Second and Third Order ENO Methods for the Eikonal Equation

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ABSTRACT

The first arrival time problem may in some cases considered as a Hamilton-Jacobi initial value problem. Such a problem can be solved by a very simple finite difference method. The second and third order essentially nonoscillatory (ENO) extensions to this method are relatively straight-forward and result in significant accuracy and efficiency gains.

Key words.travel times, eikonal equation, essentiallly nonoscillatory schemes

1. A MARCHING FORMULATION OF TRAVELTIME COMPUTATION

The first arrival time in a slowness (reciprocal velocity) field $s(x, z)$ defined on a open subset of $\Re^2$ is a *viscosity solution* of the eikonal equation (Lions, 1982):

$$\left(\frac{\partial \tau}{\partial x}\right)^2 + \left(\frac{\partial \tau}{\partial z}\right)^2 = s^2 \tag{1}$$

In the regions where $\frac{\partial \tau}{\partial z} > 0$, the eikonal equation may be rewritten as an evolution equation in z:

$$\frac{\partial \tau}{\partial z} = \sqrt{s^2 - \left(\frac{\partial \tau}{\partial x}\right)^2} \tag{2}$$

which has the form of a Hamilton-Jacobi equation in one space variable (x) and "time" z:

$$\frac{\partial \tau}{\partial z} + H\left(\frac{\partial \tau}{\partial x}\right) = 0 \tag{3}$$

with $H(u) = -\sqrt{s^2 - u^2}$. This formulation is particularly appropriate for applications in reflection seismology, where sometimes (but not always) $\frac{\partial \tau}{\partial z} > 0$ in the region of

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interest. In terms of rays (characteristic curves of the eikonal equation), $\frac{\partial \tau}{\partial z} > 0$ means that the ray velocity vector has a positive z-component, ie. the ray is "downgoing".

(Osher and Sethian, 1988) (Sethian, 1996) and others have introduced numerical methods for the Hamilton-Jacobi initial value problems based on numerical flux functions $\hat{H}$: that is

$$\tau_i^{n+1} = \tau_i^n - \Delta z \hat{H}(D^+\tau_i^n, D^-\tau_i^n) \tag{4}$$

where

$$D^+\tau_i = \frac{(\tau_{i+1} - \tau_i)}{\Delta x}, \quad D^-\tau_i = \frac{(\tau_i - \tau_{i-1})}{\Delta x}, \tag{5}$$

are the forward and backward difference approximations to $\frac{\partial \tau}{\partial x}$.

The numercial flux function $\hat{H}$ must approximate H, in some sense. In order to ensure convergence, $\hat{H}$ must be consistent. Consistency requires that $\hat{H}(u,u) = H(u)$. It is readily apparent that this condition is equivalent to the standard notion of consistency for finite difference schemes. If $\hat{H}$ is continuously differentiable and consistent, then the above scheme will be first order accurate.

In this paper, we will explore a particular class of fluxes, the so called essentially nonoscillatory (ENO) extensions of the Godunov flux, defined below. These methods are presented in (Osher and Shu, 1991). We will show that the 2nd and 3rd order ENO fluxes permit very efficient and accurate computations of traveltimes.

2. THE GODUNOV FLUX

The Godunov flux, to be defined below, is consistent, monotone, and first order accurate. $\hat{H}$ is called monotone if it is nonincreasing with respect to $D^+\tau$ and nondecreasing with respect to $D^-\tau$. The importance of monotonicity is that such schemes are proven to converge to the viscosity solutions (Lions, 1982). Viscosity solutions are a particular class of generalized or weak solutions of the H-J equation. Generally, such equations do not have smooth solutions which satisfy it in the usual sense; so one must introduce a suitable notion of generalized solution. This is done for example in (Lions, 1982); which also shows that weak solutions are not uniquely determined by their initial data. Viscosity solutions are particular weak solutions which arise as limits as $\epsilon \to \infty$ of the dissipative equation

$$\frac{\partial \tau}{\partial z} + H\left(\frac{\partial \tau}{\partial x}\right) - \epsilon \frac{\partial^2 \tau}{\partial x^2} = 0 \tag{6}$$

The last term is often said to represent "viscous damping" hence the name. Solutions of the dissipative equation are smooth and are uniquely determined by their initial data. While singularities may appear in the $\epsilon = 0$ limit, uniqueness is retained. Lions also showed that the viscosity solution of the eikonal equation is the first arrival time field. The first arrival time generally has a discontinuous gradient in regions where more than one ray passes over each point. Thus the first arrival time can be a solution to the eikonal equation only in a generalized sense. The identity of the first arrival time as the viscosity solution is responsible for the success of finite difference eikonal solvers.

The Godunov flux is also "upwind", which means that the scheme will use data that comes from backwards along characteristics. When computing discontinuous solutions, upwind methods are noticably more accurate than other stable methods of the same order (LeVeque, 1990), essentially because they compute the solutions using only the relevant information, and thus avoid using data from both sides of a discontinuity. There are many possible consistent choices for $\hat{H}$, given H. The Godunov flux is defined as:

$$\hat{H}(u,v) = \begin{cases} \min_{w\in I(u,v)} H(w) & \text{if } u \geq v \\ \max_{w\in I(u,v)} H(w) & \text{if } u < v \end{cases} \tag{7}$$

where:

$$I(a,b) = \begin{cases} (a,b) & \text{if } a < b \\ (b,a) & \text{if } a \geq b \end{cases}$$

If H is convex on the domain, this numerical flux is very easy to evaluate, because H has only one local extremum. If $I(D^-\tau, D^+\tau)$ does not include this unique local extremum, then the evaluation is nearly trivial, and it turns out in this case that:

- If H is strictly increasing on $I(D^+\tau, D^-\tau)$, then the Godunov flux uses the backward difference.
- If H is strictly decreasing on $I(D^+\tau, D^-\tau)$, then the Godunov flux uses the forward difference.

This choices turns out to be the upwind choice. To show this, let $u = \tau_x$ and differentiate $\tau_z + H(\tau_x) = 0$ to get $u_z + (H(u))_x = 0$. Let $u = u(z, x(z))$ be a characteristic of this equation. This implies that $\frac{d}{dz}u(z,x(z)) = 0$ and $u_z + u_x x' = 0$. Notice that the above equation can be rewritten as $u_z + H'(u)u_x = 0$, which means that $x' = H'(u(z,x(z)))$. Thus H' may be used to determine the upwind direction. $H' > 0$ implies that the negative x direction is upwind, and thus we should choose $D^-\tau$, and analogously, $H' < 0$ implies that we should choose $D^+\tau$, which are exactly the choices outlined earlier.

The particular example under consideration, the eikonal equation, has $H = -\sqrt{s^2 - u^2}$. This flux function is convex on the domain and thus the Godunov flux may be dissected as follows: (Note that the unique local extreme occurs at $u = 0$)

let $u_+ = D^+\tau,\ u_- = D^-\tau,$ then:
if $(u_- \leq u_+)$ then
  if $(0 \leq u_- \leq u_+)$ then
    $\hat{H} = \min(H(u)) = \min(H(u_+), H(u_-)) = H(u_-)$
  else if $(u_- \leq u_+ \leq 0)$ then
    $\hat{H} = \min(H(u)) = \min(H(u_+), H(u_-)) = H(u_+)$

else $(u_- \leq 0 \leq u_+)$
$\hat{H} = \min(H(u)) = H(0)$
else $(u_- > u_+)$
if $(0 \leq u_+ \leq u_-)$ then
$\hat{H} = \max(H(u)) = \max(H(u_+), H(u_-)) = H(u_-)$
else if $(u_+ \leq u_- \leq 0)$ then
$\hat{H} = \max(H(u)) = \max(H(u_+), H(u_-)) = H(u_+)$
else $(u_+ \leq 0 \leq u_-)$
$\hat{H} = \max(H(u)) = \max(H(u_+), H(u_-)) = H(\max(|u_+|, |u_-|))$

After some examination, the reader may clearly see that this large expression may be greatly condensed by a clever usage of maxs and mins.

$$\hat{H} = H(\mathrm{maxmod}(\max(\mathrm{u}_-, 0), \min(\mathrm{u}_+, 0))) \tag{8}$$

which always chooses u_+ when both are negative and u_- when both are positive and makes the correct choice when the have opposite signs. Pseudocode for the first-order Godunov Scheme:

let $\tau_i^n = \tau(x_i, z_n)$
initialize: $\tau_0^n = \tau(x_i, z_0)$ for $i = 1, ..., m$
$h = |x_i - x_{i-1}|, \ k = |z_j - z_{j-1}|$
for $n = 1$ to N
for $i = 1$ to $m - 1$
$u_+(i) = \frac{\tau_{i+1}^{n-1} - \tau_i^{n-1}}{h}$
for $i = 2$ to m
$u_-(i) = \frac{\tau_i^{n-1} - \tau_{i-1}^{n-1}}{h}$
$\tau_1^n = \tau_1^{n-1} - kH(u_+(1))$
for $i = 2$ to $m - 1$
$\tau_i^n = \tau_i^{n-1} - k\hat{H}(u_+(i), u_-(i))$
$\tau_m^n = \tau_m^{n-1} - kH(u_-(m))$

where $\hat{H}$ is the Godunov flux defined earlier.

3. ESSENTIALLY NONOSCILLATORY (ENO) SCHEMES

It is often the case that higher order extensions of finite difference methods yield more accurate results for a smaller number of flops. Thus it is natural to pursue a higher order extension to the Godunov first order method, in hope of developing a faster, cheaper, more accurate solution scheme. We will follow the procedure introduced by (Osher and Shu, 1991).

The key is to replace the first order accurate finite differences in the Godunov flux by higher order approximations to the derivative. The general method of achieving this goal is to use a higher order interpolating polynomial for the approximation of the derivative. The problem is that solutions with discontinuities give rise it highly oscillatory interpolates. The idea of an essentially nonoscillatory scheme is to use the interpolate locally smoothest interpolate, and thus generally choose amongst the several possible difference approximations to each derivative, the one with the smallest absolute value. From the Taylor series expansion, we find that $D^+\tau - \frac{1}{2}\Delta x\tau_{xx}$ is a second order approximation to τ_x, where there are two choices for approximating τ_{xx}: $D^+D^+\tau$ and $D^+D^-\tau$. The ENO choice is as follows: $m(D^+D^+\tau, D^+D^-\tau)$ where:

$$m(x,y) = \begin{cases} |x| & \text{if } |x| < |y| \text{ and } xy > 0 \\ |y| & \text{if } |x| \geq |y| \text{ and } xy > 0 \\ 0 & \text{if } xy < 0 \end{cases} \tag{9}$$

or as a combination of mins and maxs:

$$m(x,y) = \min(\max(x,0), \max(y,0)) + \max(\min(x,0), min(y,0)) \tag{10}$$

Also note that $m(x,y,z) = m(x, m(y,z))$. Thus the ENO scheme tells us which finite difference to plug into the Godunov numerical Hamiltonian that was derived earlier. Explicitly we use $\hat{H}(u_+, u_-)$ where:

$$D^{2,+}\tau = D^+\tau - \frac{1}{2}\Delta x\, m(D^+D^+\tau, D^+D^-\tau) \tag{11}$$

$$D^{2,-}\tau = D^-\tau + \frac{1}{2}\Delta x\, m(D^+D^-\tau, D^-D^-\tau) \tag{12}$$

The combination of this scheme with the Godunov flux, gives us a ENO second order approximation to the Hamiltonian. In order to have a second order solution scheme for the Hamilton-Jacobi equation, we also need some sort of second order method for the z step. The method of choice is a Runge-Kutta style method. The one we use is as follows:

$$k_1 = -\Delta z \hat{H}(x_i, z_n, D^{2,+}\tau_i^n, D^{2,-}\tau_i^n) \tag{13}$$

$$k_2 = -\Delta z \hat{H}(x_i, z_n + \Delta z, D^{2,+}(\tau_i^n + k_1), D^{2,-}(\tau_i^n + k_1)) \tag{14}$$

$$\tau_i^{n+1} = \tau_i^n + \frac{1}{2}k_1 + \frac{1}{2}k_2 \tag{15}$$

The Runge-Kutta schemes used in this method are derived in the usual way from quadrature methods of numerical integration. This combination of the Runge-Kutta z step and the second order numerical Godunov flux results in a 2nd order solution scheme to the H-J equation.

In order to achieve third order accuracy, two adjustments must be made. First, the finite difference approximations in the numerical Hamiltonian must all be third order accurate. Second the z step method must be third order accurate as well. The finite differences become:

$$\begin{aligned} D^{3,+}\tau &= D^+\tau - \frac{1}{2}\Delta x\, m(D^+D^+\tau - \Delta x D^+D^+D^+\tau, D^+D^-\tau) \\ &\quad -\frac{1}{6}(\Delta x)^2\, m(D^+D^+D^+\tau, D^+D^-D^+\tau, D^-D^-D^+\tau) \end{aligned} \tag{16}$$

$$\begin{aligned} D^{3,-}\tau &= D^{-}\tau + \frac{1}{2}\Delta x\, m(D^{+}D^{-}\tau, D^{-}D^{-}\tau + \Delta x D^{-}D^{-}D^{-}\tau) \\ &\quad -\frac{1}{6}(\Delta x)^2\, m(D^{+}D^{+}D^{-}\tau, D^{+}D^{-}D^{-}\tau, D^{-}D^{-}D^{-}\tau) \end{aligned} \tag{17}$$

This results in a third order accurate numerical ENO Hamiltonian. The 3rd order RK scheme is as follows:

$$k_1 = -\Delta z \hat{H}(x_i, z_n, D^{3,+}\tau_i^n, D^{3,-}\tau_i^n) \tag{18}$$

$$k_2 = -\Delta z \hat{H}(x_i, z_n + \Delta z, D^{3,+}(\tau_i^n + k_1), D^{3,-}(\tau_i^n + k_1)) \tag{19}$$

$$k_3 = -\Delta z \hat{H}(x_i, z_n + \frac{1}{2}\Delta z, D^{3,+}(\tau_i^n + \frac{1}{4}(k_1 + k_2)), D^{3,-}(\tau_i^n + \frac{1}{4}(k_1 + k_2)) \tag{20}$$

$$\tau_i^{n+1} = \tau_i^n + \frac{1}{6}k_1 + \frac{4}{6}k_3 + \frac{1}{6}k_2 \tag{21}$$

Combining these two ideas yields a third order method for computing solutions to the Hamilton-Jacobi equation.

4. PRACTICALITIES

This method, like most explicit finite difference methods, is conditionally stable. The choices of Δz and Δx determine whether the solution is stable. The CFL condition for these finite difference methods is:

$$\Delta z < \frac{\Delta x}{H'(u)} \tag{22}$$

where u is τ_x or some approximation thereof. Some analysis of the eikonal equation shows that $H'(u)$, for any particular ray, is $\tan(\theta)$ where θ is the angle that the velocity of the ray makes with the z axis. Thus if we simply determine the maximum θ of any ray in the problem, we can easily compute the proper Δz. Unfortunately, determining this maximum angle is rather difficult. Not only is it difficult, but the maximum angle may be $\frac{\pi}{2}$, which would require $\Delta z = 0$, and make the computation impossible. An alternative approach is to declare that the rays will all have velocity angles less than some $\theta_{\max}$(ie. compute times accurately only along rays that obey this restriction. In order to enforce this restriction, we modify the eikonal equation as follows:

$$\frac{\partial \tau}{\partial z} = \sqrt{\max\left(s^2 - \left(\frac{\partial \tau}{\partial x}\right)^2, s^2 \sin^2\theta_{\max}\right)} = -H_{\text{mod}} \tag{23}$$

In this case, we have:

$$H'_{\text{mod}}(u) = \begin{cases} \frac{\tau_x}{H(u)} & \text{if } \tau_x < s\cos(\theta_{\max}) \\ 0 & \text{if } \tau_x > s\cos(\theta_{\max}) \end{cases} \tag{24}$$

Note that $H'_{\text{mod}} < \tan\theta_{\max}$ so long as

$$\Delta z < \frac{\Delta x}{\tan\theta_{\max}} \tag{25}$$

This modified eikonal equation has exactly the same solution as the original at points connected to the initial condition by a unique ray whose velocity vector makes an angle less than $\theta_{\max}$ with the z axis at all points along the ray. The modification simply prevents the rays from having an angle larger than $\theta_{\max}$. The larger the allowable angle we choose, the smaller the proper z step becomes, as seen in the above formula.

5. ACCURACY TESTS

The methods were tested on several different slowness fields. The first test is a constant slowness field ($s = 1$), with a source at (0,0) and a domain of $(-0.5, 0.5) \times (0.0, 2.0)$ with a first arrival times computed at depth $z = 2.0$. The initial conditions were analytically computed at depth $z = 0.2$ and advance from that depth to the target depth. The second test is a linear slowness field, ($s = 1+0.5z$) on $(-0.5, 0.5) \times (0.2, 2.0)$, with initial conditions generated by first arrival times from a source at the origin to the depth $z = 0.2$, assuming a constant slowness field of $s = 1$ on $(-0.5, 0.5) \times (0.0, 0.2)$. Again the travel times were computed to the depth $z = 2.0$. The third test case is a sinusoidal slowness field with slowness:

$$s(x, z) = (1 + 0.2\sin(0.5\pi z)\sin(3\pi x))^{-1}$$

on the domain $(-0.6, 0.6) \times (0.2, 2.0)$. This test example used the same initial conditions as the linear slowness test and times were computed to the same depth.

The error analysis for the constant slowness field was done by comparing against the analytical soltution to the eikonal equation. In the other cases, the results were compared against the results computed using an extremely fine grid. The computations where all done in single precision and all the travel times computed are order of 1. Thus the absolute errors given may be considered as relative errors. It should be noted that all the errors for the finest test grids are around the order of round-off and in each case, the third order method converges faster than the second order method. Interestingly enough, the second order method is actually slightly more accurate on the coarsest grid, in every case. The tables below present the following information: error 2 and error 3 are the maximum differences between the computed results and the "exact" results for the second order method and third order method, respectively. Rx is the approximate order of convergence, defined by:

$$\frac{\ln\left(\frac{\text{errorx(2h)}}{\text{errorx(h)}}\right)}{\ln 2}$$

where x is the order of the method used.

Accuracy tests on constant slowness field:

h	error 2	R2	error 3	R3
0.1	.0017		.0020	
0.05	.000358	2.24	.000235	3.09
0.025	.0000846	2.08	.0000327	2.85
0.0125	.0000212	2.00	.0000129	1.34

Accuracy tests on linear slowness field:

h	error 2	R2	error 3	R3
0.1	.0021		.0023	
0.05	.00049	2.10	.00022	3.39
0.025	.00009	2.44	.00002	3.46
0.0125	.00001	3.17	.00002	0.00

Accuracy tests on a sinusoidal slowness field:

h	error 2	R2	error 3	R3
0.1	.0070		.0102	
0.05	.0030	1.22	.0021	2.28
0.025	.0004	2.91	.00012	4.13
0.0125	.00008	2.32	.00004	1.59

6. CONCLUSION

The first arrival time in a slowness field may be computed on some domains by solving the eikonal equation as an evolution equation in z. One effective finite difference scheme for solving this equation uses the Godunov flux, whose upwind property makes it particularly attractive. The higher order ENO extensions of this scheme are more accurate and efficient, giving us the capability to achieve accurate results on larger computational problems than would otherwise be possible.

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