Hamilton-Jacobi Equations and Upwind Finite Difference Schemes for Traveltime and Amplitude in Inhomogeneous Anisotropic Media:Mathematical Formulations

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ABSTRACT

By analogy with the isotropic case,we expect that the first-arrival quasi-P wave traveltime in inhomogeneous anisotropic media is the unique viscosity solution of some Hamilton-Jacobi(HJ) equation. The Essentially Nonoscillatory(ENO) Hamilton-Jacobi solver can be applied to get this viscosity solution (the first-arrival traveltime field) on a regular grid.

The eikonal and transport equation are derived based on the progressing wave expansion.Assuming that there exist nontrivial solutions to the zeroth order approximation to the elastic wave equation,the third component of the slowness vector can be expressed as a function of the other two components.Actually, there are six such functions because the characteristic equation is a sextic polynomial equation of the slowness vector.We pick out the one function among the six which corresponds to downgoing qP wave propagation.This function defines the desired Hamilton-Jacobi equation.

Starting from the HJ equation formulated by the above approach we will give a new derivation of Godunov's upwind finite difference scheme by using local ray-tracing. Then we will propose a second order Godunov-ENO-HJ difference schemes for the computation of traveltime and amplitude of qP wave in three-dimensional anisotropic media. With traveltime and amplitude at our disposal we will design new imaging methods for seismic data processing in anisotropic media.

Introduction

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As is well known,the earth's subsurface exhibits anisotropic properties (Thomsen, 1986; Backus, 1962). The processing method based on erroneous assumption of isotropy produces errors in seismic imaging (Alkhalifah, 1996; Tsvankin, 1995). So we need to take the anisotropy into consideration in the seismic processing.

To characterize the seismic body waves in inhomogeneous anisotropic media,we need to solve the corresponding elastic wave equations. However,it is very difficult to solve the general elastic wave equation in inhomogeneous anisotropic medium (Igel et al., 1995), we have to appeal to some approximate methods,e.g.,progressing wave expansion (Courant and Hilbert, 1962), which is known as ray methods or high frequency approximations (geometric optics) in seismology (Cerveny, 1972).This implies that we need to solve eikonal and transport equations to get the kinematic approximation to the elastic wave equation.

The wave modes in the inhomogeneous anisotropic media have three forms,i.e., quasi-P wave,quasi-SH wave and quasi-SV wave (Fedorov, 1968; Musgrave, 1970). If we are only interested in the first-arrival,then the first-arrival traveltime is corresponding to the qP-wave propagation.

From the viewpoint of mathematical physics,the first arrival traveltime in the isotropic media is the unique viscosity solution of the Hamilton-Jacobi equation which is derived from the corresponding eikonal equation (Lions, 1982; Crandall and Lions, 1983; Crandall et al., 1984),so that we have the uniqueness of first-arrival traveltime in the isotropic media. van Trier and Symes (van Trier and Symes, 1991) have computed this viscosity solution(the first-arrival traveltime) successfully by using upwind finite difference schemes.

By analogy with the isotropic case, we conjecture that the first-arrival traveltime in inhomogeneous anisotropic media is the unique viscosity solution of some Hamilton-Jacobi equation, so that we could compute the first-arrival traveltime in the anisotropic media uniquely.Dellinger and Symes (Dellinger, 1991; Dellinger and Symes, 1997) have investigated the possibility of using upwind finite difference schemes to compute the first-arrival traveltime in anisotropic media, but they did not give an explicit and specific approach to compute the traveltime and amplitude.

Assume that there exist nontrivial solutions to the zeroth order approximation to the elastic wave equation,the third component of the slowness vector can be expressed as a function of the other two components. Actually, there are six such functions because the characteristic equation is a sextic polynomial equation of the slowness vector.We pick out the one function among the six which corresponds to downward qP wave propagation.This function defines the desired Hamilton-Jacobi equation.By using this Hamilton-Jacobi equation,we can derive the group velocity of rays for qP wave as well.

Now the problem is how to compute the viscosity solution of the Hamilton-Jacobi equation by a stable numerical method. A widely applied class of numerical schemes for Hamilton-Jacobi equation is the upwind finite difference schemes. Crandall and

Lions (Crandall and Lions, 1984) proved the convergence of the monotone finite difference schemes to the viscosity solutions of Hamilton-Jacobi,but these are at most first-order accurate. Osher and Shu (Osher and Shu, 1991) introduced a class of high-order essentially non-oscillatory(ENO) schemes which is based on the Godunov type Hamiltonian,the numerical results reported indicate convergence to viscosity solutions of Hamilton-Jacobi equation with high order accuracy in smooth regions and sharp resolution of the discontinuous derivatives(shocks). The above methodology has already been successfully adapted to the computation of first-arrival travel time in two-dimensional and three-dimensional isotropic media by van Trier and Symes(1991),Symes et al.(1994),(El-Mageed, 1996),Kim (El-Mageed et al., 1997).

We will employ the same methodology to get the accurate and efficient approximate solutions i.e. the viscosity solution to the corresponding Hamilton-Jacobi equations in anisotropic media by constructing upwind finite difference schemes, which we expect is the first-arrival qP wave traveltime in the inhomogeneous anisotropic media. The qP wave amplitude could be obtained numerically by the similar ENO schemes.

Based on the assumption that the solution is of the form of progressing wave expansion, we will derive the eikonal and transport equation for the inhomogeneous anisotropic media and formulate the Hamilton-Jacobi equation for qP wave propagation.Then we will give a new derivation of Godunov's scheme by using the idea of local ray-tracing, next we will apply Godunov's scheme to the computation of traveltime and amplitude of qP wave in the three dimensional anisotropic media by introducing a second order ENO-HJ and TVD-RK (total variation diminishing Runge-Kutta) difference schemes.

The purpose of computation of traveltime and amplitude is to provide the necessary tool for the high frequency asymptotic imaging techniques. We will develop appropriate imaging methods such as Kirchhoff migration method for anisotropic media. At the same time we also want to prove that the viscosity solution is the first arrival traveltime corresponding to the qP wave. Finally we want to generalize DSO (differential semblance optimization) (Symes and Carazzone, 1991) for the velocity analysis to the anisotropic case.

Progressing wave expansion

The Hooke's law states that stress σ_{ij} is related to strain e_{kl} by a stiffness tensor C_{ijkl}:

$$\sigma_{ij} = C_{ijkl} e_{kl},$$

and the motion equation without body force is

$$\rho \frac{\partial^2 U}{\partial t^2} = \nabla \cdot \sigma. \qquad (1)$$

By the symmetry of the stiffness tensor,the motion (wave) equation is

$$\rho\frac{\partial^2 U_j}{\partial t^2} = \frac{\partial}{\partial x_i}(C_{ijkl}\frac{\partial U_k}{\partial x_l}) \tag{2}$$

where the Einstein summation convention is assumed.

By the progressing wave expansion(Courant and Hilbert,1962; Cerveny,1972), the solution is of the form

$$\begin{aligned} U_k(x,t) &= \sum_{n=0}^{\infty} U_k^{(n)}(x) f_n(t-\tau(x)), \\ f'_{n+1}(\theta) &= f_n(\theta), \end{aligned} \tag{3}$$

where the wave front is

$$t = \tau(x). \tag{4}$$

Substitute the above solution into the motion equation and compare the terms of the same order, especially,we are only interested in the zeroth order term and the first order term by the high frequency approximation.

The zeroth order term is the eikonal equation

$$(\Gamma_{jk} - \delta_{jk})U_k^{(0)} = 0. \tag{5}$$

and the first order term gives us the transport equation

$$p_i a_{ijkl}\frac{\partial U_k^0}{\partial x_l} + \rho^{-1}\frac{\partial}{\partial x_i}(\rho a_{ijkl} p_l U_k^{(0)}) = 0, \tag{6}$$

where

$$\Gamma_{jk} = a_{ijkl} p_i p_l, \tag{7}$$

the normalized elastic parameter

$$a_{ijkl} = \frac{C_{ijkl}}{\rho} \tag{8}$$

and the slowness vector normal to the wave front (4) is

$$p_i = \frac{\partial \tau}{\partial x_i.} \tag{9}$$

The matrix Γ_{jk} is symmetric by conservation of energy and positive definite by the requirement of the stability of material,so it has three positive eigenvalues and three orthogonal eigenvectors. Moreover,its eigenvalues,denoted as $H_m(p_i, x_i)(m = 1, 2, 3)$, are homogeneous functions of slowness vector p_i of second order,then by Euler's theorem on homogeneous functions,

$$p_i\frac{\partial H_m}{\partial p_i} = 2H_m. \tag{10}$$

The Three Wave Modes

If there exists non-trivial solution $U_k^{(0)}$ for the eikonal equation,then the following determinant should be zero:

$$det(a_{ijkl}p_i p_l - \delta_{jk}) = 0, \tag{11}$$

which means that one is one of the eigenvalues of the Kelvin-Christoffel matrix Γ_{jk}.

Introduce

$$p_i = \frac{n_i}{v_n} \tag{12}$$

where n_i is the unit normal vector to the wavefront, v_n is the normal or phase speed of the wavefront.

Then the above determinant will yield a characteristic equation of normal vector:

$$det(a_{ijkl}n_i n_l - v_n^2\delta_{jk}) = 0, \tag{13}$$

which is a cubic characteristic polynomial equation with respect to v_n^2, so it has three eigenvalues corresponding to a quasi-compressional (qP) and two quasi-shear (qSH,qSV) waves.

From the viewpoint of physics, the above equation means that for a given set of elastic parameters and a given normal n_i at a given spatial point x_i, there exists three different squared normal speed v_n^2 which correspond to three different wave propagations,and it also implies that three positive v_n's correspond to downgoing waves and three negative v_n's to upgoing waves.

In isotropic media,the direction of compressional P eigenvector is tangent to the wavefront normal, and the shear SV and SH eigenvetors are normal to the P eigenvector,the three eigenvectors form an orthogonal system in isotropic media.But as was shown by Musgrave (Musgrave, 1970), the P-eigenvetor is not in general tangent to the wavefront normal in anisotropic media,hence the prefix "quasi"(see Figure 1.) . Moreover,the experiment shows that the velocity of the qP wave is always greater than those of the qS waves (Fedorov, 1968). Hence the largest eigenvalue of equation (13) corresponding to the qP wave propagation is uniquely defined.

Hamilton-Jacobi equation for qP wave

The largest eigenvalue of the characteristic equation with normal vector is the squared phase velocity of the qP wave

$$\lambda_{max}(n) = v_{qp}^2(n_1, n_2), \tag{14}$$

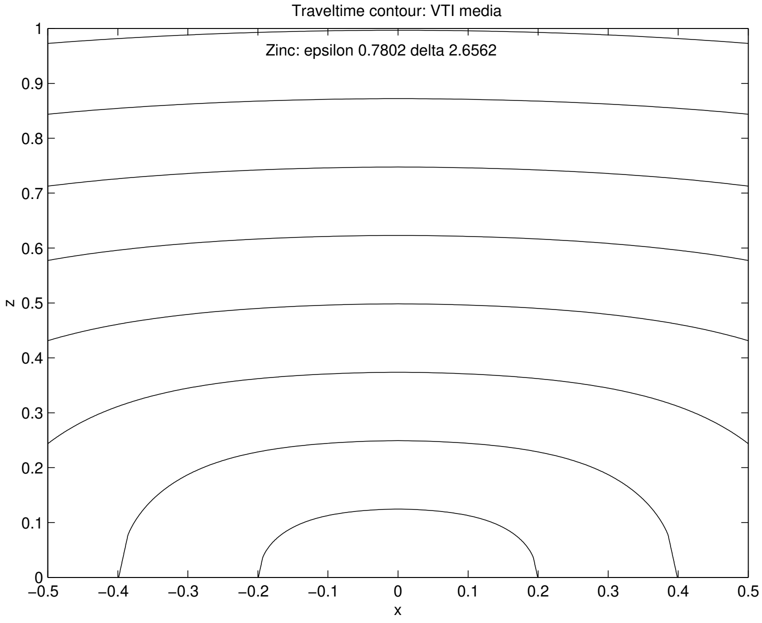

FIG. 1. The ray direction is not in general tangent to the wavefront normal in homogeneous anisotropic media: Zinc

and thus the eikonal equation for qP wave is

$$p_1^2 + p_2^2 + p_3^2 \quad = \quad \frac{1}{v_{qp}^2} \tag{15}$$

where p_i is the slowness vector of qP wave. We can use Rayleigh Quotient Iteration (RQI) to get the largest eigenvalue with cubic convergence (Golub and Van Loan, 1996).

The first-arrival traveltime is the traveltime for the qP wavefront to travel from a source point to a depth point which corresponds to downgoing qP wave propagation i.e. $p_3 > 0$. View the above eikonal equation as evolution equation in depth,we have the following Hamilton-Jacobi equation for first-arrival traveltime:

$$\begin{aligned} p_3 &= H(p_1, p_2) \\ &= \sqrt{\frac{1}{v_{qp}^2(p_1, p_2)} - p_1^2 - p_2^2}, \end{aligned} \tag{16}$$

or

$$\frac{\partial \tau}{\partial x_3} \quad = \quad H\left(\frac{\partial \tau}{\partial x_1}, \frac{\partial \tau}{\partial x_2}\right). \tag{17}$$

with some appropriate initial/boundary conditions, where τ is the traveltime.The above equation is a nonlinear first-order PDE which does not have classical solutions in general.

Assume that $p_3 > 0$,so

$$p_3 - H(p_1,p_2) \;=\; 0, \tag{18}$$

then we have the following Hamilton equation by the method of characteristics (Courant and Hilbert, 1962; Evans, 1994):

$$\frac{dx_3}{ds} \;=\; \frac{\partial}{\partial p_3}\left(p_3 - H(p_1,p_2)\right) = 1, \tag{19}$$

thus x_3 is increasing along characteristic(ray) and can be taken as the parameter along the characteristic(ray): $\{(x_1,x_2,x_3) : x_1 = x_1(x_3), x_2 = x_2(x_3)\}$. Moreover,

$$\frac{dx_1}{dx_3} \;=\; -\frac{\partial H}{\partial p_1}, \tag{20}$$

$$\frac{dx_2}{dx_3} \;=\; -\frac{\partial H}{\partial p_2}, \tag{21}$$

$$\begin{aligned} \frac{d\tau}{dx_3} &= \frac{d}{dx_3}\left(\tau(x_1(x_3),x_2(x_3),x_3)\right) \\ &= -\frac{\partial H}{\partial p_1}p_1 - \frac{\partial H}{\partial p_2}p_2 + p_3 \\ &= -\frac{\partial H}{\partial p_1}p_1 - \frac{\partial H}{\partial p_2}p_2 + H(p_1,p_2). \end{aligned} \tag{22}$$

The group velocity of ray is

$$\begin{aligned} v_g &= \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2} \\ &= \frac{dx_3}{dt}\sqrt{\left(\frac{dx_1}{dx_3}\right)^2 + \left(\frac{dx_2}{dx_3}\right)^2 + 1} \\ &= \frac{\sqrt{\left(\frac{\partial H}{\partial p_1}\right)^2 + \left(\frac{\partial H}{\partial p_2}\right)^2 + 1}}{H(p_1,p_2) - \frac{\partial H}{\partial p_1}p_1 - \frac{\partial H}{\partial p_2}p_2}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \frac{\partial H}{\partial p_1} &= \frac{\partial p_3}{\partial p_1} \\ &= \frac{\partial p_3}{\partial n_1}\frac{\partial n_1}{\partial p_1} + \frac{\partial p_3}{\partial n_2}\frac{\partial n_2}{\partial p_1} \\ &= \frac{\frac{\partial p_3}{\partial n_1}\frac{\partial p_2}{\partial n_2} - \frac{\partial p_3}{\partial n_2}\frac{\partial p_2}{\partial n_1}}{\frac{\partial p_1}{\partial n_1}\frac{\partial p_2}{\partial n_2} - \frac{\partial p_1}{\partial n_2}\frac{\partial p_2}{\partial n_1}} \\ &= \frac{p_1p_2v_{n_2} - (p_2^2+p_3^2)v_{n_1} - p_1}{p_3(1 - v_{n_1}p_1 - v_{n_2}p_2)} \end{aligned} \tag{24}$$

$$
\begin{aligned}
\frac{\partial H}{\partial p_2} &= \frac{\partial p_3}{\partial p_2} \\
&= \frac{\partial p_3}{\partial n_1}\frac{\partial n_1}{\partial p_2} + \frac{\partial p_3}{\partial n_2}\frac{\partial n_2}{\partial p_2} \\
&= \frac{-\frac{\partial p_3}{\partial n_1}\frac{\partial p_1}{\partial n_2} + \frac{\partial p_3}{\partial n_2}\frac{\partial p_1}{\partial n_1}}{\frac{\partial p_1}{\partial n_1}\frac{\partial p_2}{\partial n_2} - \frac{\partial p_1}{\partial n_2}\frac{\partial p_2}{\partial n_1}} \\
&= \frac{p_1 p_2 v_{n_1} - (p_1^2 + p_3^2) v_{n_2} - p_2}{p_3(1 - v_{n_1} p_1 - v_{n_2} p_2)}
\end{aligned} \tag{25}
$$

By

$$
\begin{aligned}
p_1 &= \frac{n_1}{v(n_1, n_2)}, \\
p_2 &= \frac{n_2}{v(n_1, n_2)}, \\
p_3 &= \frac{\sqrt{1 - n_1^2 - n_2^2}}{v(n_1, n_2)},
\end{aligned} \tag{26}
$$

where v is the normal velocity and n_i is the unit normal vector to the wavefront,the partial derivatives v_{n_1} and v_{n_2} can be found by v^2 being the eigenvalue of matrix

$$\tilde{\Gamma}_{jk} = a_{ijkl} n_i n_l \tag{27}$$

with g the corresponding unit eigenvector,then

$$v^2 = a_{ijkl} n_i n_l g_j g_k. \tag{28}$$

It follows that

$$
\begin{aligned}
v_{n_1} &= \frac{1}{2v} g^T \frac{\partial \tilde{\Gamma}}{\partial n_1} g \\
&= a_{1jkl} p_l g_j g_k - a_{3jkl} \frac{p_1}{p_3} p_l g_j g_k,
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
v_{n_2} &= \frac{1}{2v} g^T \frac{\partial \tilde{\Gamma}}{\partial n_2} g \\
&= a_{2jkl} p_l g_j g_k - a_{3jkl} \frac{p_2}{p_3} p_l g_j g_k,
\end{aligned} \tag{30}
$$

On substituting (29) and (30) into (24) and (25),and the results in (22) we obtain

$$\frac{\partial H}{\partial p_1} = -\frac{a_{1jkl} p_l g_j g_k}{a_{3jkl} p_l g_j g_k} \tag{31}$$

$$\frac{\partial H}{\partial p_2} = -\frac{a_{2jkl}p_l g_j g_k}{a_{3jkl}p_l g_j g_k} \tag{32}$$

$$\frac{d\tau}{dx_3} = \frac{1}{a_{3jkl}p_l g_j g_k} \tag{33}$$

where we have used the fact that

$$a_{ijkl}p_i p_l g_j g_k = 1. \tag{34}$$

And by further derivation,

$$g_j g_k = D_{jk}/D \tag{35}$$

where following (Cerveny, 1972),

$$\begin{aligned}
D_{11} &= (\Gamma_{22}-1)(\Gamma_{33}-1)-{\Gamma_{23}}^2,\\
D_{22} &= (\Gamma_{11}-1)(\Gamma_{33}-1)-{\Gamma_{13}}^2,\\
D_{33} &= (\Gamma_{11}-1)(\Gamma_{22}-1)-{\Gamma_{12}}^2,\\
D_{12} &= D_{21}=\Gamma_{13}\Gamma_{23}-\Gamma_{12}(\Gamma_{33}-1),\\
D_{13} &= D_{31}=\Gamma_{12}\Gamma_{23}-\Gamma_{13}(\Gamma_{22}-1),\\
D_{23} &= D_{32}=\Gamma_{12}\Gamma_{13}-\Gamma_{23}(\Gamma_{11}-1),\\
D &= D_{11}+D_{22}+D_{33}.
\end{aligned} \tag{36}$$

The Transport Equation

By the characteristic equation in slowness vector (11), the eigenvalue one corresponding to the qP wave is distinct from the other eigenvalues (Fedorov, 1968),it follows that the unit eigenvector corresponding to eigenvalue one will be unique for qP wave and smoothly depend on elastic parameters which is the same as g_k, so still denoted as g_k and could be determined once the slowness vector for qP wave is given.

From the zeroth order approximation of progressing wave expansion

$$(\Gamma_{jk}-\delta_{jk})U_k^{(0)} = 0, \tag{37}$$

we can assume that

$$U_k^{(0)} = a g_k, \tag{38}$$

where a is the magnitude of $U_k^{(0)}$ and g_k is the direction of $U_k^{(0)}$ which can be determined by (35).

So substitute the above form into the transport equation (6),we have

$$(C_{ijkl}p_i\frac{\partial g_k}{\partial x_l}+\frac{\partial}{\partial x_i}(C_{ijkl}g_kg_l))a(x)+((C_{ijkl}+C_{ljki})g_kp_i)\frac{\partial a}{\partial x_l} \quad = \quad 0. \qquad (39)$$

The above equation is a linear advection equation which coefficients involve the derivatives of the eigenvector and slowness vector.

ENO schemes

Our Hamilton Jacobi equation is of the evolution form

$$p_3 \quad = \quad H(p_1,p_2) \qquad (40)$$

or

$$\frac{\partial \tau}{\partial x_3} \quad = \quad H\left(\frac{\partial \tau}{\partial x_1},\frac{\partial \tau}{\partial x_2}\right). \qquad (41)$$

with the following initial condition

$$\tau(x_1,x_2,0) \quad = \quad \tau^0(x_1,x_2). \qquad (42)$$

The above equation is a globally nonlinear first-order PDE and it is well known that they do not have classical solutions in general,even if the Hamiltonian and the initial condition are smooth.So we have to appeal to generalized solutions. Assume both the Hamiltonian H and the initial condition τ^0 are locally Lipschitz continuous, solutions to the above Hamilton-Jacobi equation are Lipschitz continuous but may have discontinuous derivatives,regardless of the smoothness of $\tau^0(x,y)$.Such generalized solutions are not unique,Crandall and Lions (Crandall and Lions, 1983) defined the viscosity solution to single out a unique practically relevant solution among many generalized solutions. The viscosity solution is continuous but need not be differentiable everywhere.

Crandall and Lions(1984) established the convergence of a class of monotone finite difference schemes to the viscosity solution of Hamilton-Jacobi equations.The Essentially Non-Oscillatory(ENO) schemes were adapted to the Hamilton-Jacobi equation by Osher and Sethian (Osher and Sethian, 1988) and Osher and Shu (Osher and Shu, 1991). Then they were used to solve 2-D and 3-D eikonal equation in isotropic media by Symes et al.(1994),El-Mageed (El-Mageed, 1996),El-Mageed et.al (El-Mageed et al., 1997), they yielded excellent numerical results.

There are many monotone numerical Hamiltonians. Among them,the Godunov type numerical Hamiltonian is of particular interest to us because it can be used for general Hamiltonian. To get higher order ENO schemes,we just use the first order schemes as building blocks and replace the first order approximation to derivative with higher order approximation,see Shu (Shu, 1997).

Basically,to get the viscosity solutions to our Hamilton-Jacobi equation for down-going qP wave in anisotropic media,ENO-HJ numerical flux (numerical Hamiltonian) will be constructed to approximate the Hamiltonian H in discretization,TVD Runge-Kutta schemes (Shu and Osher, 1988) will be adapted to extrapolate downward along the $z-$ direction.

Because we are only interested in the first arrival traveltime which corresponds to the qP wave propagation, we expect the viscosity solution to Hamilton Jacobi equation for the qP wave is the first arrival traveltime . We will discuss some practical issues related to the numerical approximation of the viscosity solution. On the other hand,there exist cusps in the qSV wave propagation (Burridge, 1995), which implies that it is difficult to obtain the viscosity solution to Hamilton Jacobi equation for qSV wave if there is one such solution.

To be convenient we will use (x, y, z) instead of (x_1, x_2, x_3) to denote a spatial point in the following.

A New Derivation of Godunov's Scheme

At present we are only interested in the first arrival traveltime, and we will use the local ray information to construct the upwind finite difference schemes for qP wave propagation in general anisotropic media,i.e. make use of the idea of local ray-tracing. We will consider the two dimensional case to illustrate the idea,which means all out-the-plane components vanish in our equations formulated above for 3D media and all rays stay in the xz plane. The three dimensional case can be considered similarly.

So assume given a Cartesian grid in two-dimensional space, we want to seek approximation to traveltime $\tau_m^{n+1} \approx \tau((n+1)\Delta z, m\Delta x)$, where m and n are indices for x and z direction respectively, and Δx and Δz are mesh sizes along x and z direction respectively, see Figure 2.

Define the approximation of derivative τ_x at $(n\Delta z, x)$ as follows:

$$\tau_x^n \approx \begin{cases} D_x^- \tau_m^n & \text{for } (m-1)\Delta x < x < m\Delta x \\ D_x^+ \tau_m^n & \text{for } m\Delta x < x < (m+1)\Delta x \end{cases} \tag{43}$$

where D_x^- and D_x^+ are backward and forward difference quotient operator respectively :

$$D_x^{\pm}\tau_m^n = \pm\frac{\tau_{m\pm1}^n - \tau_m^n}{\Delta x}.$$

Suppose we are looking for rays crossing between $F(n\Delta z, (m-1)\Delta x)$ and $D(n\Delta z, (m+1)\Delta x)$ and hitting some point $G(n\Delta z + \Delta z_{local}, m\Delta x)$ between C and

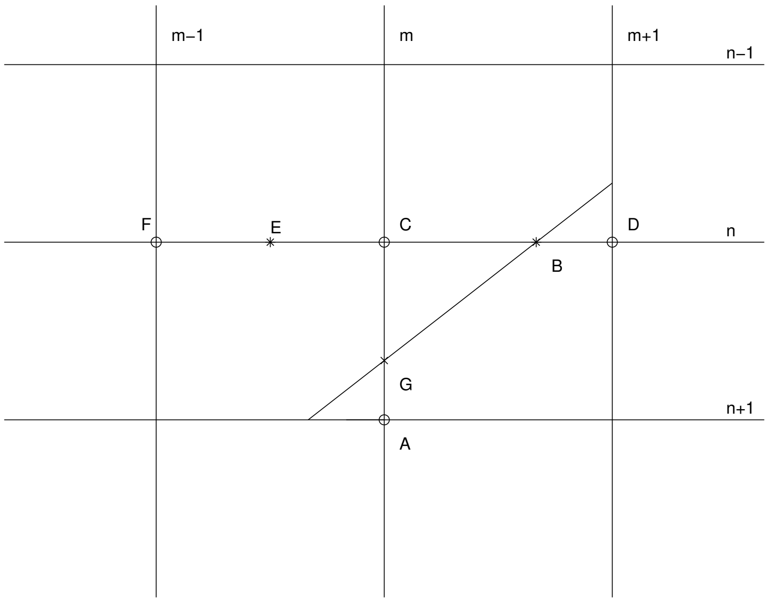

FIG. 2. Regular grid : a possible ray hitting a point G

A.If we approximate rays locally by straight lines,then the traveltime τ_G at G consists of two parts, one part being the traveltime τ_0 at B or E,another part the traveltime from B to G or E to G.

If the straight line segment $\overline{BG}$ or $\overline{EG}$ is to approximate a ray segment, its slope must be consistent with the ray equation:

$$\begin{aligned} \lambda \frac{\Delta x}{\Delta z_{local}} &= -\frac{dx}{dz} \\ &= \frac{\partial H}{\partial p_1}(\hat{p_1}) \end{aligned} \tag{44}$$

with the restriction that

$$|\lambda| \leq 1.$$

i.e.

$$\left| \frac{\Delta z_{local}}{\Delta x} \frac{\partial H}{\partial p_1}(\hat{p_1}) \right| \leq 1. \tag{45}$$

where $\hat{p_1}$ will be chosen later.

τ_0 can be approximated by linear interpolation:

$$\tau_0 \approx \tau_m^n + \lambda \Delta x \hat{p_1}. \tag{46}$$

The traveltime increment $\Delta\tau$ along the ray segment $\overline{BG}$ or $\overline{EG}$ is

$$\Delta\tau = \Delta z_{local} \frac{\sqrt{1 + \lambda^2 (\frac{\Delta x}{\Delta z_{local}})^2}}{v_g}. \tag{47}$$

where v_g is the group velocity (23) specialized in 2D case:

$$v_g \;=\; \frac{\sqrt{\left(\frac{\partial H}{\partial p_1}(\hat{p}_1)\right)^2+1}}{H(\hat{p}_1)-\hat{p}_1\frac{\partial H}{\partial p_1}(\hat{p}_1)}. \tag{48}$$

Then the traveltime at G is

$$\begin{aligned}
\tau_m^{n+1} &= \tau_0+\Delta\tau \\
&= \tau_m^n+\Delta z_{local}\hat{p}_1\frac{\partial H}{\partial p_1}(\hat{p}_1)+\Delta z_{local}\frac{\sqrt{1+\left(\frac{\partial H}{\partial p_1}(\hat{p}_1)\right)^2}}{v_g} \\
&= \tau_m^n+\Delta z_{local}\hat{p}_1\frac{\partial H}{\partial p_1}(\hat{p}_1)+\Delta z_{local}\left(H(\hat{p}_1)-\hat{p}_1\frac{\partial H}{\partial p_1}(\hat{p}_1)\right) \\
&= \tau_m^n+\Delta z_{local}H(\hat{p}_1).
\end{aligned} \tag{49}$$

From equation (44)

$$\lambda \;=\; \frac{\Delta z_{local}}{\Delta x}\frac{\partial H}{\partial p_1}(\hat{p}_1) \tag{50}$$

it follows that

1. If $\frac{\partial H}{\partial p_1}\left(D_x^{\pm}\tau_m^n\right)\geq 0$, $\lambda>0$, choose $\hat{p}_1=D_x^+\tau_m^n$.
2. If $\frac{\partial H}{\partial p_1}\left(D_x^{\pm}\tau_m^n\right)\leq 0$, $\lambda<0$, choose $\hat{p}_1=D_x^-\tau_m^n$.
3. If $\frac{\partial H}{\partial p_1}\left(D_x^{-}\tau_m^n\right)\geq 0$ and $\frac{\partial H}{\partial p_1}\left(D_x^{+}\tau_m^n\right)\leq 0$,which corresponds to rarefaction of rays, so choose $\hat{p}_1=0$.
4. If $\frac{\partial H}{\partial p_1}\left(D_x^{-}\tau_m^n\right)\leq 0$ and $\frac{\partial H}{\partial p_1}\left(D_x^{+}\tau_m^n\right)\geq 0$,which corresponds to shocks,i.e. there are two approximate rays hitting the point G. The first-arrival traveltime requires choosing one leading to least traveltime,namely,

$$H(\hat{p}_1) \;=\; \min\left\{H(D_x^+\tau_m^n),H(D_x^-\tau_m^n)\right\}. \tag{51}$$

The partial derivative $\frac{\partial H}{\partial p_1}(\hat{p}_1)$ can be computed by the formula (24).

To march one whole step Δz,we need to proceed several local steps to do so. Actually,the above restriction turns out to be CFL condition required for the stability of finite difference schemes (LeVeque, 1990).

The above scheme is monotone and of first order accuracy,higher order schemes can be constructed by using the above scheme as building blocks. We note that for isotropic case the above choice amounts to picking

$$\hat{p_1} \quad = \quad maxmod(\max(D_x^- \tau_m^n, 0), \min(D_x^+ \tau_m^n, 0)) \tag{52}$$

where maxmod returns the larger value in modulus. Our numerical results showed that it is probably true for most weakly anisotropic media.

In 3D case,the numerical Hamiltonian will be taken as (Shu, 1997)

$$H(\hat{p_1}, \hat{p_2}) \quad = \quad ext_{u \in I(D_x^- \tau, D_x^+ \tau)} ext_{v \in I(D_y^- \tau, D_y^+ \tau)} H(u, v) \tag{53}$$

where the extrema are defined by

$$ext_{u \in I(a,b)} \quad = \quad \begin{cases} max_{a \le u \le b} & \text{if } a \le b, \\ min_{b \le u \le a} & \text{if } a > b. \end{cases}$$

To determine the extrema,we need to use $\frac{\partial H}{\partial p_1}$ and $\frac{\partial H}{\partial p_2}$ to justify the upwind direction.

ENO Schemes for First-Arrival in Anisotropic Media

Given mesh sizes Δx,Δy,Δz, denote grid points as

$$(x_m, y_k, z_n) \quad = \quad (x_{\min} + (m-1)\Delta x, y_{\min} + (k-1)\Delta y, z_{\min} + (n-1)\Delta z), \tag{54}$$

the numerical approximation to the viscosity solution $\tau(x_m, y_k, z_n)$ of Hamilton-Jacobi equation (16) at the grid point (x_m, y_k, z_n) is denoted as

$$\tau_{m,k}^n \quad = \quad \tau(x_m, y_k, z_n),$$

define the backward (-) and forward (+) first order difference quotient approximations to the left and right derivatives of $\tau(x, y, z)$ at the location (x_m, y_k, z_n) with respect to x and y:

$$\begin{aligned} D_x^{\pm} \tau_{m,k}^n &= \pm \frac{\tau_{m\pm1,k}^n - \tau_{m,k}^n}{\Delta x}, \\ D_y^{\pm} \tau_{m,k}^n &= \pm \frac{\tau_{m,k\pm1}^n - \tau_{m,k}^n}{\Delta y}. \end{aligned} \tag{55}$$

We adapt Godunov's type monotone numerical Hamiltonian to approximate the Hamiltonian H in (16),the first-order upwind scheme follows:

$$\tau_{m,k}^{n+1} \quad = \quad \tau_{m,k}^n + \Delta z_{CFL} H[\tau]_{m,k}^n \tag{56}$$

where Δz_{CFL} satisfies the CFL condition and $H[\tau]^n_{m,k}$ is computed by our Godunov's scheme presented above,

$$H[\tau]^n_{m,k} \quad = \quad H\left((\hat{p_1})^n_{m,k}, (\hat{p_2})^n_{m,k}\right). \tag{57}$$

We remark that if the Hamiltonian is convex,then

$$\begin{aligned}(\hat{p_1})^n_{m,k} &= maxmod(\max(D_x^- \tau^n_{m,k}, 0), \min(D_x^+ \tau^n_{m,k}, 0)), \\ (\hat{p_2})^n_{m,k} &= maxmod(\max(D_y^- \tau^n_{m,k}, 0), \min(D_y^+ \tau^n_{m,k}, 0)),\end{aligned} \tag{58}$$

which is similar to the isotropic case.In general,we must use the derivatives $\dfrac{\partial H}{\partial p_1}$ (31) and $\dfrac{\partial H}{\partial p_2}$ (32) to justify the upwind direction.

To get higher order schemes,the above numerical Hamiltonian will be used as buliding blocks,the first order approximation to the derivative will be replaced by higher order ENO approximation. And along the z-direction the higher order TVD Runge-Kutta schemes (Shu and Osher, 1988) will be used to match the higher order ENO schemes for the other variables.

To derive a second-order scheme,use the following second order ENO finite difference to approximate the first order derivative $\dfrac{\partial \tau}{\partial x}$ (Osher and Sethian, 1988)

$$D_x^{+,2}\tau \quad = \quad D_x^+\tau - \frac{1}{2}\Delta x m(D_x^+ D_x^+ \tau, D_x^- D_x^+ \tau), \tag{59}$$

$$D_x^{-,2}\tau \quad = \quad D_x^-\tau + \frac{1}{2}\Delta x m(D_x^- D_x^- \tau, D_x^- D_x^+ \tau) \tag{60}$$

where

$$m(x,y) \quad = \quad \begin{cases} x & \text{if } |x| <= |y| \text{ and } xy > 0, \\ y & \text{if } |y| < |x| \text{ and } xy > 0, \\ 0 & \text{if } xy <= 0. \end{cases}$$

Introduce the analogues for $\dfrac{\partial \tau}{\partial y}$.

Then ENO-HJ TVD-RK second-order scheme follows :

$$\begin{aligned}\tau^{n+\frac{1}{2}}_{m,k} &= \tau^n_{m,k} + \Delta z_{CFL} H[\tau]^n_{m,k}, \\ \tau^{n+1}_{m,k} &= \frac{1}{2}(\tau^n_{m,k} + \tau^{n+\frac{1}{2}}_{m,k} + \Delta z_{CFL} H[\tau]^{n+\frac{1}{2}}_{m,k})\end{aligned} \tag{61}$$

where $H[\tau]^n_{m,k}$ is computed by our Godunov scheme with the second order ENO approximation (59) and (60) to $p_1 = \dfrac{\partial \tau}{\partial x}$ and $p_2 = \dfrac{\partial \tau}{\partial y}$, Δz_{CFL} is the CFL step,

$$H[\tau]^n_{m,k} \quad = \quad H\left((\hat{p_1})^n_{m,k}, (\hat{p_2})^n_{m,k}\right). \tag{62}$$

Now we consider the stability condition. For given consistent finite difference schemes solving evolution partial differential equations, the $Courant-Friedrichs-Levy$ (CFL) stability condition states that the numerical domain of dependence of the discretized PDE contains the physical domain of dependence of PDE (LeVeque, 1990).

Linearize Hamilton-Jacobi equation (16) for qP wave with respect to p_1, p_2 and p_3, we have

$$\delta p_3 = \frac{\partial H}{\partial p_1}\delta p_1 + \frac{\partial H}{\partial p_2}\delta p_2, \tag{63}$$

where

$$\frac{\partial H}{\partial p_1} = \frac{\partial p_3}{\partial p_1}, \tag{64}$$

$$\frac{\partial H}{\partial p_2} = \frac{\partial p_3}{\partial p_2}, \tag{65}$$

The above partial derivatives can be computed by our general formulae (31),(32).

Thus the CFL condition necessary for stability is

$$\Delta z_{CFL}\left(\max_{p_1,p_2}\sqrt{\left(\frac{\partial H}{\partial p_1}\right)^2 + \left(\frac{\partial H}{\partial p_2}\right)^2}\right) \leq \frac{\Delta x \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}. \tag{66}$$

where the maximum is taken over the relevant range of p_1 and p_2.

We should avoid choosing very small step size in the $z-$direction because a small step size increases the computational cost, and it may cause a severe numerical diffusion/dispersion since it diminishes the physical domain of dependence and therefore the numerical domain of dependence is relatively enlarged (Strikwerda, 1989).

By our assumption $p_3 > 0$, one should avoid p_3 going to zero which is achieved by enforcing the maximum angle condition.

The slowness vector for qP wave is

$$\begin{aligned} p_1 &= \frac{\sin\theta\cos\phi}{v_{qp}(\theta,\phi)}, \\ p_2 &= \frac{\sin\theta\sin\phi}{v_{qp}(\theta,\phi)}, \\ p_3 &= \frac{\cos\theta}{v_{qp}(\theta,\phi)}, \end{aligned} \tag{67}$$

where θ is the angle between the normal vector to the qP wave front and the z axis,ϕ is in between 0 and 2π.

Given θ_{max},let $|\theta| \leq \theta_{\max}$,then

$$p_3 \geq \min_{\phi} \frac{\cos(\theta_{\max})}{v_{qp}(\theta_{\max}, \phi)}, \tag{68}$$

we have a modified Hamilton Jacobi equation for qP wave

$$p_3 = bmax\left(H(p_1, p_2), \min_{\phi} \frac{\cos(\theta_{\max})}{v_{qp}(\theta_{\max}, \phi)}\right), \tag{69}$$

where $bmax$ is a piecewise quadratic function,

$$\begin{aligned} bmax(x, y) &= y \text{ if } |x| \leq 0 \\ &= y + \frac{x^2}{4y} \text{ if } 0 \leq x \leq 2y \\ &= x \text{ if } x \geq 2y. \end{aligned}$$

In effect we have modified the wavefront: the part with near vertical normals is unaffected, but the steeper part is replaced by a plane wave, so that we have a limited aperture with $2\theta_{\max}$.

What's next

To compute the traveltime and amplitude in general inhomogeneous anisotropic media,we have proposed a second order ENO-HJ and TVD-RK scheme. We have tested the algorithm in 3D transversely anisotropic media with vertical symmetry axis (VTI) which yielded efficient and accurate solutions.

When we have the traveltime and amplitude at our disposal,we want to use them in the high frequency asymptotic imaging techniques such as Kirchhoff migration method for imaging/inversion in reflection seismology (Beylkin, 1985; Symes et al., 1994; Symes, 1995).

Also with the traveltime and amplitude at hand,we want to generalize DSO (differential semblance optimization)(Symes and Carazzone, 1991; Symes, 1990) for the velocity analysis in the isotropic media to the anisotropic media.

Because all our design is based upon the assumption that the first-arrival traveltime corresponding to qP wave propagation in anisotropic media is the viscosity solution of some Hamilton-Jacobi equation (Lions, 1982; Evans, 1994),we need to prove it theoretically.

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