, then $\widehat{f}(\omega)$ must be analytic. We can thus invoke the principal of analytic continuation to state that

$$\widehat{\delta f}(\omega) = constant_1 \widehat{f}(\omega), \forall\omega, \tag{20}$$

and

$$\delta f(t) = constant_2 f(t). \tag{21}$$

Recall that by the definition of the perturbation

$$\int_{-\infty}^{\infty} f(t)\delta f(t) dt = 0. \tag{22}$$

The only way that (20) and (21) can hold is if $constant_2 = 0$. Lastly, if $constant_2 = 0$, we can recall (20) and see that

$$\delta f(t) = 0. \tag{23}$$

We have shown that when $\delta S(t) = 0$ it must be true that $\delta f(t) = 0$. If we can show that $\delta r(z) = 0$ then our analysis is complete, and the question of uniqueness for this case is affirmatively answered.

Let us return to (14).

$$\widehat{\delta f}(\omega)\widehat{r}\left(\frac{\sqrt{1 - v^2 p^2}}{v}\omega\right) + \widehat{f}(\omega)\widehat{\delta r}\left(\frac{\sqrt{1 - v^2 p^2}}{v}\omega\right) = 0. \tag{24}$$

Again, since $f(t) \neq 0$ there must exist some ω, such that $\widehat{f}(\omega) \neq 0$. Some algebra will then change the above to

$$\frac{\widehat{\delta f}}{\widehat{f}}(\omega)\widehat{r}\left(\frac{\sqrt{1-v^2p^2}}{v}\omega\right) = \widehat{\delta r}\left(\frac{\sqrt{1-v^2p^2}}{v}\omega\right). \tag{25}$$

Yet, by the previous result we know $\delta f(t) = 0$. Therefore,

$$0 = \widehat{\delta r}\left(\frac{\sqrt{1-v^2p^2}}{v}\omega\right). \tag{26}$$

Which implies $\delta r(z) = 0$.

Thus if the perturbation to the seismogram, $\delta S(t)$ is zero then it must be true that the perturbations to the source, $f(t)$, and reflectivity, $\delta r(z)$ are also zero. In less mathematical terms, given seismograms with different offsets, the source and reflectivity cannot destructively combine to consistently hide arrivals in the seismograms.

Given this result, one must ask about the realism of the above assumptions? A continuous, non-zero reflectivity of finite length support implies that there is some reflector present (a finite distance from the source), with a rheology not too different from the surrounding medium. A continuous, non-zero, finite source function implies that there is a physically realistic source of waves that does not ring on for eternity. A constant velocity medium, however, is an assumption that is difficult to sweep under the rug. Using a much more technical analysis Minkoff and Symes(95) showed that if $\frac{dv}{dz}$ is sufficiently small, the source and reflectivity can be uniquely determined.

III. The Inversion Algorithm

Simultaneous inversion for the source and reflectivity is a non-linear (bi-linear) problem. This means that inverting for the source alone is a linear problem, and inverting for the reflectivity alone is a linear problem. To avoid non-linearity issues, we use an alternating inversion scheme. The general algorithm has the following structure:

1. Make an initial estimate of the source $f_{current}$ and reflectivity $r_{current}$.
2. With $f_{current}$ fixed, use an adjoint linear least squares inversion technique to invert for the reflectivity.
3. Update the reflectivity: $r_{current} := r_{new}$.
4. With $r_{current}$ fixed, use an adjoint linear least squares inversion technique to invert for the source.

5. Update the source: $f_{current} := f_{new}$.
6. Return to 2.

Using the approach of linear least squares, when we execute step 2., we wish to find

$$min \frac{1}{2} \|Mr - d\|^2 \tag{27}$$

Where d is the data we wish to model, and M is a forward map. In this case M convolves a single source trace with reflectivity traces in the offset-time domain to produce synthetic seismograms in a layered, acoustic, constant density regime.

A minimum occurs where

$$\nabla_r \left(\frac{1}{2} \|Mr - d\|^2 \right) = M^* \left(Mr - d \right) = M^* Mr - M^* d = 0. \tag{28}$$

Where ∇_r is the Jacobian with respect to the reflectivity and M^* is the adjoint of M. We use an Euclidean inner product, so the adjoint of a matrix is simply it transpose. To find the best fitting reflectivity we must solve the normal equations

$$M^T M r_{new} = M^T d \tag{29}$$

for r_{new}. We compute the transpose of M (the adjoint map), use a conjugate gradient technique to generate

$$r_{new} = \left(M^T M \right)^{-1} M^T d, \tag{30}$$

and update the reflectivity (step 3).

With the reflectivity updated, we move on to step 4. We wish to find

$$min \frac{1}{2} \|\bar{M} f - d\|^2. \tag{31}$$

In this case, however, $\bar{M}$ is the forward map that convolves the reflectivity traces with a single-trace source to produce synthetic seismograms in an acoustic constant density regime.

The minimum we seek occurs where

$$\nabla_f \left(\frac{1}{2} \|\bar{M} f - d\|^2 \right) = \bar{M}^T \bar{M} f - \bar{M}^T d = 0. \tag{32}$$

Where ∇_f is the Jacobian with respect to the source. Again we compute the transpose (adjoint map) of $\bar{M}$, use conjugate gradient technique to generate

$$f_{new} = \left(\bar{M}^T \bar{M} \right)^{-1} \bar{M}^T d. \tag{33}$$

and update the source (step 5).

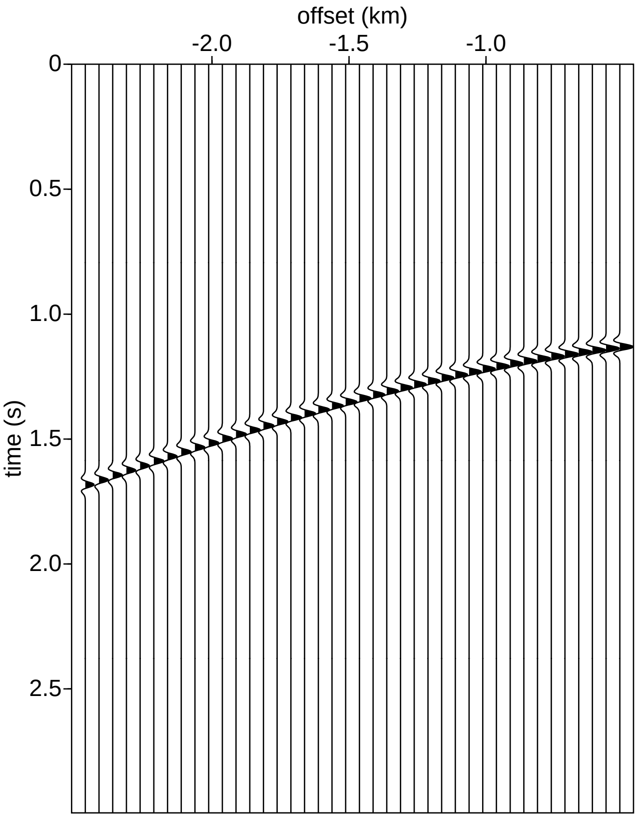

FIG. 1. The 40 "data" seismograms generated for numerical experiments. Seismograms were generated by convolving a 15 Hz Ricker wavelet with 40 offset dependent reflectivity traces derived from a layered, acoustic, constant density medium with a single reflector (depth = 1 km, reflectivity = 1.0).

IV. Synthetic Data Trials

To test the robustness of the algorithm we ran a series of synthetic data experiments. For illustrative purposes we present the results from a single reflector experiment. The data for this experiment was generated using the forward map of our algorithm. We convolved a 15 hz Ricker wavelet of negative polarity, centered at 0.1 seconds, with 40 reflectivity traces. The reflectivity traces were generated in the offset-time domain using a layered, acoustic, constant density model. In this model there is a single reflector at a depth of 1 km, with reflectivity equal to one. Figure 1 shows the data for this experiment.

For this experiment we used an initial guess for the source which was far from the source used to generate the data in many respects. The starting source model for the

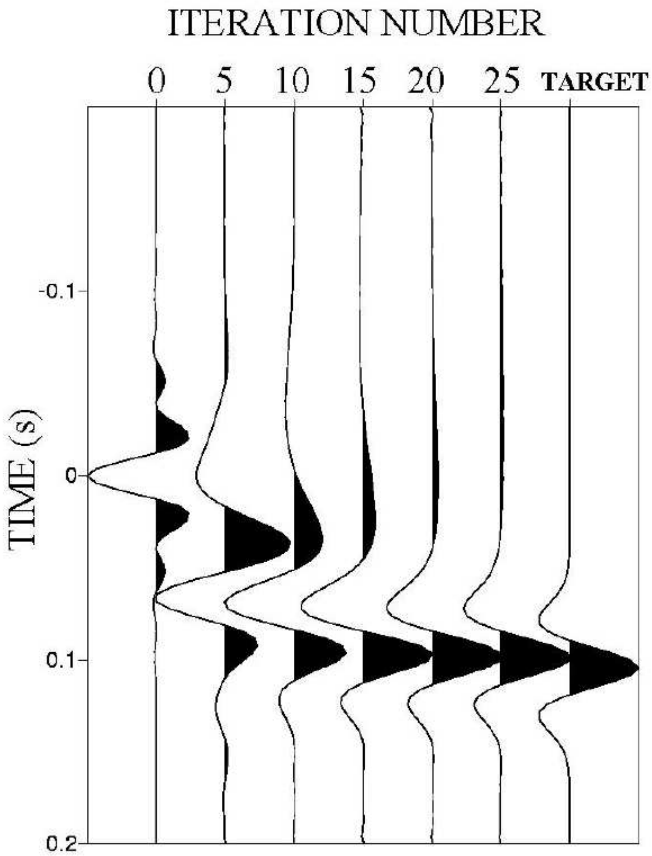

FIG. 2. The shape of the source at the completion of alternating iteration rounds. At iteration 0 (the far left trace), the initial guess for the source is a band pass filter (5-35 Hz) centered at 0.0 seconds, with a polarity opposite that of the source used to generate the data. By the 5th iteration (the second trace from the left) the source has undergone a dramatic time shift and change in shape. By the 25th iteration (second trace from the right) the source has a form very close to the source used to generate the data (the target source, or far right trace). Due to the inherent scale ambiguity (see text) all sources are scaled to have a maximum square amplitude of 1

inversion experiments is a band pass filter (5-35 Hz) centered at 0.0 seconds with a polarity opposite that of the experimental source. The far left trace in Figure 2 is a plot the initial guess for the source.

By the 5th iteration (Fig. 2, second trace from the left) the source has undergone dramatic changes in both its position in time and its shape: it has shifted by a whole wavelength, and its shape no longer resembles a band pass filter. By the 15th iteration (Fig. 2, middle trace) the calculated source closely resembles the source used to generate the data (Fig. 2, far right trace). Subsequent iterations produce much less dramatic improvements in the source. This slow improvement illustrates a problem typically encountered in alternating inversion. If the objective function to be minimized is elongated and oblique to one of the search directions, an alternation algorithm will get trapped and be forced to creep along the dominant axis of the elongated objective function. This phenomena is illustrated in Figure 3. It should be noted that this problem can be avoided by using a non-linear inversion scheme.

For this experiment, the initial guess for the reflectivity is a constant velocity medium (zero reflectivity). The far left trace in Figure 4 plots this initial guess for the reflectivity.

By the 5th iteration (Fig. 4, second trace from the left) the reflectivity has undergone dramatic changes. It has positioned a strong reflector in the proper location. Other higher frequency reflectors, however, are also present. By the 15th iteration (Fig. 4, middle trace) the calculated reflectivity closely resembles the reflectivity used to generate the data (Fig. 4, far right trace) with the exception of some low amplitude, high frequency reflectors. Again, subsequent iterations produce much less dramatic improvements in the reflectivity. By the 25th iteration (Fig. 4, second trace from the right), however, the inversion output closely resembles the reflectivity used to generate the data.

V. Summary

To image a medium with the kind of detail requested by modern applications, it is necessary to include an accurate estimate of the source in the imaging process. An accurate estimate of the source can be obtained via inversion. We presented an analysis demonstrating that it is possible to simultaneously invert for both the source and the reflectivity of a medium. This analysis was done for the case of an acoustic, constant density, constant background velocity medium. While the result of this analysis is not new (Lewis, 1989), our approach to the result is simplified and more intuitive (yet still mathematically rigorous).

We then described an alternation algorithm for source/reflectivity inversion in the offset-time domain which used an adjoint, linear least squares method. The algorithm was tested using synthetic data from a single reflector in a layered, acoustic, constant density medium. The test utilized initial guesses for the source and reflectivity which were far from the source and reflectivity used to generate the data. The results of the test were positive. In other words, the inversion scheme formed a an excellent representation of both the source and the reflectivity; confirming the mathematical analysis.

The algorithm did, however, experience a problem with slow final convergence. Slow final convergence is often encountered with alternating schemes where the scales of the parameters (source and reflectivity) are notably different.

Since the adjoint based source/reflectivity inversion algorithm was created within

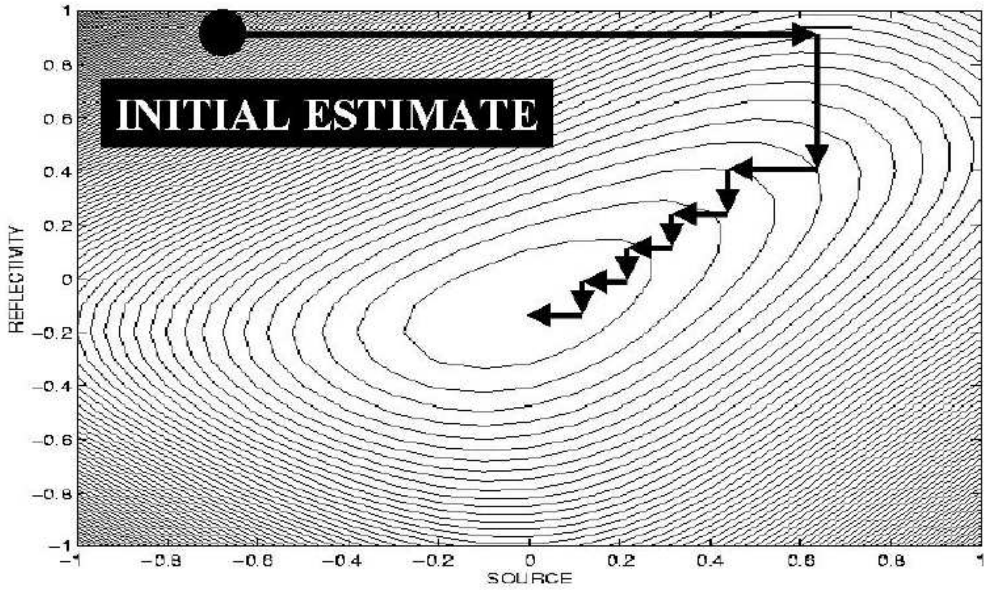

FIG. 3. The contours of an idealized objective function for a source (x axis) and reflectivity (y axis) inversion. The alternation algorithm searches for a global minimum by first searching for a "local minimum" along the constrained domain of a coordinate axis. The arrows symbolize each search for a "local minimum". In this case the objective function is elongated and slightly oblique to the source axis. This orientation causes the search procedure to get trapped along the principle axis of the objective function. The alternation algorithm is forced to take small steps in its search for the bottom of the objective function.

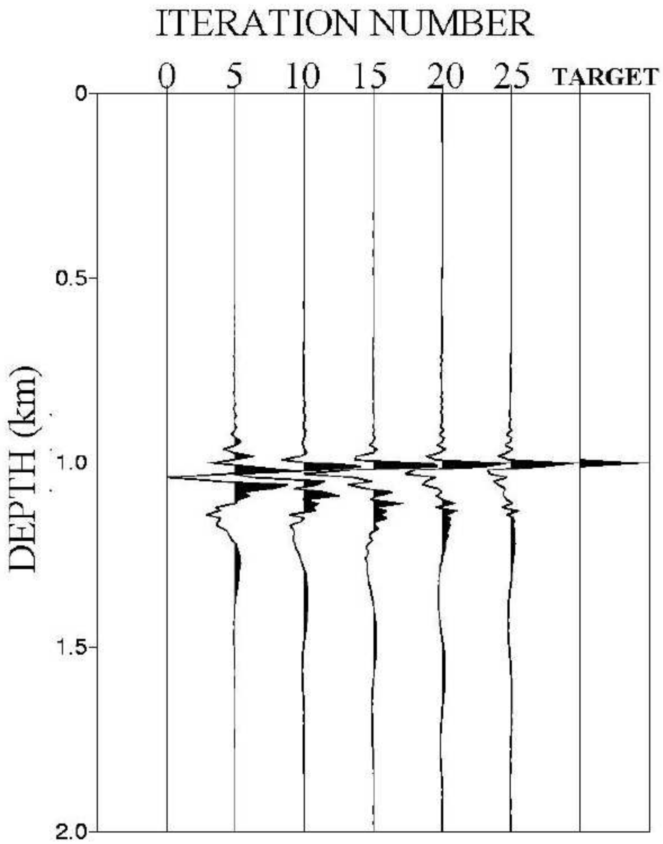

FIG. 4. The reflectivity at the completion of alternating iteration rounds. At iteration 0 (the far left trace), the initial guess for the reflectivity is a zero trace. By the 5th iteration (the second trace from the left) the reflectivity has undergone a dramatic change in shape, and has positioned a strong reflector at the proper location. By the 25th iteration (second trace from the right) the reflectivity has a form very close to the reflectivity used to generate the data (the target reflectivity or far right trace). Due to the inherent scale ambiguity (see text) all reflectivity traces are scaled to have a maximum square amplitude of 1.

the object-oriented HCL structure, the implementation of non-linear inversion techniques was greatly simplified. We attempted to solve the source/reflectivity inverse problem using non-linear inversion techniques. Specifically, we used BFGS and quasi-Newton optimization algorithms. These algorithms failed to converge. We suspect the failure is also a result of the notable difference in the scales of the parameters. We are currently investigating if this scale difference can be circumvented by the use of a non-linear, sub-space inversion technique (Kennet et al., 1988).

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