Anomalous Reflections Near a Caustic

Clifford J. Nolan

July 13, 1994

Abstract

We consider the question of boundedness of a certain mapping. This map is associated with the modeling of reflections recorded on the surface of the earth as a result of an explosion being detonated. The idea is that reflections are accounted for by the high frequency component (reflectivity) of the velocity field in the earth. We define a suitable Hilbert space for the reflectivities and another for the recordings (seismograms). This map (called the Forward map) is known to be a Fourier Integral Operator and thus can be analyzed with standard asymptotic techniques. The calculation is made possible due to an explicit representation for the Forward map in terms of the Generalized Radon Transform. We analyze the first term in the asymptotic expansion for the output of the Forward map and build a sequence of reflectivities which demonstrate the unboundedness of the Forward map. It turns out that the only situation in which the map is unbounded is when a caustic develops; the precise situation which we employ to demonstrate the unbounded nature of the map. Caustics are generally present in most realistic ray geometries and so this result has important consequences for current Linearized Inversion techniques.

1 Introduction

As an illustration we consider the experiment depicted in figure (1.1). This experiment is inspired by the work done by Cheryl Purcell in her Ph.D. thesis [1]. Here we consider performing a sequence of separate experiments labeled by an index j whereby a reflectivity (u_j) is introduced to reflect the incident bundle of rays back to the surface of the earth in the form of a plane wave. We will assume a point source located at position x_s. Before proceeding any further we define more precisely how the "reflectivity" sequence u_j is to be chosen. We will restrict our attention to the reflectivity sequence of the form

$$u_j(x) = \chi_j(x)\ e^{ik_j \cdot x} \tag{1.1}$$

where χ_j is a test function.

Next we have to describe the functions χ_j while we reserve the choice of sequence of vectors k_j until later. In order to describe the supports of the reflectivities χ_j, let us denote

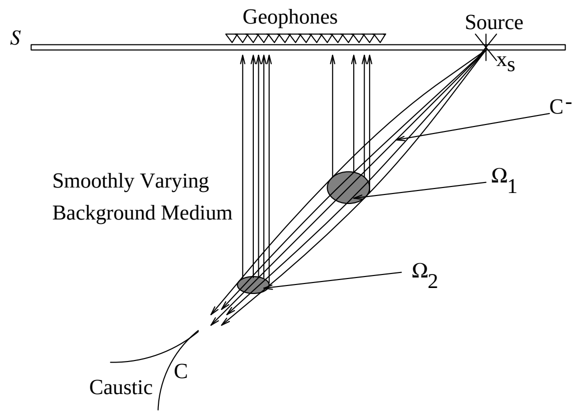

Figure 1.1: A Sequence of reflectivities

by $\mathcal{C}$ the curve prescribing a caustic that develops as a result of tracing rays through the medium from our point source x_s. Also let us denote by $\mathcal{C}^-$ the interior of the closed ray tube generated by the rays leaving the the source point x_s and travelling until the caustic curve $\mathcal{C}$ is touched by them. We will refer to $\mathcal{C}^-$ as the "precaustic region". For the general element in our sequence we set χ_j equal to the bump function supported on a disk of radius ν_j and with centre (x_j, y_j) i.e.

$$\chi_j(x) = \begin{cases} e^{-\nu_j/d_j}, \text{ if } (x-x_j)^2 + (y-y_j)^2 < \nu_j \\ \\ 0\ , \text{ if } (x-x_j)^2 + (y-y_j)^2 \geq \nu_j \\ \end{cases}$$

where

$$d_j = \nu_j - \sqrt{(x-x_j)^2 + (y-y_j)^2}.$$

Here the quantities ν_1, x_1 and y_1 are chosen so that the disk of support of χ_j does not touch the point x_s and is also contained in $\mathcal{C}^-$. Also the quantities ν_j are a strictly monotonic decreasing positive sequence, while x_j, y_j are chosen such that the distance between the centre point (x_j, y_j) and the caustic $\mathcal{C}$ is also strictly decreasing. Again we will require that these disks do not touch the point x_s and are also contained in $\mathcal{C}^-$. Furthermore we will refer to the interior of Supp χ_j as Ω_j.

In the experiment depicted in the figure (1.1) note that the same bundle of rays is intersected ("trapped") by the reflectivity u_j in each experiment (i.e. for all j). Moreover the radius of support of the reflectivity $\longrightarrow 0$ as it approaches the caustic.

The fact that it is always possible to "trap" the same bundle of rays and reflect them into a plane wave says something about the shape of the reflector in order that Snell's law be satisfied. That this experimental set up is possible (locally) is simple a consequence of the Implicit Function Theorem (see the appendix for a proof).

The experiment also suggests that the reflectivity sequence reflects arbitrarily large amounts of energy relative to the "size" of the reflector support. Recall that the energy is proportional to the L^2 norm of the amplitude of the signal. In a region close to a caustic the raytube collapses (its diameter tends to zero) and Geometrical Optics (valid here) predicts that the amplitudes "blow up".

2 Description of the Forward Map

We model the earth as a two dimensional constant density acoustic fluid. We assume that the velocity field is composed of two pieces; the background (c) and the reflectivity (u); the former being the smooth component and the latter being the rough component. Thus the pressure field (p) in the earth obeys the acoustic wave equation. Here we consider the associated fundamental solution or Green's function which we still continue to call p and satisfies by definition

$$\frac{1}{(c(x)+u(x))^2}\,\frac{\partial^2 p}{\partial t^2} - \nabla\cdot(\nabla p) = \delta(t)\delta(x-x_s). \tag{2.1}$$

Note that x_s will be a fixed point on the surface of the earth throughout this discussion.

In what follows one of the key points in producing the anticipated "blow up" is based upon how we model reflections. These are considered as arising from the perturbation in the velocity field i.e. come from the reflectivity. The reason for this is that we assume the perturbation in the Green's function for the above wave equation obeys the linearized version of the above wave equation; a formal calculation (see [2]) gives

$$\frac{1}{c^2}\frac{\partial^2 \delta p}{\partial t^2} - \nabla\cdot(\nabla \delta p) = \frac{2u}{c^3}\frac{\partial^2 p}{\partial t^2} \tag{2.2}$$

which is valid whenever the variations in the background are small relative to the wavelength of the incident field; i.e. we assume we are in the regime where the Born Approximation is valid. In fact as will be seen from our construction we will always satisfy this hypothesis since we will be working with high frequency waves ($|k_j| \to \infty$). The asymptotic expansion of the Green's function (known as the Progressing Wave Expansion [7]) is summarized by

$$p(x_s,x,t) = a(x_s,s)S(t-\tau(x_s,x)) + \text{Smoother terms.}$$

Here the amplitude a satisfies the transport equation

$$2\nabla\tau\cdot\nabla a + a\ \nabla^2\tau = 0$$

and the traveltime τ satisfies the Eikonal equation

$$|\nabla\tau|^2 = \frac{1}{c^2}$$

and S is the most singular term appearing in the (unperturbed) Green's function for a Homogeneous (c = constant) medium. So for example the Green's functions for spatial dimensions $n = 1,\ n = 2,\ n = 3$ are given by

$$n = 1: \quad G(x_s, x, t) = \rho_0 H(c_0 t - |x - x_s|)$$

$$n = 2: \quad G(x_s, x, t) = \frac{\rho_0}{2\pi} \frac{H(c_0 t - |x - x_s|)}{\sqrt{c_0^2 t^2 - |x - x_s|^2}}$$

$$n = 3: \quad G(x_s, x, t) = \frac{\rho_0 \delta(c_0 t - |x - x_s|)}{4\pi |x - x_s|}.$$

In each case pick out the most singular term (distribution) and this is what we call S - for example in one dimension S is

$$S(x_s, x, t) = H(ct \ - \ |x - x_s|)$$

etc.

This leads to the study of the perturbed greens function δp corresponding to the perturbed velocity.

This has been studied in [2] and here we will set up enough notation to quote the results needed from there. First we will define the "Forward Map" F as follows:

Let δp be the solution of equation (2.2) and then define

$$\begin{cases} F(u) = m(x_r', t_r)\ \delta p|_{\mathcal{S}} \\ \\ \delta p \equiv 0,\ t < 0. \end{cases}$$

Here $\mathcal{S}$ is the surface of the earth assumed parameterized by $x_n = 0$ when working in n dimensions. We represent vectors in $\mathbb{R}^n$ by $x = (x', x_n)$. The variable t_r is the time at which a signal is recorded on $\mathcal{S}$. The function $m \in C_0^\infty(\mathbb{R}_+^{n-1} \times R_+)$ is a cut-off or "window" function which we need to reflect the fact that we record the signal for a finite time (say T) and also only over a finite maximum offset distance on $\mathcal{S}$. We now specialize to $n = 2$ but what follows can be generalized for any dimension. Let us therefore assume *supp* $m \subset \mathcal{I} \times [0, T]$ where $\mathcal{I}$ is some interval in $\mathbb{R}$ and $T > 0$.

With this new notation what we verbalized by "blow-up" before now translates into unboundedness of the ratio

$$\frac{\|Fu_j\|_\alpha}{\|u_j\|_\beta}$$

as the experiment index $j \to \infty$. The subscripts α and β indicate two different norms; the choice of which will become clear later. Our task is to show by example unboundedness of the latter ratio and hence unboundedness of the Forward map.

3 Unboundedness of the Forward Map

The purpose of this section is to establish the following result

Theorem The Forward map defined in the last section is an unbounded Fourier Integral Operator of order $\frac{1}{2}$ iff the background produces a caustic.

Results got by Rakesh [3] show that F is a Fourier Integral Operator (FIO) [4] of order $\frac{1}{2}$ (more generally of order $\frac{n-1}{2}$). In fact it follows from the L^2 theory of FIO's ([4] page 154) that we have the following (local) property for an FIO F of order μ

$$F : H_0^s \rightarrow H_{Loc}^{s-\mu}$$

Here the zero subscript indicates compact support. It will be useful to study the so called "Normal" operator N which is simply defined by

$$N \equiv F^* F$$

where the "$*$" refers to the Hilbert (Sobolev) space adjoint of F. We now calculate F^* in terms of the transpose ${}^tF$. For this calculation we consider elements $u \in H_0^{\frac{1}{2}}(\Omega)$ and $v \in L_0^2(\mathcal{I} \times [0,T])$ where Ω is some bounded open region to be specified later. We denote by $< \cdot,\cdot >$ the L^2 inner product, $< \cdot,\cdot >_{(s,-s)}$ will denote the duality pairing for any $s \in \mathbb{R}$ and $(\cdot,\cdot)$ will denote the inner product in the Hilbert space $H_0^{\frac{1}{2}}(\Omega)$. Finally we will use the notation $\Lambda = (1 - \triangle)^{\frac{1}{2}}$ which is an operator satisfying

$$\Lambda : H_0^{\frac{1}{2}}(\Omega) \rightarrow H^{-\frac{1}{2}}(\Omega)$$

Next we have that

$$\langle Fu, v\rangle = \left\langle u,\ {}^tFv\right\rangle_{(\frac{1}{2},-\frac{1}{2})} \tag{3.1}$$

Note that the right hand side of the last equation is just a bounded linear functional on $H_0^{\frac{1}{2}}(\Omega)$ and so by the Riesz representation theorem we are guaranteed the existence of the operator F^* such that

$$\langle Fu, v\rangle = (u, F^*v) \tag{3.2}$$

Thus we get

$$(u, F^*v) = \left\langle u, {}^t Fv\right\rangle_{(\frac{1}{2},-\frac{1}{2})}$$

$$\Rightarrow \left\langle \Lambda^{\frac{1}{2}}u, \Lambda^{\frac{1}{2}}F^*v\right\rangle = \left\langle u, {}^t Fv\right\rangle_{(\frac{1}{2},-\frac{1}{2})}$$

$$\Rightarrow \langle u, \Lambda F^*v\rangle_{(\frac{1}{2},-\frac{1}{2})} = \left\langle u, {}^t Fv\right\rangle_{(\frac{1}{2},-\frac{1}{2})}$$

$$\Rightarrow N = F^*F = \Lambda^{-1}\, {}^tFF \tag{3.3}$$

We can implement all of the quantities on the right hand side of the last equality numerically and will investigate our results in this way later. It is now clear why we take our reflectivity u_j in the Sobolev space $H_0^{\frac{1}{2}}(\Omega_j)$.

In summary if we denote by F_j, N_j the restriction of the Forward and Normal maps respectively to the region $\Omega_j = \text{supp } \chi_j$, then what we have shown is the following

$$F_j : H_0^{\frac{1}{2}}(\Omega_j) \mapsto L_0^2(\mathcal{I} \times [0,T])$$

$${}^tF_j : L_0^2(\mathcal{I} \times [0,T]) \mapsto H_0^{-\frac{1}{2}}(\Omega_j)$$

$$F_j^* : L_0^2(\mathcal{I} \times [0,T]) \mapsto H_0^{\frac{1}{2}}(\Omega_j)$$

$$N_j : H_0^{\frac{1}{2}}(\Omega_j) \mapsto H_0^{\frac{1}{2}}(\Omega_j). \tag{3.4}$$

We remark that we will continue to consider $u_j \in C_0^\infty$ but note that this function space is dense in the space of distributions and in particular dense in $H_0^{\frac{1}{2}}(\Omega_j)$. The idea being that if the Forward map were to be discontinuous (unbounded) then it should be unbounded on the test functions too (by the density).

Next, we make use of the asymptotic expansion for N derived in [2]. This as we shall see later is consistent with high frequency content that will be assumed in the reflectivity sequence u_j. The main result in [2] can be summarized in following Pseudodifferential equation

$$n_j(\cdot, D)u_j(x) \;=\; N_j u_j(x) \;=\; \int_{\mathbb{R}^2} d\xi \; e^{ix\cdot\xi} n_j(x,\xi)\hat{u}_j(\xi) \tag{3.5}$$

i.e. the Normal operator is a Pseudodifferential operator (ΨDO) [5] (of order 0).

In equation (3.5) $n_j(x,\xi)$ is the *symbol* of N_j and it is also shown (in [2]) that the *principal symbol* $w_{0,j}(x,\xi)$ of $W_j \equiv {}^tF_jF_j$ is given by

$$(*)\begin{cases} w_{0,j}(x,\xi) = |\xi| \left(\frac{2m(x_r'(x,\hat{\xi}),t_r)a(x_s,x)a(x,x_r(x,\hat{\xi}))}{c^2(x)} \right)^2 E(x,\xi) \\ E(x,\hat{\xi}) \equiv |\nabla\phi(x_s,x_r(x,\hat{\xi}),x)|^{-1} \, det \begin{pmatrix} \frac{\partial}{\partial x_r'}\nabla\phi(x_s,x_r(x,\hat{\xi})) \\ {}^t\nabla\phi(x_s,x_r(x,\hat{\xi}),x) \end{pmatrix}^{-1} \end{cases}$$

Some remarks about this latter equation are in order. First $\hat{\xi}$ is the unit vector associated to ξ, and the implicit function $x_r(x,\hat{\xi})$ appears from a stationary phase approximation of the integral in (3.5). Also the function ϕ which appears is defined by the relation

$$\phi(x_s,x_r,x) = \tau(x_s,x) + \tau(x,x_r)$$

where for example $\tau(x_s,x)$ is the travel time for a ray to between the points x_s and x. Also a similar interpretation holds for the amplitudes like $a(x_s,x)$ etc. Note that the vanishing of the term $E(x,\hat{\xi})$ in $(*)$ corresponds to the situation in which the reflected ray is a continuation of the incident ray i.e. a critical refraction (see [2] for a proof of this). This is exactly the ray geometry that our hypothesis has to rule out so that we assume that $E(x,\hat{\xi})$ is always well defined and bounded strictly away from zero.

Of course the full Hilbert space normal operator which we have shown to be $\Lambda^{-1}W$ will have principal symbol equal to the product of the principal symbol of W times the pincipal symbol of Λ. Note that Λ^{-1} is itself a ΨDO of order -1 and has principal symbol equal to $|\xi|^{-1}$. Thus the principal symbol $n_{0,j}$ of N_j is given by

$$n_{0,j}(x,\hat{\xi}) = \left(\frac{2m(x_r'(x,\hat{\xi}),t_r)a(x_s,x)a(x,x_r(x,\hat{\xi}))}{c^2(x)}\right)^2 \ E(x,\hat{\xi}) \tag{3.6}$$

The significance of the principal symbol given in (3.6) above comes from the fact that we have the following expansion for the full symbol n_j of N_j (w.r.t. ξ)

$$n_j(x,\xi) = n_{0,j}(x,\hat{\xi}) + \mathcal{O}(|\xi|^{-1}) \tag{3.7}$$

which means that if we replace $n_j(x,\xi)$ in (3.5) by $n_{0,j}(x,\hat{\xi})$ then the resulting expression will be principal in the sense that the error term is smoother than this term - this is the standard use of the principal symbol of a ΨDO. Now we make an observation that will become important to us soon. Note that the quantity

$$n_{0,j}(x,\hat{\xi}) \to \infty \text{ as } j \to \infty \ , \ \forall \hat{\xi}$$

simply because geometric optics predicts that the amplitude terms a in equation (3.6) are unbounded in view of the definition of the domain sequence Ω_j and also by the assumption of no critical refractions we have that E is bounded, while all other terms in this equation are trivially bounded. Now we are ready to use this result in Gårding's Inequality [5] (page 55), which says that

$$< n_j(\cdot,D)u_j, u_j > \geq C_{1,j}\|u_j\|^2_{\frac{1}{2}} + < S_j u_j, u_j >_{(r,r)}$$

where S_j is some smoothing operator; that is we have that

$$\|F_j u_j\|_0 \ = < u_j, n_j(\cdot,D)u_j > \ \geq \ C_{1,j}\|u_j\|^2_{\frac{1}{2}} - C_{r,j}\|u_j\|^2_r \tag{3.8}$$

Here r is any real number and $C_{1,j}$ is given by

$$C_{1,j} = \frac{1}{2} \inf_{x\in\Omega_j,\ \hat{\xi}\in S^1} \ n_{0,j}(x,\xi) \tag{3.9}$$

Therefore in view of our observation and the construction of Ω_j we see that as $j \to \infty$ we get $C_{1,j} \to \infty$. Next let us fix $r = 0$.

Our goal now is to show that we can construct u_j by choosing k_j such that the first member of the right hand side of (3.8) is the dominant term. This task will be done for the two possible situations that can arise. In the first case, if $C_{0,j}$ is bounded independently of j then it is clear by the observation that the first term will become the dominant one, thus implying that the ratio

$$R_j \equiv \frac{\langle u_j, n_j(\cdot, D)u_j\rangle}{\|u_j\|^2_{\frac{1}{2}}} \;=\; \frac{\|Fu_j\|_0^2}{\|u_j\|^2_{\frac{1}{2}}}$$

is unbounded.

In the case where $C_{0,j}$ may not be bounded we can still arrange to choose the reflectivity sequence such that R_j is unbounded but we will have to work harder to show this.

Let $\epsilon > 0$ and for any $j \in \mathbb{N}$ we want to find u_j such that

$$r_j \equiv \frac{\|u_j\|_0}{\|u_j\|_{\frac{1}{2}}} C_{0,j} < \epsilon \;, \quad \forall j. \tag{3.10}$$

Having succeeded in showing this is possible then the first term on the right hand side of (3.8) is dominant and we get unboundedness of the quantity R_j as in the previous case. Now we need just prove that we can find a reflectivity sequence such that the relation (3.10) holds.

Recall that our reflectivity sequence has the form

$$u_j(x) \;=\; \chi_j(x)e^{ik_j\cdot x} \tag{3.11}$$

where $\chi_j \in C_0^\infty(\Omega_j)$. Then we have that

$$\|u_j\|^2_{\frac{1}{2}} = \int \lambda^{\frac{1}{2}}(\zeta)|\hat{u}_j(\zeta)|^2 \, d\zeta$$

$$\text{where } \lambda(\zeta) = (1+|\zeta|^2)^{\frac{1}{2}}$$

$$= \int \lambda^{\frac{1}{2}}(\zeta)\left|\int e^{ix\cdot(k_j-\zeta)}\chi_j(x)\,dx\right|^2 \, d\zeta$$

$$= \int \lambda^{\frac{1}{2}}(\zeta+k_j)\left|\int e^{-ix\cdot\zeta}\chi_j(x)\,dx\right|^2 \, d\zeta$$

$$\text{So, } \|u_j\|^2_{\frac{1}{2}} \;=\; \int \lambda^{\frac{1}{2}}(\zeta+k_j)|\hat{\chi}_j(\zeta)|^2 \, d\zeta$$

Now use Peetre's Inequality [6] (page 21)

$$\lambda^s(\omega) \le 2^{|s|}\lambda^{|s|}(\omega-\eta)\lambda^s(\eta)$$

with $s = \frac{1}{2},\ \omega = k_j,\ \eta = k_j + \zeta$ to get

$$\|u_j\|^2_{\frac{1}{2}} \geq 2^{-\frac{1}{2}}\lambda^{\frac{1}{2}}(k_j)\int \lambda^{-\frac{1}{2}}(\zeta)\ |\hat{\chi}_j(\zeta)|^2 d\zeta$$

$$= 2^{-\frac{1}{2}}\lambda^{\frac{1}{2}}(k_j)\|\chi_j\|^2_{-\frac{1}{2}}$$

therefore we have

$$r_j \leq C_{0,j}\ 2^{\frac{1}{2}}\lambda(k_j)^{-\frac{1}{2}}\frac{\|u_j\|_0^2}{\|\chi_j\|^2_{-\frac{1}{2}}}$$

$$= C_{0,j}\ 2^{\frac{1}{2}}\lambda(k_j)^{-\frac{1}{2}}\frac{\|\chi_j\|_0^2}{\|\chi_j\|^2_{-\frac{1}{2}}}$$

Next we choose the magnitude of k_j sufficiently large so that

$$2^{\frac{1}{2}}\lambda(k_j)^{-\frac{1}{2}}\ \frac{\|\chi_j\|_0^2}{\|\chi_j\|^2_{-\frac{1}{2}}}\ C_{0,j} < \epsilon,\ \forall j$$

holds; thus yielding $r_j < \epsilon$ independently of j. Note this relies on the fact that the ratio $\frac{\|\chi_j\|_0^2}{\|\chi_j\|^2_{-\frac{1}{2}}}$ is independent of k_j. This completes the second case and the proof of unboundedness of the Forward map as a global operator in the prescence of a caustic.

4 Conclusions

We have shown that the Forward map associated to the Linearized Wave equation is discontinous in the prescence of caustics. This has important consequences for Linearized Inversion techniques. Any local methods, for example gradient based algorithms, will have obvious difficulties because of the loss of differentiablity and/or continuity.

According to Rakesh [3] the Normal Operator is always a Fourier Integral Operator. Pseudodifferential Operators are a special type of the latter operators and the Normal map is such an operator when there are no caustics present. Note that Rakesh's analysis rules out any ray geometries which have turning rays present in them but it is not hard to convince oneself that wherever there is a turning ray there is also a caustic nearby. So for this reason we claim that F is bounded whenever there are no caustics present and not bounded otherwise.

As a consequence of the unboundedness of F we have that the Normal Map cannot be a Pseudodifferential Operator because if it were it would have an order m associated to it and this would mean that it would map $H_0^s(\Omega_j) \to H_{Loc}^{s-m}(\Omega_j)$ continuously but as we have just seen the operator bound grows without bound as we let Ω_j become closer to the Caustic.

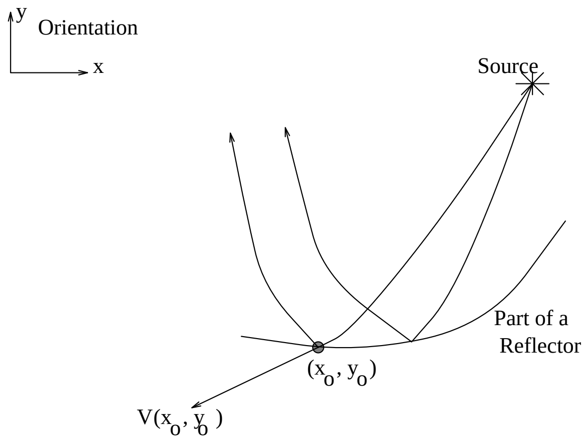

Figure 5.1: A Plane wave reflector

5 Appendix

Existence of Plane wave reflectors

In this section we will establish the intuitively obvious fact that the plane wave reflectors mentioned in the introduction do actually exist from a rigorous point of view. Consider the setup depicted in figure (5.1). First we set down some notation. Let us denote by X the two dimensional vector (x, y) as indicated in the diagram. Let V be a C^0 vector field which evaluates the unit vector parallel to the velocity vector of a ray leaving the source and passing through the point of evaluation (see diagram for example).

Now specify the direction in which you want the plane wave to be reflected i.e. specify a vector $r = (r_1, r_2)$; this choice will have to be restricted a little later but is more or less arbitrary in general. Then let $N_0 \in T_{(x_0,y_0)}\mathbb{R}^2$ be of the form

$$N_0 = N(X_0) = N(x_0, y_0) = \frac{1}{\sqrt{1+\xi_0^2}}(-\xi_0, 1)$$

(The reason of this choice is that eventually N will be the normal vector of the graph generated by $y = \psi(x)$; for some C^1 function ψ.) Also choose N in such a way that the following equation which expresses "Snell's Law of Reflection" is satisfied

$$N_0 \cdot (r + V(X_0)) = 0$$

thus we have that

$$-r_1\xi_0 - V_1(X_0)\xi_0 + r_2 + V_2(X_0) = 0 \tag{5.1}$$

In general we want this latter equation to hold with X_0 replaced by X and ξ_0 by $\xi(x)$ i.e. suppose we can have some reflecting curve given by the graph of $y = \psi(x)$, then the unit normal vector to it is given by

$$\frac{1}{\sqrt{1+\xi^2(x)}}(-\xi(x), 1)$$

where $\xi(x) = \nabla\psi(x)$. Therefore our task is complete if we can solve (locally) the equation

$$F(x,\xi) = -r_1\xi - V_1(x,y(x))\xi + r_2 + V_2(x,y(x)) = 0 \tag{5.2}$$

(with $\xi = \xi(x)$) since then all we have to do is find the function ψ by solving the ordinary differential equation

$$\begin{cases} \nabla\psi(x) = -\xi(x) \\ \\ \psi(x_0) = y_0 \end{cases}$$

The question of local solvability of the equation (5.2) is answered by the Implicit Mapping Theorem which answers "yes" to the local existence of a C^0 function $\xi(x)$ satisfying equation (5.2) provided that $\frac{\partial F}{\partial\xi}(X_0,\xi_0) \neq 0$ i.e. $r_1 + V_1(X_0) \neq 0$. Thus the choice of r_1 is restricted in a trivial way. This completes the proof of local existence of plane wave reflectors.

Acknowlegement This work was partially supported by the National Science Foundation, the Office of Naval Research, the Texas Geophysical Parallel Computation Project, the Schlumberger Foundation, and The Rice Inversion Project. TRIP Sponsors for 1994 are: Advance Geophysical, Amoco Production Co., Conoco Inc., Cray Research Inc., Exxon Production Research Co., Interactive Network Technologies, Mobil Research and Development Corp., and Texaco Inc.

References

[1] Cheryl B. Purcell. The effects of Caustics in Acoustic Inverse Scattering Experiments. Ph.D. Thesis 1989, Department of Mathematical Sciences, Rice University, Houston, Tx 77005, TR89-3.

[2] William W. Symes. Mathematical Foundations of Reflected Wave Imaging. Department of Mathematical Sciences, Rice University, Houston, Tx 77005, TR90-2.

[3] Rakesh. Commun. In Partial Differential Equations, 13(5), 573-601 (1988).

[4] J.J. Duistermaat. Fourier Integral Operators (1971), Lecture Notes, Courant Institute of Mathematical Sciences.

[5] Michael E. Taylor. Pseudodifferential Operators (1981). Princeton University Press.

[6] Xavier Saint Raymond. Elementary Introduction to the Theory of Pseudodifferential Operators (1991). Studies in Advanced Mathematics, CRC Press.

[7] R. Courant and D. Hilbert. Methods of Mathematical Physics, Vol. II (1962), published by Wiley.