An Overview of Numerical Methods for Maxwell's Equations and Ground-penetrating Radar

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ABSTRACT

The literature on numerical methods for Maxwell's equations and ground-penetrating radar (GPR) includes dicussion of finite difference, finite element, finite volume, asympotitics, pseudospectral methods. We propose further investigation of finite difference schemes using 1) nonlinear flux limiters to control dispersion and 2) special stencils to maintain accuracy at coefficient discountinuities.

INTRODUCTION

Ground-penetrating radar (GPR) records high-frequency electromagnetic waves that have reflected from subsurface contrasts in dielectric constants. In low-loss media, ground-penetrating radar, like seismic geophysical techniques, is capable of producing high-resolution images of the shallow subsurface. GPR has been applied in a variety of studies including ground water investigations, buried hazardous waste identification, soil mapping, and engineering and geotechnical investigations. GPR has also been found to be a useful tool for non-destructive archaeological investigation.

Numerical solutions to Maxwell's equations play a critical role in the modeling and inversion of GPR techniques. The problems can be formulated in the time domain, or in the frequency domain with the governing partial differential equations in the integral or in the differential form. For a given formulation, a wide variety of numerical methods may be employed, such as finite difference, finite element, finite volume, asymptotics and pseudospectral methods. These contribute to a large collection of different numerical schemes on many applications. Some of the solutions to Maxwell's equations are directly related to GPR problems.

We propose several new approaches of numerical modeling for Maxwell's wave equations are proposed. New numerical schemes provide a way to avoid the Staggered-grid. We also provide finite difference formulas of second-order accuracy on grid points near the interface, where the material properties are discontinuous due to heterogeneity. The

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formulas, combined with other standard solvers on grid points away from the interface, should model Maxwell's wave accurately in the entire domain, including points near the interface, where the solutions are discontinuous or non-smooth.

NUMERICAL SOLUTIONS TO MAXWELL'S EQUATIONS

Finite Difference Methods

Electromagnetic fields are governed by Maxwell's equations. As with many other problems, numerical solutions to Maxwell's equations can be achieved practically using today's high-powered computers, making them easier to apply to electromagnetic problems, including GPR applications.

Among numerical schemes, finite difference methods are popular due to their simplicity and relatively low cost. K. S. Kunz and R. J. Luebbers [14] offer a good of the finite difference time domain (FDTD) method for electromagnetics. Kane S. Yee [25], one of the first to solve Maxwell's equations with FDTD used Leap-Frog discretization and the staggered-grid method to replace Maxwell's equations with a set of finite difference equations using Leap-Frog discretization, with staggered-grid method. Yee provided difference equations for the full Maxwell's equation in three dimensions and a numerical example for transverse magnetic (TM) waves.

Yee analyzed Maxwell's equations, in an isotropic media, of the form

$$\begin{aligned} \frac{\partial B}{\partial t} + \nabla \times E &= 0 \\ \frac{\partial D}{\partial t} - \nabla \times H &= J \\ B &= \mu H \\ D &= \epsilon E, \end{aligned} \tag{1}$$

where, E is electric field intensity, D is electric flux density, H is magnetic field intensity, B is magnetic flux density, J is the current density, μ is the permeability and ϵ is the permittivity. In order to discretize the derivatives, Yee employed the Leap-Frog method, which uses centered difference to discretize and evaluate values of different field components at *different* sets of grid points that do not intersect each other. Denoting a grid point of the space as

$$(i, j, k) = (i\Delta x, j\Delta y, k\Delta z) \tag{2}$$

and putting

$$F(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = F^n(i, j, k) \tag{3}$$

for any function of space and time allowed Yee to obtain a set of finite difference equations corresponding to Maxwell's equations. For example, for perfectly conducting boundary conditions, equation

$$-\frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \tag{4}$$

the first scalar equation of the first vector equation in (1), was replaced by the difference equation

$$\begin{aligned}
&\frac{1}{\Delta t}\left(B_x^{n+1/2}(i,j+\frac{1}{2},k+\frac{1}{2})-B_x^{n-1/2}(i,j+\frac{1}{2},k+\frac{1}{2})\right)\\
&\quad=\frac{1}{\Delta z}\left(E_y^n(i,j+\frac{1}{2},k+1)-E_y^n(i,j+\frac{1}{2},k)\right)\\
&\quad-\frac{1}{\Delta y}\left(E_z^n(i,j+1,k+\frac{1}{2})-E_z^n(i,j,k+\frac{1}{2})\right).
\end{aligned} \tag{5}$$

We can see that B_x is evaluated at the grid point of $(i,j+\frac{1}{2},k+\frac{1}{2})$ at time step $n+\frac{1}{2}$, E_y is evaluated at $(i,j+\frac{1}{2},k+1)$, and E_z is evaluated at $(i,j+1,k+\frac{1}{2})$, both at time step n. Yee continued forms of finite difference for other fields similarly by taking values at different sets of grid points. Because the Leap-frog method involves values at three different time level, Yee employed the staggered grid method which decouples the physical fields into disjoint computational fields, saving a lot of computational work.

Yee's method has been widely used in numerical computations of Maxwell's equations. Raymond J. Luebbers *et al.* [17, 18] used a new FDTD formulation for frequency-dependent materials ($(\mathrm{FD})^2\mathrm{TD}$), which allows the permittivity and conductivity to be frequency dependent, to compute wide-band reflection from an air-water interface over a frequency range where the complex permittivity of water varied significantly with frequency. They also showed that ($(\mathrm{FD})^2\mathrm{TD}$) can be applied to compute transient propagation in plasma that is characterized by a complex frequency-dependent permittivity. For a linear dispersive medium, the relationship between $D(t)$ and $E(t)$ is given by a convolution form, which is discretized for the computation.

Wang *et al.* [23] developed a finite-difference solution for three-dimensional (3-D) transient electromagnetic problems. They also chose to use a staggered-grid, discretizing, for example, the partial differential equation (4)

$$-\frac{\partial B_x}{\partial t}=\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}$$

by

$$\begin{aligned}
&B_x^{n+1/2}(i,j+1/2,k+1/2)\\
&\quad=B_x^{n-1/2}(i,j+1/2,k+1/2)-\frac{1}{2}(\Delta t_{n-1}+\Delta t_{n+1})\\
&\quad\times\left[\frac{E_z^n(i,j+1,k+1/2)-E_z^n(i,j,k+1/2)}{\Delta y_j}\right.\\
&\quad\left.-\frac{E_y^n(i,j+1/2,k+1)-E_y^n(i,j+1/2,k)}{\Delta z_k}\right]
\end{aligned}$$

Wang's values of different fields, though, are at different grid points from those of Yee's in (5). Wang's time stepping uses a modified version of the DuFort-Frankel method [6] which is explicit and always stable. Both conductivity and magnetic permeability can be functions of space, and the model geometry can be arbitrarily complicated. Wang's solution provides both electric and magnetic field responses throughout the earth.

Wang *et al.* [24] also developed a FDTD solution to Maxwell's equations that simulates electromagnetic wave propagation in 3-D media. They considered EM fields in a linear, isotropic medium, which is governed by Maxwell's equations

$$\begin{gathered} \frac{\partial B}{\partial t} + \nabla \times E = 0 \\ \varepsilon \frac{\partial E}{\partial t} + \sigma E - \nabla \times H = -J_i \\ \nabla \cdot B = 0 \end{gathered} \tag{6}$$

with the constitutive relation

$$B - \mu H = \mu_0 M_i, \tag{7}$$

where σ, ε and μ denote the electric conductivity, the electric permittivity and the magnetic permeability of the medium, respectively, and μ_0 is the free-space magnetic permeability. M_i and J_i denote the impressed magnetic and electric current density, respectively.

Wang's solution used Yee's staggered grid technique to sample the E and H fields. Wang designed a new optimized second-order difference scheme to approximate the spatial derivatives. By applying the second-order scheme to equations (6), for B_x and E_y components, and replacing the time derivatives in the equations by central differences, they obtained

$$\begin{aligned} &B_x^{n+1/2}(i, j+\tfrac{1}{2}, k+\tfrac{1}{2}) = B_x^{n-1/2}(i, j+\tfrac{1}{2}, k+\tfrac{1}{2}) \\ , \quad &-\frac{\Delta t}{\Delta} \sum_{l=-1}^{2} a_l \left[E_z^n(i, j+l, k+\frac{1}{2}) - E_y^n(i, j+\frac{1}{2}, k+l) \right] \end{aligned} \tag{8}$$

and

$$\begin{aligned} E_y^{n+1}(i, j+\frac{1}{2}, k) &= \frac{2\varepsilon_{i,j+1/2,k} - \Delta t \sigma_{i,j+1/2,k}}{2\varepsilon_{i,j+1/2,k} + \Delta t \sigma_{i,j+1/2,k}} E_y^n(i, j+\frac{1}{2}, k) \\ &+ \frac{2}{2\varepsilon_{i,j+\frac{1}{2},k} + \Delta t \sigma_{i,j+\frac{1}{2},k}} \frac{\Delta t}{\Delta} \sum_{l=-1}^{2} a_l \left[H_x^{n+1/2}(i, j+\frac{1}{2}, k+l-\frac{1}{2}) \right. \\ &\left. - H_z^{n+1/2}(i-l+\frac{1}{2}, j+\frac{1}{2}, k) \right], \end{aligned} \tag{9}$$

where Δ is uniform grid spacing, $a_{-1} = -a_2 = 0.05679$, and $a_0 = -a_1 = -1.17036$, as obtained by optimization that takes into account both formal truncation errors and wavenumber characteristics. $\varepsilon_{i,j+1/2,k}$ is the average permittivity evaluated at the grid node $(i, j+\frac{1}{2}, k)$, given by

$$\varepsilon_{i,j+1/2,k} = \frac{1}{4}(\varepsilon_{i-1,j,k-1} + \varepsilon_{i,j,k-1} + \varepsilon_{i-1,j,k} + \varepsilon_{i,j,k}).$$

$\sigma_{i,j+1/2,k}$ was computed similarly. Near the computational domain boundary, they discretized equations (6) with conventional second-order finite difference. Therefore, the boundary conditions remained the same as a conventional second-order finite difference scheme.

The second-order optimized scheme needed four discrete values to calculate a single derivative, as a fourth-order scheme does. However, the optimized scheme was accurate over a wider wavenumber range. The optimized scheme allowed coarse grids, despite imposing stricter limitations on the time step sizes, compared with the fourth-order scheme. Overall, optimization yielded more efficiency in terms of computation time and memory requirement than the fourth-order scheme. Notice that the temporal derivatives were approximated by second-order central differences throughout.

Mackie *et al.* [19] used a difference-equation algorithm that is based on the integral forms of Maxwell's equations rather than the differential forms. This formulation does not require approximating derivatives of earth properties or electromagnetic fields, as happens when using the second-order vector diffusion equation. Rather, Mackie's formulation requires one determine how to compute averages. Mackie's formulation thus resembles to a finite-volume method.

Stability is considered infrequently in the literature. Bergmann *et al.* [1] derived numerical stability and dispersion criteria for two common one-dimensional (1-D) finite-difference solutions to Maxwell's equations in an isotropic heterogeneous media. In one finite-difference scheme Bergmann used one-sided differences to approximate the conducting term; in the other the team employed centered differences. This is because stability is governed both by the well-known CFL criterion and, using a one-side difference conducting term, by a condition for the diffusion component of Maxwell's equations. Approximating the conducting term by one-sided or centered finite differences results in remarkably different stability and grid dispersion properties. Centered differences generally yield superior numerical results. Bergmann found that the grid dispersion curves for the centered O(2,4) (2nd-order time and 4th-order space) and O(2,2) schemes were identical to those of the non-conducting case and of the acoustic wave equation.

Finite-Element Methods

Finite-element methods have also been used to solve Maxwell's equations, particularly those that require flexible handling of complicated boundaries or interfaces. One disadvantage of finite element methods is that they have much higher computational costs compared with finite-difference methods. Unstructured triangulation, for example, can be costly in some boundary or interfaces, as can solving for mass matrix or stiff matrix, which will usually need quadrature of other means of approximations. These costs must be considered when applying finite-element methods.

Madsen *et al.* [20] have presented several different methods for solving Maxwell's equations in the time-domain by using irregular non-orthogonal grids. Several of their methods are finite-element derived methods, which result from their application of Galerkin approximation procedure.

Madsen's team started with Maxwell's curl equations of on an irregular two-dimensional (2-D) domain R that has a boundary denoted by S. They was also assumed that the domain R had been discretized into quadrilaterals (or triangles). Maxwell's curl equations are given by

$$\begin{aligned}\varepsilon\frac{\partial E}{\partial t} &= \nabla\times H \\ \mu\frac{\partial H}{\partial t} &= -\nabla\times E\end{aligned} \tag{10}$$

where, for the two-dimensional setting, E and H are $(E_x, E_y, 0)$ and $(0, 0, H_z)$ respectively. In order to apply a Galerkin procedure to obtain discrete approximations for Maxwell's equations, they assumed that the electric and magnetic fields could be approximated as piecewise polynomials

$$\begin{aligned}E &= \sum_{i=1}^{N_E} E_i(t)\phi_i(x) \\ H &= \sum_{i=1}^{N_H} H_i(t)\psi_i(x),\end{aligned} \tag{11}$$

where ϕ and ψ are basis functions associated with the ith node or element of the domain discretization. They obtained discrete Galerkin equations by multiplying the two curl equations in (10) by ϕ and ψ respectively, and then integrating the resulting equations over the domain R. The Galerkin equations are

$$\begin{aligned}\int_R \varepsilon\phi_i\frac{\partial E}{\partial t} &= \int_R \phi_i\nabla\times H, \quad i = 1, 2, \ldots, N_E \\ \int_R \mu\psi_i\frac{\partial H}{\partial t} &= -\int_R \psi_i\nabla\times E, \quad i = 1, 2, \ldots, N_H.\end{aligned} \tag{12}$$

Substituting the basis-function expansions (11) into (12) gives the following coupled set of ordinary differential equations:

$$\begin{aligned}M_E\frac{dE_x}{dt} &= C_y H_z \\ M_E\frac{dE_y}{dt} &= -C_x H_z \\ M_H\frac{dH_z}{dt} &= B_y E_x - B_x E_y,\end{aligned} \tag{13}$$

where the nodal amplitude coefficients E_x and E_y are vectors of dimension N_E and H_z is a vector of length N_H. The matrices M_E, M_H, C_x, C_y, B_x, and B_y are given by

$$\begin{aligned}[M_E]_{ij} &= \int \varepsilon\phi_i\phi_j, \quad [C_Y]_{ij} = \int \phi_i\partial_y\psi_j, \quad [B_Y]_{ij} = \int \psi_i\partial_y\phi_j, \\ [M_H]_{ij} &= \int \mu\psi_i\psi_j, \quad [C_x]_{ij} = \int \phi_i\partial_x\psi_j, \quad [B_x]_{ij} = \int \psi_i\partial_x\phi_j.\end{aligned}$$

In addition to these equations, Madsen's team specified exterior boundary conditions and discussed the choices of the basis functions ϕ_i and ψ_j. Different choices of polynomials of the same order (equal-order interpolation) and of differing order (mixed-interpolation) were used, effectively yielding different methods.

Another example of finite-element method was used by Unsworth *et al.* [22] to solve how a 2-D conductivity structure responde to excitation by a 3-D current source. This so called 2.5-D problem requires the fields to be Fourier transformed in the invariant conductivity direction and then expressed in a variational form. Unsworth's team considered Maxwell's equations

$$\nabla \cdot B = 0 \tag{14}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{15}$$

and

$$\nabla \times B - \mu_0 \sigma E = \mu_0 J_s \tag{16}$$

where σ is the electrical conductivity which is assumed to vary in two dimensions and μ_0 is the magnetic permeability of free space. For a point electromagnetic source, J_s is singular and difficult to represent accurately by a discrete formulation. Unsworth's team overcame this problem by separating the total electric and magnetic fields into their primary and secondary components. They first calculated the primary fields for a simple, 1-D conductivity structure, $\sigma_0(z)$. Equations (15) and (16) also govern primary fields, when $\sigma = \sigma(z)$ and when subtraction equations obtained for the secondary fields are

$$\nabla \times E^s = -\frac{\partial B^s}{\partial t} \tag{17}$$

$$\nabla \times B^s - \mu_0 \sigma E^s = \mu_0 \Delta \sigma E^p, \tag{18}$$

where $\Delta\sigma = \sigma - \sigma_0$ is the difference between the total 2-D conductivity and the 1-D background conductivity used to calculate the primary fields. Providing that $\Delta\sigma = 0$ at the source, the singular source term can be replaced by the nonsingular function $\Delta\sigma E^p$, which is distributed throughout the solution space.

A Cartesian coordinate system was defined with $\hat{z}$ vertically upwards and the conductivity invariant in the $\hat{x}$-direction. Since the conductivity varies in only two dimensions, Unsworth further simplified the solution by Fourier transforming each field component in the $\hat{x}$-direction, with the Fourier transform of each component defined as

$$\hat{F}(k_x, y, z) = \int_{-\infty}^{\infty} F(x, y, z) e^{ik_x x} dx. \tag{19}$$

Equations (17) and (18) can be written in Cartesian form as

$$\begin{aligned}
-ik_x \hat{E}^s_y - \partial_y \hat{E}^s_x - i\omega \hat{B}^s_z &= 0 \\
\partial_y \hat{B}^s_z - \partial_z \hat{B}^s_y - \mu_0 \sigma \hat{E}^s_x &= \mu_0 \Delta\sigma \Delta \hat{E}^p_x \\
\partial_z \hat{E}^s_x + ik_x \hat{E}^s_z - iw \hat{B}^s_y &= 0 \\
-ik_x \hat{B}^s_y - \partial_y \hat{B}^s_x - \mu_0 \sigma \hat{E}^s_z &= \mu_0 \Delta\sigma \hat{E}^p_z \\
\partial_y \hat{E}^s_z - \partial_z \hat{E}^s_y - i\omega \hat{B}^s_x &= 0 \\
\partial_z \hat{B}^s_x + ik_x \hat{B}^s_z - \mu_0 \sigma \hat{E} y^s &= \mu_0 \Delta\sigma \hat{E}^p_y
\end{aligned} \tag{20}$$

Equations (20) may be rearranged to yield coupled equations for $\hat{E}_x^s$ and $\hat{B}_x^s$, the secondary, along-strike fields in the Fourier transform domain

$$
\begin{aligned}
\nabla \cdot \left(\frac{\mu_0 \sigma \nabla \hat{E}_x^s}{\gamma^2}\right) - \mu_0 \sigma \nabla \hat{E}_x^s &= \mu_0 \Delta\sigma \hat{E}_x^p - ik_x \nabla \cdot \left(\frac{\Delta \hat{E}^p}{\gamma^2}\right) \\
&\quad + ik_x \left[\nabla \hat{B}_x^s \times \nabla \left(\frac{1}{\gamma^2}\right)\right] \cdot \hat{x} \\
\nabla \cdot \left(\frac{i\omega \nabla \hat{B}_x^s}{\gamma^2}\right) - i\omega \hat{B}_x^s &= i\omega\mu_0 \nabla \times \left(\frac{\Delta\sigma \hat{E}^p}{\gamma^2}\right) \cdot \hat{x} \\
&\quad + ik_x \left[\nabla \hat{E}_x^s \times \nabla \left(\frac{1}{\gamma^2}\right)\right] \cdot \hat{x},
\end{aligned} \tag{21}
$$

where $\gamma^2 = k_x^2 - i\omega\mu_0\sigma$, $\hat{x}$ is a unit vector in the x-direction, and $\nabla = (0, \partial_y, \partial_z)$. The coupling term is proportional to k_x so that at zero wavenumber the modes are independent. With a nonzero wavenumber, a minimum of two field components must be calculated at a point to represent the 3-D electromagnetic fields. In the zero-wavenumber case, one suffices. Equations (21) are of the form

$$
-\nabla \cdot (a(y,z)\nabla V) + b(y,z)V = f(y,z), \tag{22}
$$

which has a solution equivalent to that of minimizing

$$
\mathcal{L} = \int^{\Omega} L(V, \partial_y V, \partial_z V)\,dy\,dz, \tag{23}
$$

with

$$
L = \frac{a}{2}(\partial_y V)^2 + \frac{a}{2}(\partial_z V)^2 + \frac{bV^2}{2} + fV.
$$

The team then solved equation (23) with the finite-element method. When combined with an iterative solution for the along-strike electromagnetic fields, this Unsworth's method considerably reduces computation costs.

Finite-Volume Methods

While finite-volume methods are used infrequently, some approaches relate to finite-volume methods. Finite-volume techniques are usually derived by integrating the basic equations to be solved over grid elements and then forming discrete approximations for the resulting integral quantities. Madsen *et al.*, for example, developed a finite volume technique on irregular grids-quadrilateral grids. For each quadrilateral element, they defined the barycenter and constructed the dual grid by connecting barycenters with straight lines passing through each of the interior sides of the original grid.

Newman *et al.* formulated the 3-D transient electromagnetic scattering problem in frequency domain in terms of an electric field volume integral equation. They then Fourier transformed the 3-D responses to the time domain using techniques similar to finite-volume methods. For the formulated integral equations in terms of Green's functions, they approximated the integral by discretizing it in the summation of the layers. They then used the fields values at the center of cells to approximate their average values, ending up with algebraic linear equations.

Because finite-volume methods employ averaged values over certain domain by some approximation, they avoid approximating derivative values. They also have the flexibility to treat relatively complicated geometries. Computational cost are higher than those of finite difference methods, however. In addition the formulations are not as simple as finite-difference methods.

Asymoptotics Methods

Asymoptotics analysis can provide good insight into the qualitative and quantitative behaviors of certain problems. It is also a fine approximation method for certain parameter ranges. Roberts *et al.* [21] examined electromagnetic pulse propagation in anomalous dispersive media, using the Debye model as an example. They used asymptotic methods to derive short-pulse, long-pulse, short-time, and long-time approximations and amplitude rate-of-decay estimates. They considered the electric field incident on the half-space from the air side ($z < 0$) to be a plane pulse $E_y^{\mathrm{inc}}(x, z, t) = f(t - x \sin\phi_{\mathrm{inc}}/c - z\cos\phi_{\mathrm{inc}}/c)$ of duration T_p. Defining the time-like variable $\xi = t - x/v$, and changing coordinates $(x, z, t) \to (z, \xi)$ in the resulting one-dimensional system, they obtained a single third-order partial differential equation for the electric field $E = \tilde{E}_y(z, \xi) = E_y(0, z, \xi) = E_y(x, z, t)$

$$\partial_\xi(\partial_\xi - c_0\partial_z)(\partial_\xi + c_0\partial_z)E + \frac{\beta}{\tau}(\partial_\xi - c_1\partial_z)(\partial_\xi + c_1\partial_z)E = 0, \qquad z > 0, \tag{24}$$

where $c_0 = c/(\sqrt{\varepsilon_\infty}\cos\phi_{inc})$, $\beta = 1 + \Delta\varepsilon/(\varepsilon_\infty\cos^2\phi_{inc})$ and $c_1 = c_0/\sqrt{\beta}$. Their asymptotic analysis was conducted on the magnitude of the *dimensional* quantity β/τ. From experimentally obtained data typical of tissue/water media $\tau = O(10^{12})$ sec and $\beta = O(10)$, β/τ is large, or equavlently, τ/β is small, introducing a small parameter. By defining $\epsilon = \tau/\beta$ equation (24) yielded

$$(\partial_\xi^2 - c_0^2\partial_z^2)(\epsilon\partial_\xi E) + (\partial_\xi^2 - c_1^2\partial_z^2)E = 0, \qquad z > 0, \tag{25}$$

where ϵ has units of time, and the operator $\epsilon\partial_\xi$ is dimensionless.

With (25) Roberts' team was able to describe qualitatively how pulse propagation is governed by the speeds c_0 and c_1 in mutually exclusive spatial regions. Due to the presence of $\epsilon\partial_\xi$, the two wave equations in (25) will not balance each other throughout the domain $z > 0$. They called those disturbances described by the principal part of the operator in (25) in regions where the action of $\epsilon\partial_\xi$ is large high-order waves, while calling those described by the remaining operator (in regions where the action of $\epsilon\partial_\xi$ is small) low-order waves. According to this definition, the action of $\epsilon\partial_\xi$ is larger for incident pulses with rapid variation until such variation is smoothed out.

Roberts' team proceeded to describe the high-order wave propagation by considering a general $g(\xi)$, not necessarily continuous at $\xi = 0$. The possible discontinuity in g was used to model rapid variations in the incident pulse. Then they set everywhere in (24) $\partial_\xi \sim -c_0\partial_z$ except in the operator $\partial_\xi + c_0\partial_z$, since it expresses the propagation of these waves into the region $z > 0$. They found that the high-order waves are modeled by

$$(\partial_\xi + c_0\partial_z)E + \frac{\beta}{\tau}\frac{c_0^2 - c_1^2}{2c_0^2}E = 0; \qquad z < c_0\tau, \tag{26}$$

subject to the boundary condition $E(0,\xi) = g(\xi)$. The solution of (26) in the region $\xi \to z/c_0$ from above was

$$E = g(0^+)\exp\left[-\left(\frac{\varepsilon_s/\varepsilon_\infty - 1}{2\cos^2\phi_{\text{inc}}}\right)\left(\frac{z}{c_0\tau}\right)\right]. \tag{27}$$

They obtained qualitative properties of the high-order wave from (27). They then performed similar analyses on the low-order wave. The asymptotic solutions they obtained for the high-order wave and low-order wave were *not* uniformly valid in depth (equivalently, time), though they are valid in disjoint spatial (time) regions.

Asymptotics provide good insight into the qualitative behavior of the solutions, and they may give ideal approximations quantitatively without involving much computation. However, they are usually limited to some range of the parameters, and their validity is often not uniform.

Pseudospectral Methods

Pseudospectral algorithms are also quite popular among some researchers. Casper *et al.* [5] developed a pseudospectral forward-modeling algorithm for ground-penetrating radar based on explicitly solving a 2-D lossy electromagnetic wave equation. The method accommodated complex soil structures with heterogeneous spatial distributions of both wave velocity and electrical conductivity. The Casper algorithm used a Gaussian line source with uniform directivity, along with conductive buffer regions around the soil model to approximate absorbing boundary conditions. The wave equations for the problem were separated into three vector components, one of which, for example, was

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 E_y}{\partial t^2} + \mu_0\sigma\frac{\partial E_y}{\partial t} \tag{28}$$

The approach to solve equation (28) was analogous to pseudospectral forward-modeling methods developed for seismic waves [8, 13]. Casper's team expanded the second-order time derivative in equation (28) with a centered difference and the first-order time derivative with a forward difference. They then calculated the spatial derivatives by multiplying by the wavenumbers in the frequency domain. This creates an expression for $E(x,z,t+\Delta t)$ in terms of $E(x,z,t)$ and $E(x,z,t-\Delta t)$ as

$$\begin{aligned} E(x,z,t+\Delta t) &= C_1[FFT^-_{x,z}(k_x^2 + k_z^2)E(k_x,k_z,t)] \\ &\quad + C_2E(x,z,t) + C_3E(x,z,t-\Delta t), \end{aligned} \tag{29}$$

where $E \equiv E_y$, and

$$C_1 = \left[\frac{1}{[v(x,z)]^2\Delta t^2} + \frac{\mu_0\sigma(x,z)}{\Delta t}\right]^{-1},$$
$$C_2 = C_1\left[\frac{2}{[v(x,z)]^2\Delta t^2} + \frac{\mu_0\sigma(x,z)}{\Delta t}\right],$$
$$C_3 = C_1\left[\frac{1}{[v(x,z)]^2\Delta t^2}\right].$$

Equation (29) then can be used to update the solution by one time step. Notice that this equation needs initial conditions at the first two time steps. Casper's team determined these initial conditions by defining an electric field pattern at $t = 0$ and using a Taylar series expansion to evaluate the field pattern at $t = \Delta t$. Three soil models were used to illustrate different aspects of radar-wave propagation. The team found that higher frequency GPR would produce a sharper wavelet and could map soil layering structures with high resolution. In a conductive soil, however, higher frequencies attenuate more rapidly and the radar could not detect deeper layers.

Using Fourier transformation to solve the spatial derivatives increases the numerical accuracy of the simulation. However, it imposes periodic boundary conditions at the edges of the computational grid, limiting its applicability.

Other Approaches

In overview paper of this scope, it is impossible to list every type of Maxwell equation solution employed. Many schemes, for example, combine techniques. Karlsson *et al.* [12] used a time-domain technique to analyze the wave propagation of electromagnetic pulses in media that are periodic in the direction perpendicular to the direction of propagation. Karlsson's team used a Fourier expansion in the transverse spatial direction and a wave-splitting technique to reduce the wave equation to a system of 1-D first-order time-dependent hyperbolic equations They started with the internal electric field ($z > 0$) wave equation

$$\nabla^2 E(x,z,t) - \frac{1}{c^2(x)}\partial_t^2 E(x,z,t) = 0. \tag{30}$$

Since the geometry they discussed was periodic in x with period p, they expended c^{-2} and E as a Fourier series in x. The functions

$$\exp(i\lambda_n x), \quad n = \ldots, -2, -1, 0, 1, 2, \ldots \tag{31}$$

with corresponding values $\lambda_n = 2n\pi/p$, form a basis in the interval $-p/2 < x < p/2$. If

$$\begin{aligned} c^{-2}(x) &= \sum_{n=-\infty}^{\infty} a_n \exp(i\lambda_n x) \\ E(x,z,t) &= \sum_{n=-\infty}^{\infty} \psi_n(x,t)\exp(i\lambda_n x), \end{aligned} \tag{32}$$

where

$$a_n = \frac{1}{p}\int_{-p/2}^{p/2} c^{-2}(x)\exp(i\lambda_n x)dx$$
$$\psi_n(z,t) = \frac{1}{p}\int_{-p/2}^{p/2} E(x,z,t)\exp(i\lambda_n x)dx, \tag{33}$$

the wave equation is converted to

$$\partial_z^2\Psi(z,t) - T^2\partial_t^2\Psi(z,t) - \Lambda^2\Psi(z,t) = 0, \tag{34}$$

where $\Psi(z,t)$ is the vector of $\psi(z,t)$ coefficients, Λ^2 is the diagonal matrix

$$\Lambda^2_{nm} = \lambda_n^2\delta_{nm}, \tag{35}$$

where δ_{nm} is the Kronecker delta, and T^2 is the Toeplitz matrix

$$T^2_{nm} = a_{m-n}. \tag{36}$$

Karlsson's team showed that T^2 is positive definite (therefore had this notation). Wave equation (34) is a 1-D hyperbolic scalar equation, which can be more easily handled, for example, by wave splitting or diagnoalization techniques. By introducing a new variable, $\partial_z\Psi$, to get a vector variable $\begin{pmatrix}\Psi \\ \partial_z\Psi\end{pmatrix}$, Karlsson's team converted equation (34) into a system of first-order equations. Using a similar (matrix) transformation, they diagonalized and solved the system of first-order equations. The method is especially useful for the propagating broad pulses.

GPR APPLICATIONS

During the last decade, ground-penetrating radar (GPR) has become an important tool for exploring the shallow subsurface. This high-frequency EM technique is conceptually similar to seismic reflection methods [3] and, therefore, many of the algorithms used in oil exploration can be applied, with minor modification, to the interpretation of georadargrams. In particular, forward modeling in heterogeneous media is essential to validating GPR. Numerical simulation of GPR has enjoyed fast-growth recently. However, GPR wave modeling in the literature is, in general, 1-D, only very recently, has Goodman [9] proposed a 2-D wave simulation method based on ray-tracing techniques, but his modeling suffers the disadvantages of asymptotic ray methods.

Other successful techniques used in the treatment of seismic problems can also be applied or extended to GPR applications. Grasmueck [10] developed and tested 3-D GRP (georadar) techniques suitable for geological engineering applications. Grasmueck conducted initial experiments on the floor of a quarry in southern Switzerland from which ornamental gneissic rock is extracted. During a brief two-day period, he recorded constant-offset georadar data over a 650 m^2 area with a grid cell size of 0.1 m $\times$ 0.2 m. He then estimated georadar velocities from the results of expanding spread surveys. All georadar data and associated geometry files were recorded automatically in seismic industry formats. The experimental georadar data set was processed, image-enhanced, and interpreted using 3-D seismic reflection software operating on a workstation. Grasmueck concluded that 3-D georadar methods have the potential to resolve a wide range of engineering and environmental problems.

The common midpoint (CMP) processing technique has been shown to be effective in improving the results of GPR profiling. When radar data are collected with the CMP multioffset geometry, stacking increases the signal-to-noise ratio of subsurface radar reflections and results in an improved subsurface image. An important aspect of CMP processing is normal-moveout velocity analysis. Greaves [11] showed that variation in radar velocity can be quite significant and that the stacked profile improves as the number of velocity analysis locations is increased.

Goodman [9] developed forward modeling of ground penetration radar using exact ray-tracing techniques. Structural boundaries for a ground model are incorporated via a discrete grid with interfaces described by splines and polynomials, and in the case of special structures such as circular objects, the boundaries are given in terms of their functional formula.

Marckie [19] developed an algorithm for computing the magnetotelluric response of 3-D earth models. His difference equation algorithm on a staggered grid is based on the integral forms of Maxwell's equations rather than the differential forms. The finite-difference equations are first-order equations, so this formulation does not require approximating derivatives of earth properties or electromagnetic fields. Rather, the method computes their averages. To solve the 3-D system of equations, Marckie propagated an impedance matrix that relates all the horizontal H fields in that same layer up through the earth model.

Carcione [4] presented two engineering applications of a novel forward-modeling algorithm for the simulation of georadargrams. His method was based on solving the time-domain Maxwell's equations using a direct grid method. Carcione modified the standard TM differential equations in order to include out-of-phase conduction currents, which contribute to the effective permittivity, and dielectric attenuation processes, such as free and bound water relaxation. The convolution integral introduced by the relaxation formulation is circumvented by defining new *hidden* field variables that correspond to dielectric relaxation processes. Carcione solved the equations numerically by using the Fourier-pseudospectral operator for computing the spatial derivatives and a new time-splitting integration algorithm that obviates the stiffness of the differential equations.

Carcione's two applications illustrate the potential of the modeling algorithm. In the first simulation, he computed a radargram corresponding to three cylindrical objects. Since the objects were small compared with the dominant wavelength of the electromagnetic pulse, they acted as secondary sources, according to Huygen's principle, and generated typical hyperbolic responses. Despite the small size of the cylinders, the effects of the material composition was observed in the amplitude and phase of the reflected wave fields.

The second model represented an old concrete road overlying a cavity. The different layers and the cavity, which was 30 cm deep, was resolved with a 850 MHz transmitting antenna but not with a 500 MHz antenna. Carcione's model also analyzed the influence of the water relaxation phenomenon by considering the cavity filled with slat water. Its presence strongly attenuate the reflections generated at the bottom of the cavity and at the layer below.

APPROACHES WE PROPOSE

High-Resolution Flux-Limiter Method

Finite-difference methods are popular because of their simplicity and efficiency. However, they usually generate numerical dispersion, as well as nonphysical oscillations, especially for second-order methods. We propose employing conservation laws and flux limiters to reduce numerical dispersion and eliminate nonphysical oscillations. Originally developed for fluid dynamics and later used in some other applications [15, 16], these methods have proven to be effective in the modeling of acoustic and elastic waves [26, 27, 28], and we believe they could provide similar advantages for solutions to Maxwell's wave equations.

In this approach, a hyperbolic Equation of the form

$$U_t + f_x(U) + g_y(U) = 0 \tag{37}$$

in two spatial dimensions could be solved with the high-resolution flux-limter method. Where U is some vector variable, depending on the application, $f(U)$ and $g(U)$ are vector functions of U. The waves will be decomposed based on Godunov's method. On each time step, waves are split based on their local characteristics and for each wave flux limiter can be employed optionally.

Flux-corrected transport, another technique originally used in fluid dynamics, could also be a good candidate for modeling Maxwell's wave equations. This technique has been employed to eliminate numerical dispersion in finite-difference modeling and migration for acoustic and elastic waves [7]. Diffusive fluxes were introduced on top of conventional finite-difference methods and then anti-difussion terms were selectively applied.

Immersed-Interface Method

The immersed-interface method has been used in the modeling of acoustic and elastic waves in heterogeneous media [26, 27, 28]. These studies analyzed linear hyperbolic systems of equations of acoustics and elasticity. Typical equations were of the form

$$U_t + AU_x + BU_y = 0, \tag{38}$$

in two spatial dimensions, where U was a vector variable consisting of velocity components and, for acoustics, the pressure, or for elasticity, the stress components, A and B are matrices of compatible rank whose elements were functions of the material properties, such as the density, compression bulk modulus (for acoustics), or Lamé constants (for elasticity). These material properties were discontinuous for heterogeneous media across some interface. We solved equation (38) with some solver for homogeneous media, and at points near the interface, a special finite difference formula of the form

$$U^{n+1}(i,j) = U^n(i,j) + \frac{\Delta t}{\Delta} \sum_{l=1}^{6} \Gamma_l(i,j) U_l^n(i,j). \tag{39}$$

was used. Where $\Gamma_l(i,j)$ $(l = 1, 2, \ldots, 6)$ were matrices of the same rank as that of matrix A, $U_l^n(i,j)$ $(l = 1, 2, \ldots, 6)$ were values at grid points of the standard 5-point stencil together with one additional corner point. Our purpose was to solve for the Γ's, advancing the solution by one time-step. With this method, the waves should be modeled to second-order accuracy even on grid points near the interface. The advantages of the immersed interface method include low computation cost, high accuracy even near the interface, and flexibility (at points away from the interface, any standard solver can be used).

The immersed-interface method could be applied to Maxwell's equations. For example, in cylindrical coordinates one can decompose any electromagnetic field in two spatial dimensions into "transverse electric" and "transverse magnetic" fields if ε and μ are constants. The two modes of eloectromagnetic waves are characterized by
1) transverse electric wave (TE)

$$\begin{aligned} & H_x = H_y = 0, \qquad E_z = 0, \\ & -\mu \frac{\partial H_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}, \\ & \varepsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y}, \qquad \varepsilon \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}. \end{aligned} \tag{40}$$

and
2) transverse magnetic wave (TM)

$$\begin{aligned} & E_x = E_y = 0, \qquad H_z = 0, \\ & \varepsilon \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}, \\ & \mu \frac{\partial H_x}{\partial t} = -\frac{\partial E_z}{\partial y}, \qquad \mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x}. \end{aligned} \tag{41}$$

Both equations (40) and (41) are of the form of (38), and similar techniques could be employed. The main challenge would be to determine the jump conditions at the interface for the vector variables and their derivatives, and thus obtain special formulas for grid points near the interface. The immersed-interface method has been derived in the form of finite-difference method, however, it has been successfully merged with a finite-volume based method for problems of acoustics and elasticity [26, 27, 28].

CONCLUSION

This paper has reviewed numerical simulations of the Maxwell's equations, including GPR problems. The specific application will very much affect and even dictact the choice of the simulation. Significant number of the schemes use FDTD because of its simplicity and efficiency, even though these approaches are limited to simple geometries. Finite-element methods excel in dealing with complicated boundaries/interfaces, but are expensive computationally. Finite-volume methods can avoid evaluating derivatives of earth properties or electric or magnetic fields, and are not limited by the complexity of interfaces/boundaries. But as with finite-element methods, finite-volume methods are expensive computationally. Asymptotics work well for some range of parameters; giving simple, quick approximation and good qualitative analysis. But these strategies are not uniformly valid. Pseudospectral methods usually give good accuracy of the simulation, but often impose periodic boundary conditions at the edge of the computational grid, limiting their applicability.

We propose the high-resolution flux-limiter method to model Maxwell's waves. Flux limiters will help reduce numerical dispersion and eliminate nonphysical oscillations. We also suggest the immersed-interface method to solve Maxwell's wave equations in heterogeneous materials. This could contribute a method of high accuracy and low computational cost, as is shown in problems of acoustics and elasticity.

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