A Fourth Order Method for Acoustics Waves in Heterogeneous Media

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ABSTRACT

We present a fourth order finite difference method for the simulation of acoustic waves in heterogeneous media. We derive the numerical jump conditions from the physical jump conditins at the interface of the heterogeneous media, and build the numerical jump conditions into a special stencil to obtain fourth order methods.

key words: *fourth order method, wave propagation, heterogeneous media, immersed interface method, finite difference method.*

1. INTRODUCTION

High accuracy simulations of wave propagation in heterogeneous media draw considerable interest in many applications. Given the computational resource and the availability of the data, fourth order methods are often highly desired, sometimes even required for practical reasons. However, the presence of the interface of the discontinuous makes the implementation difficult.

Many researchers have worked on acoustic wave equations and other wave equations in heterogeneous media (*e.g.* F. B. Jensen *et al.*, G. L Wojcik *et al.*). To model arbitrarily oriented interfaces, one either accepts "stair-step" representations or uses schemes for averaging discontinuous coefficients to obtain intermediate grid values. This naturally limits the accuracy to first order. Some other methods use unstructured grids to handle the interface, usually having high computational cost. We list some of the works which cover a broad range of different techniques by R. Madariaga, P. P. Delsanto *et al.*, D. L. Brown, . J. LeVeque, M. Bouchon *et al.*, John Lysmer *et al.*, and Todd Dupont. First order accuracy were achieved in the above work.

Card and Allen obtained second order accuracy, at the expense of sacrificing the ability to use explicit time-stepping. Zhang and LeVeque used the immersed interface

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method to obtain second order accuracy on complicated interfaces.

In this paper we present a method that gives fourth order accuracy at all grid points, including those near the interface. This is an extension of the second order method for wave equations in heterogeneous media presented by Zhang and LeVeque. At grid points away from the interface, any standard finite difference method can be used. This flexibility will essentially maintain the efficiency of the computation on the majority of computational grids. At grid points near the interface, numerical jump conditions (derived from the physical jump conditions together with the wave equations) are built into a special stencil. Once the numerical jump conditions are derived, the coefficients used in the special finite difference method are computed as a preprocessor. These coefficients then can be used in every time step.

This work is presented in acoustics. However, the fourth order method could be extended to elasticity, and in some cases, to Maxwell's wave equations.

2. ONE DIMENSIONAL ACOUSTIC WAVE EQUATIONS

We will start with one dimensional acoustic wave equations. Let us consider

$$\begin{aligned} \rho u_t + p_x &= 0 \\ p_t + \kappa u_x &= 0, \end{aligned} \tag{1}$$

where ρ is the density of the medium, u is the velocity, p is the pressure and κ is the compression (bulk) modulus. For the case of a heterogeneous medium, ρ and/or κ can be discontinuous at the interface between different media. Jump conditions are then given by

$$[u] = 0, \quad [p] = 0 \tag{2}$$

at the interface $x = \alpha$, i.e., both the velocity and the pressure are continuous. The coefficients are discontinuous across the interface $x = \alpha$:

$$(\rho,\ \kappa) = \begin{cases} (\rho^-,\ \kappa^-) & \text{if } x < \alpha, \\ (\rho^+,\ \kappa^+) & \text{if } x > \alpha. \end{cases}$$

The spacial coordinate x represents the depth underneath the surface of the Earth (z is often used). The discontinuities in the coefficients occur at interfaces in a vertically stratified Earth.

Let

$$U = \begin{pmatrix} u(x,t) \\ p(x,t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & (1/\rho) \\ \kappa & 0 \end{pmatrix}.$$

This yields a system of first order equations

$$U_t + AU_x = 0. \tag{3}$$

In the region where A is smooth, any standard solver can be used. We can use, for example, a fourth order method using a 5-point stencil:

$$
\begin{aligned}
u_j^{n+1} \;=\; u_j & -\frac{1}{12}\frac{\nu}{\rho_j}(p_{j-2}-8p_{j-1}+8p_{j+1}-p_{j+2}) \\
& +\frac{1}{24}\nu^2 c_j^2(-u_{j-2}+16u_{j-1}-30u_j+16u_{j+1}-u_{j+2}) \\
& -\frac{1}{12}\frac{\nu^3 c_j^2}{\rho_j}(-p_{j-2}+2p_{j-1}-2p_{j+1}+p_{j+2}) \\
& +\frac{1}{24}\nu^4 c_j^4(u_{j-2}-4u_{j-1}+6u_j-4u_{j+1}+u_{j+2}) \\
p_j^{n+1} \;=\; p_j & -\frac{1}{12}\nu\kappa_j(u_{j-2}-8u_{j-1}+8u_{j+1}-u_{j+2}) \\
& +\frac{1}{24}\nu^2 c_j^2(-p_{j-2}+16p_{j-1}-30p_j+16p_{j+1}-p_{j+2}) \\
& -\frac{1}{12}\nu^3 c_j^2\kappa_j(-u_{j-2}+2u_{j-1}-2u_{j+1}+u_{j+2}) \\
& +\frac{1}{24}\nu^4 c_j^4(p_{j-2}-4p_{j-1}+6p_j-4p_{j+1}+p_{j+2}),
\end{aligned} \tag{4}
$$

where u_j^{n+1} (or p) is the numerical approximation to $u(x_j, t_{n+1})$ (or p). u and p terms on the right hand side of the equations are at time n, while the superscript n is omitted for simplicity. $\nu = k/h$, where $k = \Delta t$, the time step, and $h = \Delta x$, the grid size; and ρ_j, κ_j and c_j are ρ, κ and $c = \sqrt{\kappa/\rho}$ (wave speed) values at x_j. This formula can be used at all points $j \neq J-1,\ J,\ J+1,\ J+2$, with J such that

$$x_J \leq \alpha < x_{J+1}.$$

i.e., α is the interface location in between of points x_J and x_{J+1}.

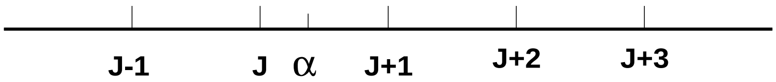

FIG. 1. 1-D interface

The points of the stencil are on the same side of the interface are called *regular points*. The points of the stencil that are on different sides of the interface are called *irregular points*. At the four points $J-1$, J, $J+1$ and $J+2$, which are irregular points, we can no longer use the fourth order scheme because of the discontinuity at the interface. Special formulas are needed at these points.

To start, we rewrite equation (4) in matrix form as:

$$U_j^{n+1} = U_j + \sum_{l=1}^{5} \Gamma_{j,l} U_{j+l-3}, \quad j \neq J-1, J, J+1, J+2 \tag{5}$$

where the coefficients $\Gamma_{j,l}$ $(l = 1, 2, 3, 4, 5)$ are 2×2 matrices and explicit expressions can easily be found. For example,

$$\Gamma_{j,1} = \frac{1}{24}\nu(\nu^2 c_j^2 - 1)\begin{pmatrix} \nu c_j^2 & 2/\rho_j \\ \nu c_j^2 & 2\kappa_j \end{pmatrix}.$$

$\Gamma_{j,2}$, $\Gamma_{j,3}$, $\Gamma_{j,4}$, $\Gamma_{j,5}$, are equally simple. For the four irregular points near the interface, we hope to keep the same form as equation (4). We will work on point J to demonstrate the procedure. The formula is given by

$$U_J^{n+1} = U_J + \sum_{l=1}^{5} \Gamma_{J,l} U_{J+l-3}, \tag{6}$$

We want to choose the Γ's so that fourth order accuracy is maintained. Consider the local truncation error at J:

$$\begin{aligned} L &= \frac{1}{k}\sum_{l=1}^{5}\Gamma_{J,1}U_{J-3+l} + \Big(AU_x - \frac{1}{2}kA^2U_{xx} \\ &\quad + \frac{1}{6}k^2A^3U_{xxx} - \frac{1}{24}k^3A^4U_{xxxx}\Big)_J + \mathrm{O}(k^4). \end{aligned} \tag{7}$$

This can be simplified using the fact that $A^2 = c^2 I$, where I is a 2×2 identity matrix,

$$\begin{aligned} L &= \frac{1}{k}\sum_{l=1}^{5}\Gamma_{J,1}U_{J-3+l} + \Big(AU_x - \frac{1}{2}kc^2U_{xx} \\ &\quad + \frac{1}{6}k^2c^2AU_{xxx} - \frac{1}{24}k^3c^4U_{xxxx}\Big)_J + \mathrm{O}(k^4). \end{aligned} \tag{8}$$

Let

$$r_l = x_{J-3+l} - \alpha, \quad (l = 1, 2, 3, 4, 5),$$

Taylor series expansion of each U at $x = \alpha$ gives

$$\begin{aligned} L &= \frac{1}{k}\sum_{l=l}^{5}\Gamma_{J,1}\left(U^* + r_lU_x^* + \frac{1}{2}r_l^2U_{xx}^* + \frac{1}{6}r_l^3U_{xxx}^* + \frac{1}{24}r_l^4U_{xxxx}^*\right) \\ &\quad + A\left(U_x^- + r_3U_{xx}^- + \frac{1}{2}r_3^2U_{xxx}^- + \frac{1}{6}r_3U_{xxxx}^-\right) \\ &\quad - \frac{1}{2}kc^2\left(U_{xx}^- + r_3U_{xxx}^- + \frac{1}{2}r_3^2U_{xxxx}^-\right) \\ &\quad + \frac{1}{6}k^2c^2A\left(U_{xxx}^- + r_3U_{xxxx}^-\right) - \frac{1}{24}k^3c^4U_{xxxx}^- + \mathrm{O}(k^4). \end{aligned} \tag{9}$$

Here, the superscripts $*$ denotes $-$ or $+$ depending on whether the lth point is on the $-$ or $+$ side of the interface. In this case, $*$ denotes $-$ for $l = 1, 2, 3$ and $+$ for $l = 4, 5$. Expression (9) shows that the local truncation error L is a linear function of U^*, U_x^*, U_{xx}^*, U_{xxx}^* and U_{xxxx}^*. Since $*$ may be $-$ or $+$ depending on which side a point lies, L depends on ten terms: the values of the above five terms on each side of the interface. However, the ten terms are not independent, and later we will see that we have the following jump condition (see Section 4)

$$\begin{aligned} U^+ &= D_1 U^- \\ U_x^+ &= D_2 U_x^- \\ U_{xx}^+ &= D_3 U_{xx}^- \\ U_{xxx}^+ &= D_4 U_{xxx}^- \\ U_{xxxx}^+ &= D_5 U_{xxxx}^-, \end{aligned} \tag{10}$$

where $D_1,\ D_2, ..., D_5$ are 2×2 matrices given by

$$\begin{aligned} &D_1 = I, \quad D_2 = \mathrm{diag}\left(\kappa^-/\kappa^+,\ \rho^+/\rho^-\right), \\ &D_3 = \left(c^-/c^+\right)^2 I, \quad D_4 = \left(c^-/c^+\right)^2 D_2, \quad D_5 = \left(c^-/c^+\right)^4 I. \end{aligned} \tag{11}$$

Eliminating these values on the $+$ side with $-$ side values in equation (9), we obtain

$$\begin{aligned} L \;=\; & \frac{1}{k}\sum_{l=3}^{5}\Gamma_{J,1}\left(U^- + r_l U_x^- + \frac{1}{2}r_l^2 U_{xx}^- + \frac{1}{6}r_l^3 U_{xxx}^- + \frac{1}{24}r_l^4 U_{xxxx}^-\right) \\ & \frac{1}{k}\sum_{l=4}^{5}\Gamma_{J,1}\left(D_1 U^- + r_l D_2 U_x^- + \frac{1}{2}r_l^2 D_3 U_{xx}^-\right. \\ & \left. + \frac{1}{6}r_l^3 D_4 U_{xxx}^- + \frac{1}{24}r_l^4 D_5 U_{xxxx}^-\right) \\ & + A\left(U_x^- + r_3 U_{xx}^- + \frac{1}{2}r_3^2 U_{xxx}^- + \frac{1}{6}r_3 U_{xxxx}^-\right) \\ & - \frac{1}{2}kc^2\left(U_{xx}^- + r_3 U_{xxx}^- + \frac{1}{2}r_3^2 U_{xxxx}^-\right) \\ & + \frac{1}{6}k^2c^2 A\left(U_{xxx}^- + r_3 U_{xxxx}^-\right) - \frac{1}{24}k^3c^4 U_{xxxx}^- + \mathrm{O}(k^4). \end{aligned} \tag{12}$$

L depends on the five terms: U^-, U_x^-, U_{xx}^-, U_{xxx}^- and U_{xxxx}^-. It will suffice for fourth order accuracy if we force the coefficients of U^-, U_x^-, U_{xx}^-, U_{xxx}^- and U_{xxxx}^- to be zero, which leads to a system of matrix equations:

$$\sum_{1=1}^{3}\alpha_{il}\Gamma_{J,l} + \sum_{1=4}^{5}\alpha_{il}\Gamma_{J,l}D_i = F_i, \quad (i = 1,2,3,4,5). \tag{13}$$

Here,

$$\alpha_{il} = (r_l/h)^{i-1}, \quad (i = 1,2,3,4,5; \quad l = 1,2,3,4,5),$$

and

$$\begin{aligned} F_1 &= 0 \\ F_2 &= -\nu A^- \\ F_3 &= -2\nu\alpha_{23}A^- + \nu^2(c^-)^2 \\ F_4 &= -3\nu\alpha_{23}^2 A^- + 3\nu^2\alpha_{23}(c^-)^2 - \nu^3(c^-)^3 \\ F_5 &= -4\nu\alpha_{23}^3 A^- + 6\nu^2\alpha_{23}^2(c^-)^2 - 4\nu^3\alpha_{23}(c^-)^2 A^- + \nu^4(c^-)^4. \end{aligned} \tag{14}$$

The systems of linear equation (13) for Γs depend on the geometry, the media properties and the jump condition (from the physical application). They do not depend on the solution of the problem. Therefore we can compute Γs once before time stepping for the problem of non-moving interface, and use the same Γs for every time step.

For the other three irregular points, the same derivation can be carried on. We list the final linear systems of equations here:

At irregular point $J-1$:

$$\sum_{1=1}^{4} \alpha_{il}\Gamma_{J-1,l} + \alpha_{i5}\Gamma_{J-1,5}D_i = F_i, \quad (i = 1,2,3,4,5), \tag{15}$$

where

$$\alpha_{il} = \left(\frac{r_l}{h} - 1\right)^{i-1}, \quad (i = 1,2,3,4,5; \;\; l = 1,2,3,4,5),$$

and F_i $(i = 1,2,3,4,5)$ are unchanged.

At irregular point $J+1$:

$$\sum_{1=1}^{2} \alpha_{il}\Gamma_{J-1,l}D_i + \sum_{1=3}^{5} \alpha_{i5}\Gamma_{J-1,5} = F_i, \quad (i = 1,2,3,4,5), \tag{16}$$

where

$$\alpha_{il} = \left(\frac{r_l}{h} + 1\right)^{i-1}, \quad (i = 1,2,3,4,5; \;\; l = 1,2,3,4,5),$$

and in F_i $(i = 1,2,3,4,5)$, only A^- and c^- are replaced by A^+ and c^+ respectively.

At irregular point $J+2$:

$$\alpha_{i1}\Gamma_{J+2,1}D_i + \sum_{1=2}^{5} \alpha_{i5}\Gamma_{J+2,5} = F_i, \quad (i = 1,2,3,4,5), \tag{17}$$

where

$$\alpha_{il} = \left(\frac{r_l}{h} + 2\right)^{i-1}, \quad (i = 1,2,3,4,5; \;\; l = 1,2,3,4,5),$$

F_i $(i = 1,2,3,4,5)$ are the same as at point $J+1$.

Equations (13), (15), (16), (17), can be solved for the Γ's. Fortunately, because the matrices D_1, D_2, D_3, D_4 and D_5 are diagonal, each system (i.e. each of (13), (15) (16) and (17)) can be decoupled into four disjoint 5×5 systems of equations for the four components of the Γ matrices, each of which is 5×5. It is then very easy to get the solutions of Γ's.

3. TWO DIMENSIONAL ACOUSTIC WAVE EQUATIONS

We now consider the two-dimensional acoustic wave equations

$$\begin{aligned} \rho u_t + p_x &= 0 \\ \rho v_t + p_y &= 0 \\ p_t + \kappa(u_x + v_y) &= 0, \end{aligned} \tag{18}$$

where u and v are the components of the velocity in the x and y directions and p is the pressure. The density, ρ, and the bulk modulus, κ, are discontinuous

$$(\rho,\ \kappa) = \begin{cases} (\rho^-,\ \kappa^-) & \text{if } \mathcal{F}_{\text{int}}(x,y) < 0, \\ (\rho^+,\ \kappa^+) & \text{if } \mathcal{F}_{\text{int}}(x,y) > 0 \end{cases}$$

at the interface $\mathcal{C}$ defined by $\mathcal{F}_{\text{int}}(x,y) = 0$. The physical jump conditions are similar to (2) from the one dimensional case:

$$[\vec{n} \cdot \vec{u}] = [p] = 0, \tag{19}$$

where $\vec{n}$ is the unit vector in the direction normal to the interface. That is, the velocity normal to the interface and the pressure are continuous across the interface. In contrast to the one-dimensional problem, where the number of the physical jump conditions equals the number of variables in the problem, here we have only two physical jump conditions and three variables. We will leave the discussion of the jump conditions to Section 4.

For equation (18), let

$$U = \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \\ p(x,y,t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & (1/\rho) \\ 0 & 0 & 0 \\ \kappa & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (1/\rho) \\ 0 & \kappa & 0 \end{pmatrix}.$$

We have the 3-variable first order system

$$U_t + AU_x + BU_y = 0. \tag{20}$$

In the region where A and B are smooth, we can use standard numerical schemes. As in the one dimensional case, the fourth order method can be used:

$$U_{ij}^{n+1} \quad = \quad U_{ij} + \text{update terms} \tag{21}$$

Again, the superscript n on the right hand side is omitted for simplicity. The update terms can be from any standard solver, typically with a 17-point stencil (the dot and o in Figure 2). This is valid at regular points, where all *seventeen* points in the stencil lie on the same side of the interface.

At irregular points, where some of the points of the 17-point stencil are on opposite sides of the interface, the fourth order method is no longer valid. At these points a 15-point difference scheme of the form

$$U_{ij}^{n+1} = U_{ij} + \sum_{l=1}^{15} \Gamma_{ij,l} U_{ij,l} \tag{22}$$

is used.

The Γ's are 3×3 matrices, and the $U_{ij,l}$ $(l = 1, 2, ..., 15)$ are values at some set of grid points neighboring the point (i,j). Why do we choose a 15-point stencil formula

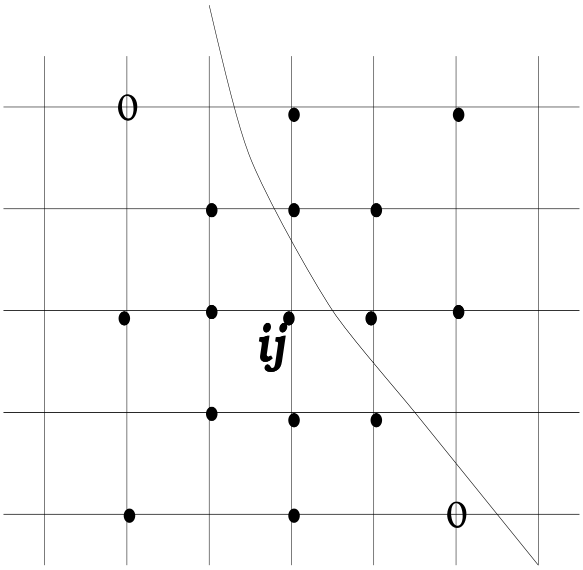

FIG. 2. 2-D interface and stencil

for irregular points? In one dimension, we use a 5-point stencil for irregular points, which enables us to eliminate all the terms up to fourth order derivatives in the local truncation error. In two dimensions, there are 15 terms of up to fourth order derivatives, which are

$$U, U_x, U_y, U_{xx}, U_{xy}, U_{yy}, U_{xxx}, U_{xxy}, U_{xyy}, U_{yyy}, U_{xxxx}, U_{xxxy}, U_{xxyy}, U_{xyyy}, U_{yyyy}. \quad (23)$$

In order to obtain a fourth order method, at least a 15-point stencil is required. One of the choices of the fifteen points can be the dots in Figure 2.

We want to choose the Γ's so that fourth order accuracy is maintained. Consider the local truncation error

$$L = \frac{1}{k}\sum_{l=1}^{15}\Gamma_{ij,l}U_l - \left(U_t + \frac{1}{2}kU_{tt} + \frac{1}{6}k^2U_{ttt} + \frac{1}{24}k^3U_{tttt}\right)_{ij} + \mathrm{O}(k^4). \quad (24)$$

The subscript ij indicates that the terms are evaluated at point (i,j). There will be a different set of fifteen Γ matrices for each point (i,j). These must be derived at each point, based on the geometry of the interface near this point. We will now concentrate on a particular point (i,j) and discuss the derivation of the corresponding Γ matrices. Since the jump conditions are specified in the directions normal and tangential to the interface (see Section 4), we need to introduce local coordinates, whose orientations are normal and tangential to the interface, respectively. We write the coordinate transformation as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad (25)$$

where ξ and η are in the directions normal and tangential to the interface at point (x_0, y_0) respectively, where $(x_0, y_0) \in \mathcal{C}$ is a point on the interface near the point (i, j), and β is the rotation angle. Let

$$\overline{U}(\xi,\eta,t) = \begin{pmatrix} \overline{u}(\xi,\eta,t) \\ \overline{v}(\xi,\eta,t) \\ p(\xi,\eta,t) \end{pmatrix}, \quad Q_0 = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{26}$$

Then we have

$$U(x,y,t) = Q_0 \overline{U}(\xi,\eta,t). \tag{27}$$

and (20) gives the transformed equation

$$\overline{U}_t + A\overline{U}_\xi + B\overline{U}_\eta = 0. \tag{28}$$

We want to also transform (24) to ξ-η coordinates. Now, expressing the derivatives of U with respect to t in terms of those of $\overline{U}$ with respect to ξ and η, we obtain

$$\begin{aligned} U_t &= Q_0\overline{U}_t = -Q_0\left(A\overline{U}_\xi + B\overline{U}_\eta\right) \\ U_{tt} &= Q_0\overline{U}_{tt} = c^2 Q_0\left(\overline{U}_{\xi\xi} + \overline{U}_{\eta\eta}\right) \\ U_{ttt} &= Q_0\overline{U}_{ttt} = -c^2 Q_0\left((A\overline{U}_\xi + B\overline{U}_\eta)_{\xi\xi} + (A\overline{U}_\xi + B\overline{U}_\eta)_{\eta\eta}\right) \\ U_{tttt} &= Q_0\overline{U}_{tttt} = c^4 Q_0\left(\overline{U}_{\xi\xi\xi\xi} + 2\overline{U}_{\xi\xi\eta\eta} + \overline{U}_{\eta\eta\eta\eta}\right). \end{aligned} \tag{29}$$

Also, we have

$$\sum_{l=1}^{15} \Gamma_{ij,l} U_{ij,l} = \sum_{l=1}^{15} \Gamma_{ij,l} Q_0 \overline{U}_{ij,l}. \tag{30}$$

To simplify the notation, we will now concentrate on a particular point (i, j) and drop these subscripts. We also incorporate the necessary rotation into the Γ's, now defining the following Γ's and U's as shorthand:

$$\Gamma_l = Q_0^{-1}\Gamma_{ij,l}Q_0, \quad U_l = \overline{U}_{ij,l}. \tag{31}$$

Then with (29), (30) and (31) the local truncation error (24) at point (i, j) has the form

$$L = Q_0\left\{\frac{1}{k}\sum_{l=1}^{15}\Gamma_l U_l - \left(\overline{U}_t + \frac{1}{2}k\overline{U}_{tt} + \frac{1}{6}k^2\overline{U}_{ttt} + \frac{1}{24}k^3\overline{U}_{tttt}\right)\right\} + \mathrm{O}(k^4). \tag{32}$$

Expanding U_l ($l = 1, 2, ..., 15$) about point (x_0, y_0) and setting $U^* = \overline{U}(x_0, y_0)$, we get

$$\begin{aligned} U_l \;=\; & U^* + (\xi_l U^*_\xi + \eta_l U^*_\eta) + \frac{1}{2}\left(\xi_l^2 U^*_{\xi\xi} + 2\xi_l\eta_l U^*_{\xi\eta} + \eta_l^2 U^*_{\eta\eta}\right) \\ & + \frac{1}{6}\left(\xi_l^3 U^*_{\xi\xi\xi} + 3\xi_l^2\eta_l U^*_{\xi\xi\eta} + 3\xi_l\eta_l^2 U^*_{\xi\eta\eta} + \eta_l^3 U^*_{\eta\eta\eta}\right) \\ & + \frac{1}{24}\left(\xi_l^4 U^*_{\xi\xi\xi\xi} + 4\xi_l^3\eta_l U^*_{\xi\xi\xi\eta} + 6\xi_l^2\eta_l^2 U^*_{\xi\xi\eta\eta}\right. \\ & \left. + 4\xi_l\eta_l^3 U^*_{\xi\eta\eta\eta} + \eta^4 U^*_{\eta\eta\eta\eta}\right) + \mathrm{O}(k^5), \end{aligned} \tag{33}$$

where ξ_l and η_l are coordinates of point l under ξ-η coordinates, and $*$ denotes $-$ or $+$ depending on whether the lth point is on the $-$ or on the $+$ side of the interface. Also, with (29), expanding its right hand side about point (x_0, y_0), we have

$$
\begin{aligned}
\overline{U}_t &= -AU^*_\xi - BU^*_\eta - \left(\xi_c \left(AU^*_\xi + BU^*_\eta\right)_\xi + \eta_c \left(AU^*_\xi + BU^*_\eta\right)_\eta\right) \\
&\quad - \left(\xi_c^2 \left(AU^*_\xi + BU^*_\eta\right)_{\xi\xi} + 2\xi_c\eta_c \left(AU^*_\xi + BU^*_\eta\right)_{\xi\eta} + \eta_c^2 \left(AU^*_\xi + BU^*_\eta\right)_{\eta\eta}\right) \\
&\quad - \left(\xi_c^3 \left(AU^*_\xi + BU^*_\eta\right)_{\xi\xi\xi} + 3\xi_c^2\eta_c \left(AU^*_\xi + BU^*_\eta\right)_{\xi\xi\eta}\right. \\
&\quad \left. +3\xi_c\eta_c^2 \left(AU^*_\xi + BU^*_\eta\right)_{\xi\eta\eta} + \eta_c^3 \left(AU^*_\xi + BU^*_\eta\right)_{\eta\eta\eta}\right) + \mathrm{O}(k^4) \\
\overline{U}_{tt} &= c^2 \left(\left(U^*_{\xi\xi} + U^*_{\eta\eta}\right) + \xi_c \left(U^*_{\xi\xi} + U^*_{\eta\eta}\right)_\xi + \eta_c \left(U^*_{\xi\xi} + U^*_{\eta\eta}\right)_\eta\right. \\
&\quad \left. +\xi_c^2 \left(U^*_{\xi\xi} + U^*_{\eta\eta}\right)_{\xi\xi} + 2\xi_c\eta_c \left(U^*_{\xi\xi} + U^*_{\eta\eta}\right)_{\xi\eta} + \eta_c^2 \left(U^*_{\xi\xi} + U^*_{\eta\eta}\right)_{\eta\eta}\right) + \mathrm{O}(k^3) \\
\overline{U}_{ttt} &= -c^2 \left(\left(AU^*_\xi + BU^*_\eta\right)_{\xi\xi} + \left(AU^*_\xi + BU^*_\eta\right)_{\eta\eta}\right. \\
&\quad +\xi_c \left(\left(AU^*_\xi + BU^*_\eta\right)_{\xi\xi} + \left(AU^*_\xi + BU^*_\eta\right)_{\eta\eta}\right)_\xi \\
&\quad \left. +\eta_c \left(\left(AU^*_\xi + BU^*_\eta\right)_{\xi\xi} + \left(AU^*_\xi + BU^*_\eta\right)_{\eta\eta}\right)_\eta\right) + \mathrm{O}(k^2) \\
\overline{U}_{tttt} &= c^4 \left(U^*_{\xi\xi\xi\xi} + 2U^*_{\xi\xi\eta\eta} + U^*_{\eta\eta\eta\eta}\right) + \mathrm{O}(k).
\end{aligned}
\tag{34}
$$

The subscript c in ξ and η means the center point, point (i, j). With equations (33) and (34) we can see from (32) that the local truncation error L is a linear function of fifteen terms listed in (23). Since for a irregular point near the interface, some of the points of the stencil are on different sides, L depends on thirty terms: the values of the fifteen terms on *each* side of the interface. We can express these values on one side of the interface in terms of those on the other side. With the jump conditions specified (see Section 4), we have the following

$$
\begin{aligned}
U^+ &= Q_1 U^- \\
U^+_\eta &= Q_1 U^-_\eta \\
U^+_\xi &= Q_2 U^-_\xi + Q_3 U^-_\eta \\
U^+_{\eta\eta} &= Q_1 U^-_{\eta\eta} \\
U^+_{\xi\xi} &= \left(c^-/c^+\right)^2 Q_1 U^-_{\xi\xi} + \left(\left(c^-/c^+\right)^2 - 1\right) Q_1 U^-_{\eta\eta} \\
U^+_{\xi\eta} &= Q_2 U^-_{\xi\eta} + Q_3 U^-_{\eta\eta} \\
U^+_{\eta\eta\eta} &= Q_1 U^-_{\eta\eta\eta} \\
U^+_{\xi\eta\eta} &= Q_2 U^-_{\xi\eta\eta} + Q_3 U^-_{\eta\eta\eta} \\
U^+_{\xi\xi\eta} &= \left(c^-/c^+\right)^2 Q_1 U^-_{\xi\xi\eta} + \left(\left(c^-/c^+\right)^2 - 1\right) Q_1 U^-_{\eta\eta\eta} \\
U^+_{\xi\xi\xi} &= Q_4 U^-_{\xi\xi\xi} + Q_5 U_{\xi\xi\eta} + Q_6 U_{\xi\eta\eta} + Q_7 U_{\eta\eta\eta} \\
U^+_{\eta\eta\eta\eta} &= Q_1 U^-_{\eta\eta\eta\eta} \\
U^+_{\xi\eta\eta\eta} &= Q_2 U^-_{\xi\eta\eta\eta} + Q_3 U^-_{\eta\eta\eta\eta} \\
U^+_{\xi\xi\eta\eta} &= \left(c^-/c^+\right)^2 Q_1 U^-_{\xi\xi\eta\eta} + \left(\left(c^-/c^+\right)^2 - 1\right) Q_1 U^-_{\eta\eta\eta\eta} \\
U^+_{\xi\xi\xi\eta} &= Q_4 U^-_{\xi\xi\xi\eta} + Q_5 U_{\xi\xi\eta\eta} + Q_6 U_{\xi\eta\eta\eta} + Q_7 U_{\eta\eta\eta\eta} \\
U^+_{\xi\xi\xi\xi} &= U^-_{\xi\xi\xi\xi} + 2\left(c^-/c^+\right)^2 \left(\left(c^-/c^+\right)^2 - Q_1\right) U^-_{\xi\xi\eta\eta} \\
&\quad + \left(\left(c^-/c^+\right)^4 - 2\left(c^-/c^+\right)^2 Q_1 + Q_1\right) U^-_{\eta\eta\eta\eta},
\end{aligned}
\tag{35}
$$

where the Q's are given as follows

$$
\begin{aligned}
&Q_1 = \mathrm{diag}\left(1,\ \rho^-/\rho^+,\ 1\right)\\
&Q_2 = \mathrm{diag}\left(\kappa^-/\kappa^+,\ 1,\ \rho^+/\rho^-\right)\\
&Q_3 = (\rho^-/\rho^+)\left(\left(c^-/c^+\right)^2 - 1\right)\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\
&Q_4 = \mathrm{diag}\left(\frac{\kappa^-}{\kappa^+}, 0, \frac{\rho^+}{\rho^-}\left(\frac{c^-}{c^+}\right)^2\right)\\
&Q_5 = \left(\frac{c^-}{c^+}\right)^2\begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\
&Q_6 = \frac{\rho^+}{\rho^-}\left(\left(\frac{c^-}{c^+}\right)^2 - 1\right)\mathrm{diag}\,(0,0,1)\\
&Q_7 = \left(\left(\frac{c^-}{c^+}\right)^2 - 1\right)\begin{pmatrix} 0 & \rho^-/\rho^+ & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.
\end{aligned}
\tag{36}
$$

Similar to the one-dimensional case, substituting (33) and (34) into (32), and replacing $U^+, U^+_\eta, ..., U^+_{\xi\xi\xi\xi}$ with the jump condition (35), we obtain the local truncation error expressed in terms of $U^-, U^-_\eta, ..., U^-_{\xi\xi\xi\xi}$, reducing the dependent variables to the fifteen terms. For fourth order accuracy it will suffice to make all the coefficients of terms listed in (23) equal zero in the local truncation error L. Then we have the following fifteen equations

$$\sum_{l=1}^{15} \Gamma_l \hat{Q}_{il} = F_i \qquad (i = 1, 2, ..., 15), \tag{37}$$

where the $\hat{Q}$'s are 3×3 matrices. If point l is on the same side as the center point, then

$$
\begin{aligned}
&\hat{Q}_{1l} = I,\ \hat{Q}_{2l} = \xi_l I,\ \hat{Q}_{3l} = \eta_l I,\ \hat{Q}_{4l} = \xi_l^2 I,\ \hat{Q}_{5l} = 2\xi_l\eta_l I,\ \hat{Q}_{6l} = \eta_l^2 I,\\
&\hat{Q}_{7l} = \xi_l^3 I,\ \hat{Q}_{8l} = 3\xi_l^2\eta_l I,\ \hat{Q}_{9l} = 3\xi_l\eta_l^2 I,\ \hat{Q}_{10,l} = \eta_l^3 I,\\
&\hat{Q}_{11,l} = \xi_l^4 I,\ \hat{Q}_{12,l} = 4\xi_l^3\eta_l I,\ \hat{Q}_{13,l} = 6\xi_l^2\eta_l^2 I,\ \hat{Q}_{14,l} = 4\xi_l\eta_l^3 I,\ \hat{Q}_{15,l} = \eta_l^4 I;
\end{aligned}
\tag{38}
$$

Otherwise

$$
\begin{aligned}
\hat{Q}_{1l} &= Q_1 \\
\hat{Q}_{2l} &= \xi_l Q_2 \\
\hat{Q}_{3l} &= \eta_l Q_1 + \xi_l Q_3 \\
\hat{Q}_{4l} &= \xi_l^2 \left(c^-/c^+\right)^2 Q_1 \\
\hat{Q}_{5l} &= 2\xi_l \eta_l Q_2 \\
\hat{Q}_{6l} &= \eta_l^2 Q_1 + 2\xi_l \eta_l Q_3 + \xi_l^2 \left(\left(c^-/c^+\right)^2 - 1\right) Q_1, \\
\hat{Q}_{7l} &= \xi_l^3 Q_4 \\
\hat{Q}_{8l} &= \xi_l^3 Q_5 + 3\xi_l^2 \eta_l \left(c^-/c^+\right)^2 Q_1 \\
\hat{Q}_{9l} &= \xi_l^3 Q_6 + 3\xi_l \eta_l^2 Q_2 \\
\hat{Q}_{10,l} &= \xi_l^3 Q_7 + 3\xi_l^2 \eta_l \left(\left(c^-/c^+\right) - 1\right) Q_1 + 3\xi_l \eta_l^2 Q_3 + \eta_l^3 Q_1 \\
\hat{Q}_{11,l} &= \xi_l^4 I \\
\hat{Q}_{12,l} &= 4\xi_l^3 \eta_l Q_4 \\
\hat{Q}_{13,l} &= 2\left(c^-/c^+\right)^2 \left(\left(c^-/c^+\right)^2 - Q_1\right) + 4\xi_l^3 \eta_l Q_5 + 6\xi_l^2 \eta_l^2 \left(c^-/c^+\right)^2 Q_1 \\
\hat{Q}_{14,l} &= 4\xi_l^3 \eta_l Q_6 + 4\xi_l \eta_l^3 Q_2 \\
\hat{Q}_{15,l} &= \left(\left(c^-/c^+\right)^4 - 2\left(c^-/c^+\right)^2 Q_1 + Q_1\right) + 4\xi_l^3 \eta_l Q_7 \\
&\quad + 6\xi_l^2 \eta_l^2 \left(\left(c^-/c^+\right)^2 - Q_1\right) Q_1 + 4\xi_l \eta_l^3 Q_3 + \eta_l^4 Q_1,
\end{aligned}
\tag{39}
$$

and F_i $(i = 1, 2, ..., 15)$ are given by

$$
\begin{aligned}
F_1 &= 0 \\
F_2 &= -\nu A \\
F_3 &= -\nu B \\
F_4 &= \nu^2 c^2 - 2\nu\xi_c A \\
F_5 &= -2\nu(\eta_c A + \xi_c B) \\
F_6 &= \nu^2 c^2 - 2\nu\eta_c B \\
F_7 &= -\nu^3 c^2 A + 3\nu^2 c^2 \xi_c - 6\nu\xi_c^2 B \\
F_8 &= -\nu^3 c^2 B + 3\nu^2 c^2 \eta_c - 6\nu\xi_c(\xi_c B + 2\eta_c A) \\
F_9 &= -\nu^3 c^2 A + 3\nu^2 c^2 \xi_c - 6\nu\eta_c(\eta_c A + 2\xi_c B) \\
F_{10} &= -\nu^3 c^2 B + 3\nu^2 c^2 \eta_c - 6\nu\eta_c^2 B \\
F_{11} &= \nu^4 c^4 - 4\nu^3 c^2 \xi_c A + 12\nu^2 c^2 \xi_c^2 - 24\nu\xi_c^3 A \\
F_{12} &= -4\nu^3 c^2(\xi_c B + \eta_c A) + 24\nu^2 c^2 \xi_c\eta_c - 24\nu\xi_c^2(\xi_c B + 3\eta_c A) \\
F_{13} &= 2\nu^4 c^4 - 4\nu^3 c^2(\xi_c A + \eta_c B) + 12\nu^2 c^2(\xi_c^2 + \eta_c^2) - 72\nu\xi_c\eta_c(\xi_c B + \eta_c A) \\
F_{14} &= -4\nu^3 c^2(\xi_c B + \eta_c A) + 24\nu^2 c^2 \xi_c\eta_c - 24\nu\eta_c^2(3\xi_c B + \eta_c A) \\
F_{15} &= \nu^4 c^4 - 4\nu^3 c^2 \eta_c B + 12\nu^2 c^2 \eta_c^2 - 24\nu\eta_c^3 B.
\end{aligned}
\tag{40}
$$

Note that ξ_l, η_l, ξ_c and η_c have been non-dimensionalized by dividing the original ξ_l, η_l, ξ_c and η_c by h, the grid size. This is similar to $\alpha_{2l} = r_l/h$ in one dimensional case, though we still use the same notation here. This non-dimensionalization process gives terms in $\hat{Q}$s and Fs expression of order one (O(1)), yielding a simple but effective pre-conditioning.

The fifteen matrix equations can be used to solve for Γ_l $(l = 1, ..., 15)$. Since each Γ is 3×3, we have 135 equations for 135 unknowns. Normally we do not need to solve for the 135 unknowns simultaneously, we can decouple the system into smaller systems of equations. Taking the transpose of the system (37), we get

$$\sum_{l=1}^{15} \hat{Q}'_{il}\Gamma'_l = F'_i, \quad i = 1, 2, ..., 15$$

which is equivalent to

$$\hat{Q}\hat{\Gamma} = \hat{F}. \tag{41}$$

Here $\hat{Q}$ is a 15×15 block matrix, with block element (i, l) being $\hat{Q}'_{il}$, and $\hat{\Gamma}$ and $\hat{F}$ are block matrices of dimension 16×1, with block element l of $\hat{\Gamma}$ and i of $\hat{F}$ being Γ'_l and F'_i, respectively. Solving the system of equation (41) is equivalent to solving 3 systems of 45 linear equations, with the 3 systems each having the same coefficient matrix $\hat{Q}$.

The treatment of the interface is local, this means that at each irregular point, the size of the system we need to solve for the Γ's does not depend on the mesh size.

Also note that the calculation of the Γ's for each irregular point is independent of all other points, lending itself to parallel computing. More importantly, since the forward time stepping is preserved, the procedure of determining the finite difference formulas at the irregular points need be done only once as a preprocessing stage. The same formulas can then be used in every time step in solving the wave equation.

4. JUMP CONDITIONS AT THE INTERFACE FOR ACOUSTICS

In Sections 2 and 3 we used the numerical jump conditions for vector variable U and its derivatives up to fourth order. We have already seen that those jump conditions are extremely important in formulating the special finite difference scheme at the irregular points near the interface. Here we will give the derivation of the jump conditions in the case of one spatial dimension first, followed by those for two spatial dimensions with a linear interface.

4.1. One Dimensional Problem

We will derive the one dimensional jump condition (10) first. The jump conditions (2) give

$$[U] = 0 \quad \Longrightarrow \quad U^+ = D_1 U^-. \tag{42}$$

Here, the pair of square of brackets represents the jump of the quantity they enclose. From (2) we get

$$[u] = 0 \quad \Longrightarrow \quad [u_t] = 0 \quad \Longrightarrow \quad \left[\frac{1}{\rho} p_x\right] = 0$$

$$[p] = 0 \quad \Longrightarrow \quad [p_t] = 0 \quad \Longrightarrow \quad [\kappa u_x] = 0,$$

or,

$$U_x^+ = D_2 U_x^-. \tag{43}$$

Using $U_t + AU_x = 0$ and therefore $U_{tt} = c^2 U_{xx}$, we have

$$[U] = 0 \quad \Longrightarrow \quad [U_{tt}] = 0 \quad \Longrightarrow \quad [c^2 U_{xx}] = 0 \quad \Longrightarrow \quad U_{xx}^+ = D_3 U_{xx}^-. \tag{44}$$

$$[U] = 0 \quad \Longrightarrow \quad [U_{ttt}] = 0 \quad \Longrightarrow \quad [c^2 A U_{xxx}] = 0 \quad \Longrightarrow \quad U_{xxx}^+ = D_4 U_{xxx}^-. \tag{45}$$

$$[U] = 0 \quad \Longrightarrow \quad [U_{tttt}] = 0 \quad \Longrightarrow \quad [c^4 U_{xxxx}] = 0 \quad \Longrightarrow \quad U_{xxxx}^+ = D_5 U_{xxxx}^-. \tag{46}$$

The combination of (42), (43), (44), (45) and (46) gives (10), with D_1, D_2, D_3, D_4 and D_5 given in (11).

4.2. Two Dimensional Problem

Now we derive the jump conditions in two spatial dimensions with a linear interface. We will take advantage of the jump conditions available for a second order method, which are given by Zhang and LeVeque:

$$\begin{aligned}
U^+ &= Q_1 U^- \\
U_\eta^+ &= Q_1 U_\eta^- \\
U_\xi^+ &= Q_2 U_\xi^- + Q_3 U_\eta^- \\
U_{\eta\eta}^+ &= Q_1 U_{\eta\eta}^- \\
U_{\xi\xi}^+ &= \left(c^-/c^+\right)^2 Q_1 U_{\xi\xi}^- + \left(\left(c^-/c^+\right)^2 - 1\right) Q_1 U_{\eta\eta}^- \\
U_{\xi\eta}^+ &= Q_2 U_{\xi\eta}^- + Q_3 U_{\eta\eta}^- \\
U_{\xi\eta}^+ &= Q_2 U_{\xi\eta}^- + Q_3 U_{\eta\eta}^- .
\end{aligned} \tag{47}$$

The Qs are defined in (36). Note that the definition and derivation are based on the local coordinate system ξ-η, Recall that ξ and η are in the directions normal and tangential to the interface. Directly differentiating the above jump conditions with respect to η up to four times gives

$$\begin{aligned}
U_{\eta\eta\eta}^+ &= Q_1 U_{\eta\eta\eta}^- \\
U_{\xi\eta\eta}^+ &= Q_2 U_{\xi\eta\eta}^- + Q_3 U_{\eta\eta\eta}^- \\
U_{\xi\xi\eta}^+ &= \left(c^-/c^+\right)^2 Q_1 U_{\xi\xi\eta}^- + \left(\left(c^-/c^+\right)^2 - 1\right) Q_1 U_{\eta\eta\eta}^- \\
U_{\eta\eta\eta\eta}^+ &= Q_1 U_{\eta\eta\eta\eta}^- \\
U_{\xi\eta\eta\eta}^+ &= Q_2 U_{\xi\eta\eta\eta}^- + Q_3 U_{\eta\eta\eta\eta}^- \\
U_{\xi\xi\eta\eta}^+ &= \left(c^-/c^+\right)^2 Q_1 U_{\xi\xi\eta\eta}^- + \left(\left(c^-/c^+\right)^2 - 1\right) Q_1 U_{\eta\eta\eta\eta}^- .
\end{aligned} \tag{48}$$

This leaves us jump conditions of $U_{\xi\xi\xi}$, $U_{\xi\xi\xi\eta}$ and $U_{\xi\xi\xi\xi}$ to work out.

For $U_{\xi\xi\xi}$, we have

$$\begin{aligned}
[u_{ttt}] &= \left[c^2\left(u_{t\xi\xi} + u_{t\eta\eta}\right)\right] = 0 \\
[p_{ttt}] &= \left[\kappa\left(p_{t\xi\xi} + p_{t\eta\eta}\right)\right] = 0,
\end{aligned}$$

which give

$$\begin{aligned}
&\left[\left(\frac{c^2}{\rho}\right)\left(p_{\xi\xi\xi} + p_{\eta\eta\eta}\right)\right] = 0 \\
&\left[\kappa\left(\left(u_\xi + v_\eta\right)_{\xi\xi} + \left(u_\xi + v_\eta\right)_{\eta\eta}\right)\right] = 0.
\end{aligned}$$

These lead to

$$\begin{aligned}
p_{\xi\xi\xi}^+ &= \frac{\rho^+}{\rho^-}\left(\frac{c^-}{c^+}\right)^2\left(p_{\xi\xi\xi}^- + p_{\xi\eta\eta}^-\right) - p_{\xi\eta\eta}^+ \\
u_{\xi\xi\xi}^+ &= \frac{\kappa^-}{\kappa^+}\left(u_{\xi\xi\xi}^- + u_{\xi\eta\eta}^- + v_{\xi\xi\eta}^- + v_{\eta\eta\eta}^-\right) - u_{\xi\eta\eta}^+ + v_{\xi\xi\eta}^+ + v_{\eta\eta\eta}^+ .
\end{aligned} \tag{49}$$

Replacing the + superscript terms on the right hand side in (49) with $-$ superscript terms using (47) and (48), we obtain

$$\begin{aligned} p^+_{\xi\xi\xi} &= \frac{\rho^+}{\rho^-}\left(\frac{c^-}{c^+}\right)^2 p^-_{\xi\xi\xi} + \frac{\rho^+}{\rho^-}\left(\left(\frac{c^-}{c^+}\right)^2 - 1\right) p^-_{\xi\eta\eta} \\ u^+_{\xi\xi\xi} &= \frac{\kappa^-}{\kappa^+} u^-_{\xi\xi\xi} + \frac{\rho^-}{\rho^+}\left(\left(\frac{c^-}{c^+}\right)^2 - 1\right) v_{\eta\eta\eta} \end{aligned} \tag{50}$$

Also, from $v_\xi = u_\eta$, as has been shown by Zhang and LeVeque, we have $v^+_{\xi\xi\xi} = u^+_{\xi\xi\eta}$. Using (48) gives

$$v^+_{\xi\xi\xi} = \left(\frac{c^-}{c^+}\right)^2 u_{\xi\xi\eta} + \left(\left(\frac{c^-}{c^+}\right)^2 - 1\right) u_{\eta\eta\eta}. \tag{51}$$

Equations (50) and (51) complete jump condition for $U_{\xi\xi\xi}$:

$$U^+_{\xi\xi\xi} = Q_4 U^-_{\xi\xi\xi} + Q_5 U_{\xi\xi\eta} + Q_6 U_{\xi\eta\eta} + Q_7 U_{\eta\eta\eta}, \tag{52}$$

where,

$$\begin{aligned} Q_4 &= \operatorname{diag}\left(\frac{\kappa^-}{\kappa^+}, 0, \frac{\rho^+}{\rho^-}\left(\frac{c^-}{c^+}\right)^2\right), \qquad Q_6 = \frac{\rho^+}{\rho^-}\left(\left(\frac{c^-}{c^+}\right)^2 - 1\right)\operatorname{diag}(0,0,1) \\ Q_5 &= \left(\frac{c^-}{c^+}\right)^2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_7 = \left(\left(\frac{c^-}{c^+}\right)^2 - 1\right)\begin{pmatrix} 0 & \rho^-/\rho^+ & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{53}$$

Differentiating (52) with respective η yields

$$U^+_{\xi\xi\xi\eta} = Q_4 U^-_{\xi\xi\xi\eta} + Q_5 U_{\xi\xi\eta\eta} + Q_6 U_{\xi\eta\eta\eta} + Q_7 U_{\eta\eta\eta\eta}. \tag{54}$$

For $U_{\xi\xi\xi\xi}$, starting with

$$[U_{tttt}] = 0 \quad \Longrightarrow \quad \left[c^4\left(U_{\xi\xi\xi\xi} + 2U_{\xi\xi\eta\eta} + U_{\eta\eta\eta\eta}\right)\right] = 0,$$

and this gives

$$U^+_{\xi\xi\xi\xi} = \left(c^-/c^+\right)^4\left(U^-_{\xi\xi\xi\xi} + 2U^-_{\xi\xi\eta\eta} + U^-_{\eta\eta\eta\eta}\right) - 2U^+_{\xi\xi\eta\eta} - U^+_{\eta\eta\eta\eta}. \tag{55}$$

Substituting $U^+_{\xi\xi\eta\eta}$ and $U^+_{\eta\eta\eta\eta}$ from (48) into (55), we have

$$\begin{aligned} U^+_{\xi\xi\xi\xi} &= U^-_{\xi\xi\xi\xi} + 2\left(c^-/c^+\right)^2\left(\left(c^-/c^+\right)^2 - Q_1\right)U^-_{\xi\xi\eta\eta} \\ &\quad + \left(\left(c^-/c^+\right)^4 - 2\left(c^-/c^+\right)^2 Q_1 + Q_1\right)U^-_{\eta\eta\eta\eta} \end{aligned} \tag{56}$$

The combination of equations (47), (48), (52), (54) and (56) gives (35), which is the complete numerical jump conditions for up to fourth order derivatives.

5. CONCLUSIONS

High accuracy simulations of wave equations in heterogeneous media are highly desired in many applications. However, the presence of an interface across the heterogeneous media makes obtaining the solution difficult. In this paper we have presented a fourth order finite difference method for the simulation of the acoustic wave equation in one- and two-dimensions. At regular points away form the interface, any standard formula (preferably fourth order) can be used. At irregular points near the interface, the method builds the numerical jump conditions, which we derive from the physical jump conditions, into special finite difference stencils. This special formula gives fourth order accuracy even on points near the discontinuity. To the extent of our knowledge, this is the first fourth order method of wave propagation in heterogeneous media.

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REFERENCES

M. Bouchon, M. Campillo, and S. Gaffet. A boundary integral equation – discrete wavenumber representation method to study wave propagation in multilayered media having irregular interfaces. *Geophysics*, 54:1134–1140, 1989.

D. L. Brown. A note on the numerical solution of the wave equation with piecewise smooth coefficients. *Math. Comput.*, 42:369–391, 1984.

C. L. Card and M. B. Allen. Difference approximations for wave equations via finite elements: I. Construction and performance. *Numer. Meth. PDE*, 11:127–146, 1995.

C. L. Card and M. B. Allen. Difference approximations for wave equations via finite elements: II. Error analysis. *Numer. Meth. PDE*, 11:147–163, 1995.

P. P. Delsanto, R. S. Schecter, H. H. Chaskelis, R. B. Mignogna, and R. Kline. Connection machine simulation of ultrasonic wave propagation in materials. ii: the two-dimensional case. *Wave Motion*, 20:295–314, 1994.

Todd Dupont. l^2-estimates for galerkin methods for second order hyperbolic equations. *SIAM J. Num. Anal.*, 10:880, 1973.

A. Q. Filho E. Tessmer A. Behle D. Kosloff, D. Kessler and R. Strahilevitz. Solution of the equations of dynamic elasticity by a chebychev spectral method. *Geophys. J. Int.*, 55:734–748, 1990.

F. B. Jensen, W. A. Kuperman, M. B. Porter, and H. Schmidt. *Computational Ocean Acoustics.* American Institute of Physics, New York, 1993.

R. J. LeVeque. Wave propagation algorithms for multi-dimensional hyperbolic systems. submitted, 1995.

John Lysmer and Lawrence A. Drake. A finite element for seismology. Methods of Computational Physics, II , B.A. Bolt, Ed. Academic Press, New York, 1972.

R. Madariaga. Dynamics of an expanding circular fault. *Bull. Seism. Soc. Am.*, 66:163–182, 1976.

W.W. Symes and A. Sei. Error analysis of numerical schemes for the wave equation in heterogeneous media. *Applied Numerical Mathematics*, 15:465–480, 1994.

G. Wojcik, J. Mould, Jr., R. Ferguson, and K. K. Low. Some image modeling issues for I-line, 5x phase shifting masks. In *SPIE Proceedings Vol 2197: Optical/Laser Microlithography VII.* SPIE, 1994.

G. L Wojcik, D. K. Vaughan, N. Abboud, and J. Mould, Jr. Electromechanical modeling using explicit time-domain finite elements. In *IEEE Ultrasonics Symposium Proceedings.* IEEE, 1993.

Chaoming Zhang. Immersed interface methods for hyperbolic systems of partial differential equations with discontinuous coefficients. *PhD Dissertation*, University of Washington, 1996
(`ftp://amath.washington.edu/pub/chaoming/dissertation.ps.Z`).

Chaoming Zhang and Randall J. LeVeque. The immersed interface method for acoustic wave equations with discontinuous coefficients. *Wave Motion*, 25:237–263, 1997.