On Crosswell Traveltime Tomography

Hua Song* **William W. Symes***

Abstract

In this paper, we give a theoretical analysis of the crosswell traveltime tomography. In the single arrival case, we prove that the objective function is smooth, and every critical point is a kinematically correct velocity model. A numerical example is presented to demonstrate the properties we prove and the difference between traveltime tomography and waveform inversion via the output least squares. These results are extended to a local formulation of traveltime tomography in the presence of multiple arrivals.

1 Introduction

In crosswell seismic experiments, seismic sources are fired in one well, and the wavefields generated are measured in another well. The goal of the crosswell seismology is to find physical parameters, especially the velocities, of the rocks between the wells from these measurements. This amounts to the mathematical problem of solving a coefficient inverse problem of the multidimensional acoustic wave equation.

Crosswell inverse problems are important in reservoir characterisation and enhanced oil recovery (Justice *et al*., 1989). The more or less conventional method to resolve these problems is called *traveltime tomography*. One extracts the traveltime information from the seismic data, and looks for a model to fit the traveltime data in the least-squares sense.

Some theoretical analyses on linearized traveltime tomography from reflection data are found in Bube (1992) and Delprat-Jannaud and Lailly (1991). Bube studied the uniqueness of determining reflector depths and non-uniqueness of determining velocity using infinitely many sources and receivers. For that particular problem, he gave a characterization of the null space of velocity , which has an immediate analogue for traveltime tomography in the crosswell setting. Hence for many reasonable experiment configurations, both the traveltimes as a function of velocity and its derivative as a function of velocity perturbation is not injective, and the least squares problem has many solutions when it has one. This situation occurs when the data is noise-free. Since this non-uniqueness is intrinsic to traveltime tomography, by a "solution" we will mean any velocity model wich produces the right traveltimes, and we will say this velocity model is *kinematically correct.* Since the above mentioned least squares problem is ill-posed, some kind of regularization is need. Delprat-Jannaud and Lailly suggested to regularize the problem by asking more smoothness of the model, and studied approximations of the regularized problem in finite dimensional spaces. For nonlinear traveltime tomography, except the above mentioned non-uniqueness, very little is known theoretically.

*The Rice Inversion Project, Department of Computational and Applied Mathematics, Rice University, Houston TX 77251-1892

All the above mentioned results assume either there is only one arrival time at a receiver location or only the first arrival is used in the presence of multiple arrivals. The difficulty in dealing with multiple arrivals lies in the fact that the traveltime data are parameterized by reveiver location, hence the traveltime function is not single-valued when multiple arrivals are in presence. Lailly *et al* (1992) propose a new formulation of the traditional traveltime tomography which split the traveltime data into two groups: traveltimes and emerging locations, and reparameterizes these two sets of data with the ray parameters. These two functions are always single valued since for a single ray, there is at most one traveltime and emerging point at the receiver surface. The least squares criterion is suggested to match traveltimes and emerging locations for an estimated velocity model with those calculated from the traveltime data with the same ray parameter. The key point in this new formulation is that for a receiver location and a traveltime, the ray parameter of the corresponding ray can be easily calculated from the data as long as there are a densely placed sources. We have not seen any analysis of this formulation.

As far as numerical experiments are concerned, a lot of work has shown that traveltime tomography behaves well and Newton-like methods converge in a large range of the model. For numerical experiments, see Bishop *et al 1985*, Chiu and Stewart 1987, Scales 1987, Herman 1992, and Lailly *et al* 1992. The main drawback of this approach to the inverse problem is that the traveltimes are not primary data of the seismic experiments. Traveltimes must be picked from the waveform data. This picking procedure is often manual, and can be difficult when the structure of the medium is complex. Phase errors are difficult to avoid in the presence of multi-arrivals. Full waveform inversion using output least squares has been suggested by Tarantola 1986, Kolb *et al* 1986, Gauthier *et al* 1986, etc.. Unfortunately, the output least squares objective function is not a smooth function of the velocity, and usually it has many local minimizers (see Herman 1992), hence expensive global optimization methods are needed to locate the global minimizer. For more discussion on waveform inversion, see Song and Symes 1994.

Although traveltime tomography has its shortcomings, it is still an effective inversion method in some cases. Also, a better understanding of this method, and the kinematics of wave propagations in general, is helpful in designing and analyzing new waveform inversion methods (see Song and Symes 1994). In this paper, we give a theoretical analysis of the crosswell traveltime tomography. In the single arrival case, we prove that the objective function is smooth, and every critical point is a kinematically correct velocity model. A numerical example is presented to demonstrate the properties we prove and the difference between traveltime tomography and the output least squares waveform inversion. These results are extended to a local formulation of traveltime tomography in the presence of multiple arrivals.

2 Single Arrival Case

2.1 Formulation

Typical crosswell seismic experiments involve at least two vertical wells in the earth. Sources (e.g., dynamite, air guns, etc.) are fired in some wells, and the wavefields generated are recorded in other wells. Suppose that there is only one source well and one receiver well, and only one source is fired in the source well (see Fig. 2.1). If we use the notation $X = (x, y, z)$ for a point in "flat" earth with z being the depth, $v(X)$ for the velocity field, $\rho(X)$ for the density field, t for time, $p(X, t)$ for pressure field, $f(t)$ for source wavelet, and X_s for a source location , then the pressure field generated by the source with source wavelet $f(t)$ at location

X_s is approximately governed by the acoustic wave equation

$$\frac{1}{v(X)^2\rho(X)}\frac{\partial^2 p}{\partial t^2} - \nabla\cdot(\frac{1}{\rho(X)}\nabla p) = f(t)\delta(X - X_s)$$

where δ is the Dirac delta function, with causality intitial condition

$$p(X,t) = 0 \qquad \text{for } t << 0$$

and an appropriate boundary condition. In order to make the idea simple, we will ignore the boundary, hence the refelections from the boundary. Since it is very under-determined to find the function v from traveltime data, we assume that the source-receiver plane is perpendicular to the y-axis, and v is independent of y. A consequence of this assumption is that all rays which start with directions within the source-receiver plane will always stay in this plane. This fact will be used later in our analysis of the gradient of the objective function.

With our non-caustic assumption about the velocity model, the retarded Green's function is

$$g(X, X_s, t) = a(X)\delta(t - \phi(X)) + \text{smoother terms}$$

where ϕ is the traveltime function which satisfies the eiconal equation

$$|\nabla\phi|^2 = \frac{1}{v^2}$$

and a is the amplitude which satisfies the transport equation

$$\nabla a\cdot\nabla\phi + \frac{1}{2}a\Delta\phi = 0$$

By convolving the source wavelet f and the retarded Green's function in t, we get an expansion of the wavefield

$$p(X,t) = a(X)f(t - \phi(X)) + \text{smoother terms}$$

Usually f is oscillatory, hence the first term in the above expansion contains most energy in the seismogram, and the data (seismogram) recorded at the receiver well will be approximated as

$$p(X_r, t) = a(X_r)f(t - \phi(X_r))$$

where $X_r = (x_r, y_r, z_r)$ is the receiver location, x_r and y_r are fixed, and $0 \le z_r \le Z$. Since usually the source wavelet has a distinct peak, the graph of the data also has distinct peaks, sometimes they can be picked out, which gives the traveltime data. Traveltime tomography tries to find a velocity model which gives the best fit in traveltimes, using some criterion , e.g., least squares. Be aware that traveltimes are not always easy to pick. This is the a shortcoming of this method and the main reason that other methods which use the raw (waveform) data directly might be desirable.

Let $\tau_d(z_r) : 0 \le z_r \le Z$ be the traveltime data extracted from the seismogram:

$$\tau_d(z_r) = \phi_d(X_r),$$

where $\phi_d(X_r)$ is the traveltime field generated by the true velocity model, and $\tau[v](z_r) : 0 \le z_r \le Z$ be traveltime for a given velocity v:

$$\tau[v](z_r) = \phi[v](X_r),$$

where $\phi[v](X_r)$ is the traveltime field generated by the velocity model v. A simple least squares formulation is

$$\min_{v \in V} J[v]$$

where V is a bounded set of velocites which will be defined below, and

$$J[v] = \frac{1}{2}\|\tau[v] - \tau_d\|^2_{L^2([0,Z])}$$

The source-receivers plane is the $x - z$ plane. The velocities depend on x and z only. We will study the smoothness and the gradient and hessian of the objective function. The cases involving multiple wells, multiple sources, can be studied similarly.

We will use the following finite dimensional set for velocity, which simplifies the analysis without hurting our goal. We take a linearly independent subset $\{v_i : i = 1, 2, ..., n\}$ of $C^\infty(\Omega)$, where $\Omega = [x_s, x_r] \times [0, Z]$, here $v_i, i = 1, ..., n$, are positive and bounded above and below away from zero, and define a set

$$V = \{\sum_{i=1}^{n} c_i v_i : c_i \in I_i \subset R, i = 1, 2, ..., n\}$$

where $I_i, i = 1, 2, ..., n$, are bounded intervals such that for some constants $0 < v_{min} < v_{max}$, $v \in V$ implies $v_{min} \leq v(x, y, z) \leq v_{max}$. We also assume that V contains the velocity model v^* which produces the traveltime data τ_d, and that v_i 's and I_i 's are chosen in such a way that for each $v \in V$, all the rays connecting the source location and the receiver locations stay in the region $(x_s, x_r) \times [0, Z]$ and do not intersect. The existence of such a set V will become clear in the proof of smooth dependence of the traveltime function on velocity. Under these assumptions, $a[v]$ and $\tau[v]$, as solutions of the eiconal equation and transport equation respectively, are uniformly bounded above and below by positive constants, and their derivatives are uniformly bounded above. We regard all the functions of the velocity as defined on V, and by smoothness of their dependence on v we mean they are smooth as functions of $c = (c_1, c_2, ..., c_n) \in R^n$ through v, although we will not treat them as functions of c explicitly. Implicitly we identify V as subset of R^n and equip it with the euclidean norm.

2.2 Smoothness

All the analysis in this section depends on the theory of propagation of singularities. An indispensable tool in this theory is the Hamilton system. For the purpose of later use, we introduce one version of the Hamiltonian system governing the rays associated to the acoustic wave propagation and some terminology. Let the Hamiltonian be

$$H(x, z, t, \xi, \eta, \tau) = \frac{1}{2}\{v(x, z)^2(\xi^2 + \eta^2 + \zeta^2) - \tau^2\}.$$

Then the Hamiltonian system is

$$\begin{aligned}
\dot{x} &= \partial_\xi H = v(x, z)^2 \xi \\
\dot{y} &= \partial_\eta H = v(x, z)^2 \eta \\
\dot{z} &= \partial_\zeta H = v(x, z)^2 \zeta \\
\dot{t} &= -\tau \\
\dot{\xi} &= -\partial_x H = -v(x, z)\partial_x v(x, z)(\xi^2 + \eta^2 + \zeta^2)
\end{aligned}$$

$$
\begin{aligned}
\dot{\eta} &= -\partial_y H = -v(x,z)\partial_y v(x,z)(\xi^2+\eta^2+\zeta^2) \\
\dot{\zeta} &= -\partial_z H = -v(x,z)\partial_z v(x,z)(\xi^2+\eta^2+\zeta^2) \\
\dot{\tau} &= 0
\end{aligned}
$$

where " $\dot{}$ " means $\frac{d}{ds}$ and s is a parameter along a ray. The trajectories of this ordinary differential system are called *bicharacteristic strips.* It is easy to check that H is constant on the bicharacteristic strips. The bicharacteristic strips on which $H = 0$ are called *null bicharacteristic strips.* The projections of the null bicharacteristic strips into the (x, y, z) space are called $(characteristic)$ *rays.* Since v is independent of y in our case, from the sixth and the second equations we know that the rays which start with directions in the $x - z$ plane will stay in the plane forever, and for these rays, $\eta = 0, y = 0$. Hence we can treat the problem as a two-dimensional one, and delete the second and sixth equations. From the last equation in the above system we see that τ is a constant, and we only consider the problem forward in time, we can take $\tau = -1$. Now the Hamiltonian system becomes

$$
\begin{aligned}
\frac{dx}{dt} &= v(x,z)^2\xi \\
\frac{dz}{dt} &= v(x,z)^2\zeta \\
\frac{d\xi}{dt} &= -v(x,z)\partial_x v(x,z)(\xi^2+\zeta^2) \\
\frac{d\zeta}{dt} &= -v(x,z)\partial_z v(x,z)(\xi^2+\zeta^2)
\end{aligned}
$$

which in fact is the Hamiltonian system with Hamiltonian $\frac{1}{2}\{v(x,z)^2(\xi^2+\zeta^2)-1\}$. Under our assumption that v does not produce caustics between the wells, $\dot{x} = \frac{dx}{dt}$ is never zero between the two wells. Hence we can reparametrize the rays by x. Also note that for the null-characteristics,

$$\xi^2+\zeta^2 = 1/v(x,z)^2$$

The fact $\dot{x} \neq 0$ also implies that ξ does not change sign. Since for all rays connecting the source and receivers, the initial values of ξ are positive, we can solve ξ from the identity above. Hence

$$\xi = \sqrt{1/v(x,z)^2-\zeta^2}$$

The Hamiltonian system above can then be rewriten as

$$
\begin{aligned}
\frac{dt}{dx} &= \frac{1}{v(x,z)^2\sqrt{1/v(x,z)^2-\zeta^2}} \\
\frac{dz}{dx} &= \frac{\zeta}{\sqrt{1/v(x,z)^2-\zeta^2}} \\
\frac{d\zeta}{dx} &= -\frac{v_z(x,z)}{v(x,z)^3\sqrt{1/v(x,z)^2-\zeta^2}}
\end{aligned}
$$

Given initial conditions, say,

$$
\begin{aligned}
t(0) &= t_0 \\
z(0) &= z_0 \\
\zeta(0) &= \zeta_0
\end{aligned}
$$

we can trace the ray with the starting position (x_0, z_0) and starting direction $(\xi_0 = \sqrt{1/v(x_0,z_0)^2 - \zeta_0^2}, \zeta_0)$ through the medium by solving the initial value problem of the Hamilton system.

If the receiver is located at (x_r, z_r), then solve a two-point boundary value problem, if it is solvable, with boundary conditions

$$\begin{aligned} t(0) &= 0 \\ z(0) &= z_s \\ z(x_r) &= z_r \end{aligned}$$

the traveltime at the receiver is

$$\tau[v](z_r) = t(x_r)$$

In order to use the smooth dependence on parameters and initial values results of initial value problems of ODEs, we try to find an initial value for ζ such that the initial value problem with initial conditions

$$\begin{aligned} t(0) &= 0 \\ z(0) &= z_s \\ \zeta(0) &= \zeta_s \end{aligned}$$

is equivalent to the above two point boundary value problem. Define map

$$F(\zeta(0)) = z(x_r)$$

where $z(x_r)$ is part of the solution to the two point boundary value problem. The smooth invertibility of this map corresponds to the absense of caustics.

Apply the smooth dependence result of solutions of ODEs on parameters (Lang 1993, Chapter XIV, Theorem 4.3) , we get

proposition 2.1 *$\tau[v](z_r)$ is $C^p, p \geq 1$ is an integer, as a function of $v \in V, z_r \in [0, Z]$ in a neighborhood of v^*.*

Consepuently, we have

Theorem 2.1 *$J[v]$ is smooth in $v \in V$.*

2.3 Gradient and A Characterization of Critical Points

In this subsetion, we investigate the gradient of the objective function J, and give a characteriztion of the critical points. In order to analyze the gradiet of $J[v]$, we need the following lemmas.

Lemma 2.1 *The directional derivative of τ with respect to v in direction δv is*

$$D\tau[v]\delta v(z_r) = \oint_{X_s}^{X_r} \left(-\frac{\delta v}{v^2}\right)(x(s), z(s))ds,$$

where the integration path is the raypath from X_s to X_r, and s is the ray-length from the source location X_s.

Proof. From the Eiconal equation

$$|\nabla\phi[v]|^2 = \frac{1}{v^2},$$

we have

$$\nabla D\phi[v]\delta v \cdot \nabla\phi[v] = -\frac{\delta v}{v^3},$$

i.e.,

$$\nabla D\phi[v]\delta v \cdot (v\nabla\phi[v]) = -\frac{\delta v}{v^2}.$$

Since $v\nabla\phi[v]$ is the unit tangential direction along the ray, we have

$$D\phi[v]\delta v(X) = \oint_{X_s}^{X} \left(-\frac{\delta v}{v^2}\right)(x(s), z(s))ds,$$

hence,

$$\begin{aligned} D\tau[v]\delta v(z_r) &= D\phi[v]\delta v(X_r) \\ &= \oint_{X_s}^{X_r} \left(-\frac{\delta v}{v^2}\right)(x(s), z(s))ds. \end{aligned}$$

Q.E.D.

Now we analyze the gradient. As a preparation result, first we prove that, unlike $D\tau[v]$, its L^2 adjoint $(D\tau[v])^*$ is injective under our assumptions about V. In fact, we have

Lemma 2.2 *The adjoint operator $(D\tau[v])^*$ maps $L^2([0, Z])$ into $L^2(\Omega)$, and there exists a positive constant C_0 such that for any $v \in V$ and any $g \in L^2([0, Z])$,*

$$\|(D\tau[v])^* g\| \geq C_0\|g\|. \tag{2.1}$$

Proof. First we derive the formula for $(D\tau[v])^*g$. For any $z_r \in [0, Z]$, there is a ray linking the source X_s and the point X_r. For any point on the ray with x-axis $x \in [x_s, x_r]$, we denote its z-coordinate by $z(x; z_r)$. Then we have

$$\begin{aligned} \int_0^Z (D\tau\delta v)(z_r)g(z_r)dz_r &= \int_0^Z \left\{\oint_{X_s}^{X_r} \left(-\frac{\delta v}{v^2}\right)(x(s), z(s))ds\right\} g(z_r)dz_r \\ &= \int_0^Z \left\{\int_{x_s}^{x_r} \left(-\frac{\delta v}{v^2}\right)(x, z(x; z_r))J_1(x; z_r)dx\right\} g(z_r)dz_r \end{aligned}$$

where $J_1(x; z_r)$ is the jacobian of the change of variable from s to x on the ray linking the source X_s and the receiver X_r. Now make another change of variable from $z_r \in [0, Z]$ to z for each fixed x, we get

$$\int_0^Z (D\tau\delta v)(z_r)g(z_r)dz_r = \int_0^Z \int_{x_s}^{x_r} \left(-\frac{\delta v}{v^2}\right)(x, z)J_1(x; z_r(z; x))g(z_r(x, z))J_2(z; x)dxdz \tag{2.2}$$

where J_2 is the jacobian of the change of variable from z_r to z from a fixed x. Note that this change of variables is defined only in the part of the region covered by rays. Now it is clear that $(D\tau[v])^*$ is defined by

$$(D\tau[v])^* g(x, z) = \left(-\frac{1}{v^2}\right)(x, z)J_1(x; z_r(z; x))g(z_r(x, z))J_2(z; x)\chi(x, z)$$

where χ is the characteristic function of the region covered by rays. This formula is interpreted as spreading the value of g at z along the ray linking X_r and X_s, or "backward projection", on the part covered by rays. Hence,

$$\begin{aligned}\|(D\tau[v])^*g\|^2 &= \int_0^Z \int_{x_s}^{x_r} \{(D\tau[v])^*g(x,z)\}^2 dxdz \\ &= \int_0^Z \int_{x_s}^{x_r} \left\{\left(-\frac{1}{v^2}\right)(x,z)J_1(x;z_r(z;x))g(z_r(x,z))J_2(z;x)\chi(x,z)\right\}^2 dxdz.\end{aligned}$$

Make an inverse change of variable from z to z_r for each fixed x, we get

$$\|(D\tau[v])^*g\|^2 = \int_0^Z \int_{x_s}^{x_r} \left\{\left(-\frac{1}{v^2}\right)(x,z(z_r;x))J_1(x;z_r)g(z_r)J_2(z(z_r;x);x)\right\}^2 dxdz_r\ .$$

We know that v is bounded below, and $J_1(x;z_r) = |\frac{dx}{ds}|^{-1} = (2|\nabla\phi|)^{-1} = \frac{v}{2}$ is bounded below. The "non-caustic" assumption implies that $J_2(z(z_r;x);x)$ is bounded below. Hence, there is a constant C_0, which is independent of g, such that

$$\|(D\tau[v])^*g\| \geq C_0\|g\|.$$

q.e.d.

Let $\tau_d(z_r)$ be the noise-free traveltime data extracted from the waveform data $S_{data}(z,t)$ generated by velocity model v^* , i.e., $\tau_d = \tau[v^*]$. It can be calculated that the gradient of $J[v]$ is defined by

$$DJ[v]\delta v = \langle D\tau[v]\delta v, \tau[v] - \tau[v^*]\rangle = \langle \delta v, (D\tau[v])^*(\tau[v] - \tau[v^*])\rangle$$

Hence the L^2 gradient of J is

$$GradJ[v] = (D\tau[v])^*(\tau[v] - \tau[v^*])$$

Thus, according to the previous lemma,

$$\|GradJ[v]\| \geq C\|\tau[v] - \tau[v^*]\|$$

where C is independent of v, under the non-caustics assumption. This means that all critical points of J give the same traveltimes, hence they belong to the set of all kinematically correct velocity models. It must be noticed that, with a single, or sometimes even with multiple sources, one can not expect this set to be a singleton, or in other words, the "null space" of the traveltime tomography to be empty (see the first section).We summarize the above discussion as

Theorem 2.2 *All local extrema interior to V of $J[v]$ are global minimizers, and they are kinematically correct velocities.*

We use a simple numerical example to demonstrate the properties of the objective function and the difference between traveltime tomography and output least squares waveform inversion. The distance between the two wells is 100 meters. Source is located at the depth 125 meters and 21 receivers are placed evenly in the receiver well, 12.5 meters apart, from the surface of the earth down to depth 250 meters. The sample interval is .25 mini seconds, and there are 501 samples per trace. Fig. 2.2 is a Ricker source wavelet with peak frequency 200 hz. Fig. 2.3 is the seismogram using the above source wavelet and the homogeneous velocity model $v = 2000m/s$. Fig. 2.4 is the output least squares objective function. Fig. 2.5 is traveltime data from the same experiment. Fig. 2.6 is traveltime tomography objective function, which is smooth and convex, hence local minimization methods, i.e., Newton-like methods, can be used to locate the global minimizer.

2.4 Multiple Sources

If more than one sources are used in the sources well, we can formulate and analyze the traveltime inversion in a similar fashion. We parametrize a source by an index $s \in S$, where S is the finite set of all source indices. We still use the velocity space V defined in Section 2.1, and assume that the assumptions for the single source case in Section 2.1 holds for each source in the multiple source case. Let τ_d^s be the traveltime data generated by the source located at X_s, and let $\tau^s[v]$ be simulated trveltime data with velocity v and the source located at X_s. The traveltime inversion is formulated as a minimization problem with objective function

$$J[v] = \frac{1}{2}\sum_{s\in S}\|\tau^s[v] - \tau_d^s\|^2$$

Similar calculation as in Section 2.3 leads to

$$\begin{aligned} Grad J[v](x,z) &= \sum_{s\in S}(D\tau^s[v])^*(\tau^s[v] - \tau_d^s) \\ &= \sum_{s\in S}\left(-\frac{1}{v^2}\right)(x,z)J_1^s(x;z_r(z;x))J_2^s(z;x)(\tau^s[v] - \tau_d^s)(z_r(x,z))\chi^s(x,z) \end{aligned}$$

where J_1^s and J_2^s are the two jacobians in the proof of Lemma 2.2, respectively, for the source at X_s, and χ^s is the characteristic function of the region covered by rays connecting the source location X_s and all the receiver locations (see Fig 2.7). Since these regions have disjoint subregions near the source well, we can find $x_c \in (x_s, x_r)$ such that

$$Grad J[v] = 0$$

implies

$$\left(-\frac{1}{v^2}\right)(x,z)J_1^s(x;z_r(z;x))J_2^s(z;x)(\tau^s[v] - \tau_d^s)(z_r(x,z))\chi^s(x,z) = 0$$

for all $s \in S$, for $x_s < x < x_c$, and all $z \in [0, Z]$, which in turn implies that

$$(\tau^s[v] - \tau_d^s)(z_r) = 0,$$

for all $s \in S$, and all $z_r \in [0, Z]$. Hence v is a global minimizer.

3 Multiple Arrivals – The Beam-Forming Idea

Traditional traveltime tomography either assumes that data do not contain multiple arrivals (e.g., triplications) caused by the presence of caustics (envolopes of rays), as in Chapter 2, or gets rid of the multi-valued feature of the traveltime function by using only the first arrival. Traveltime tomography using only the first arrival can be analyzed in a similar fashion as in Chapter 2, but it is not satisfactory since the first arrivals are usually very weak and do not carry information of the most interesting features of the medium (Geoltrain and Brac, 1992). A simple example is a medium which is homogeneous except a local low velosity zone. If the receiver line is far enough away from the low velocity zone, a triplication will be seen in the traveltime curve, and most rays which go through the low velocity zone give later arriving traveltimes. Hence traveltime inversion using the first arrivals will not be able to give a reasonable image of the low velocity zone (see Figures 3.1 - 3.3).

The difficulty in dealing with multiple arrivals lies in the fact that the traveltime data are parameterized by receiver location, hence the traveltime function is not single-valued when multiple arrivals are in presence. In this section, we will present a way to deal with this difficulty.

The "beam-forming" approach is based on the idea that different branches in the traveltime data correspond to different groups of set-off angles of rays, hence the rays can be grouped in a way that each group generates a single arrival. We are only be able to formulate this approach locally, and will use the triplication as a simple example to illustrate the idea. We assume that the data contains a simple triplication, and the velocity model which produces this data is v^*. According to the smooth dependence result of solutions of ODEs on parameters, any velocity in a neighorhood of v^* will also produce a triplication in data. The three branches in the data can be seperated, and for a given velocity model in the neighborhood of v^*, all the rays can be grouped into three groups by raytracing in the following way: begin ray-tracing with the starting direction turning clockwise, a group is formed and the next group is started when the emerging location of the ray begin to turn back (see Fig. 3.4 – 3.6). Each group of rays formed in this way will produce a single valued branch. We denote the three branches in the data by $\tau_d^i, i = 1, 2, 3$, respectively, from up down, and the three branches for a given velocity model v by $\tau^i[v], i = 1, 2, 3$, in the same order. Since τ_d^i and $\tau^i[v]$ may have different supports, we shrink the supports, which are closed intervals, of τ_d^i, i=1,2,3, at the ends on caustics properly such that for any v in the above-mentioned neighborhood of v^*, support of $\tau^i[v]$ contains the shrinked support of τ_d^i. We denote these shrinked closed intervals by $I^i, i = 1, 2, 3.$, then all the traveltime functions defined above are single-valued smooth functions of the receiver location z in one of the intervals I^i's. The objective function is

$$J[v] = \sum_{i=1}^{3} \frac{1}{2} \|\tau^i[v] - \tau_d^i\|^2_{L^2(I^i)}$$

The directional derivative in direction δv is

$$DJ[v]\delta v = \langle \delta v, \sum_{i=1}^{3} (D\tau^i[v])^*(\tau^i[v] - \tau_d^i)\rangle$$

where

$$(D\tau^i[v])^*(\tau^i[v] - \tau_d^i)(x, z) =$$
$$\left(-\frac{1}{v^2}\right)(x, z) J_1^i(x; z_r(z; x)) J_2^i(z; x)(\tau^i[v] - \tau_d^i)(z_r(x, z))\chi^i(x, z), \quad , i = 1, 3$$

where J_1^i and J_2^i are the two jacobians in the proof of Lemma 2.2, respectively, for ith group of rays , and χ^i is the characteristic function of the region covered by the ith group of rays. Now we consider $(D\tau^2[v])^*$. We know that

$$D\tau^2[v]\delta v(z_r) = \oint_{X_s}^{X_r} \left(-\frac{\delta v}{v^2}\right)(x(s), z(s))ds,$$

and

$$\langle D\tau^2[v]\delta v, \tau^2[v] - \tau_d^2\rangle = \int_0^Z \left\{ \oint_{X_s}^{X_r} \left(-\frac{\delta v}{v^2}\right)(x(s), z(s))ds \right\} (\tau^2[v] - \tau_d^2)(z_r)dz_r$$

As in the proof of Lemma 2.2, we can do change of variable from s to x,

$$\langle D\tau^2[v]\delta v, \tau^2[v] - \tau_d^2\rangle = \int_0^Z \left\{ \int_{x_s}^{x_r} \left(-\frac{\delta v}{v^2}\right)(x, z(x; z_r))dx J_1^2(x; z_r)dx \right\} (\tau^2[v] - \tau_d^2)(z_r)dz_r$$

For all fixed x except the $x = x^*$ corresponding to the cusped caustic (see Fig. 3.7) , we can do change of variable from z_r to z. This change of variable is regular if x is to the left of x^* and multi-valued, and singular on caustics if x is to the right of x^*. The caustics except the cusp are folding singularities, and the multi-value nature of this change of variable is caused by this folding. This multi-valued change of variable can be written down more precisely, but since it is not a key point in our reasoning, we will ignore this multi-valuedness and write it as a single change of variable. We denote the jacobian of this change of variable by $J_2^2(z;x)$, which is singular on caustics, then

$$\begin{aligned}&\langle D\tau^2[v]\delta v, \tau^2[v] - \tau_d^2\rangle \\ &= \int_0^Z \int_{x_s}^{x_r} \left(-\frac{\delta v}{v^2}\right)(x,z) J_1^2(x; z_r(z;x)) J_2^2(z;x)(\tau^2[v] - \tau_d^2)(z_r(x,z))\chi^2(x,z)\,dx\,dz\end{aligned}$$

where χ^2 is the characteristic function of the region covered by the second group of rays. Hence the L^2 gradient of $J[v]$ is

$$\begin{aligned} Grad J[v](x,z) &= \sum_{i=1}^{3} (D\tau^i[v])^*(\tau^i[v] - \tau_d^i)(x,z) \\ &= \sum_{i=1}^{3} \left(-\frac{1}{v^2}\right)(x,z) J_1^i(x; z_r(z;x)) J_2^i(z;x)(\tau^i[v] - \tau_d^i)(z_r(x,z))\chi^i(x,z) \end{aligned}$$

When x_0 is close enough to x_s, the vertical line with x-coordinate x intersects the three regions in three disjoint line segments, hence

$$grad J[v](x_0, z) = 0$$

implies

$$\left(-\frac{1}{v^2}\right)(x,z) J_1^i(x; z_r(z;x)) J_2^i(z;x)(\tau^i[v] - \tau_d^i)(z_r(x,z))\chi^i(x_0,z) = 0, \quad i = 1,2,3$$

which in turn implies

$$(\tau^i[v] - \tau_d^i)(z_r) = 0 \quad \text{for } z_r \in I^i,\ i = 1,2,3$$

Hence the synthetic traveltime data generated by velocity model v coincide with the data on the intervals $I^i, i = 1,2,3$. Hence this local beam-forming method has similar properties as the traveltime tomography with single arrivals, i.e., the objective function is smooth and locally every critical point is a velocity model which reproduces the traveltime data, though only mostly.

Acknowlegement This work was partially supported by the National Science Foundation, the Office of Naval Research, the Texas Geophysical Parallel Computation Project, the Schlumberger Foundation, and The Rice Inversion Project. TRIP Sponsors for 1994 are: Advance Geophysical, Amoco Production Co., Conoco Inc., Cray Research Inc., Exxon Production Research Co., Interactive Network Technologies, Mobil Research and Development Corp., and Texaco Inc.

Bibliography

1. T.N.BISHOP, K.P.BUBE,R.T.CUTLER,R.T.LAINGAN,P.L.LOVE, J.R.RESNICK, R.T.SHUEY,D.A.SPINDLER,and H.W.WYLD. Tomographic determination of velocity and depth in laterally varying media. *Geophysics*, 50:903-923,1987.
2. K.P.BUBE. Uniqueness of reflector depths and characterization of the slowness null space in linearized seismic reflection tomography. Manuscript, 1992.
3. S.CHIU and R.STEWART. Tomographic determination of three-dimensional seismic velocity structure using well logs, vertical seismic profiles, and surface seismic data. *Geophysics*, 52:1085-1098,1987.
4. F.DELPRAT-JANNAUD and PATRICK LAILLY. What information is contained in reflection tomography data? Part I. Manuscript, 1991.
5. O.GAUTHIER, A.TARANTOLA, and J. VIRIEUX. Two-dimensional nonlinear inversion of seismic waveforms. *Geophysics*,51:1387-1403,1986.
6. G.C.HERMAN. Generalization of traveltime inversion. *Geophysics*, 57:9-14,1992.
7. J.H.JUSTICE,A.A.VASSILIOU,S.SINGH,J.D.LOGEL,P.A.HANSEN, B.R.HALL, P.R.HUTT,J.J.SOLANKI. Acoustic tomography for monitoring enhanced oil recovery. *Geophysics: The Leading Edge of Exploration*,1989.
8. P. KOLB, F.COLLINO, and P. LAILLY. Prestack inversion of a 1d medium. *Proceedings of IEEE* 74, papes 498-506, 1986.
9. P. LAILLY, F. DELPRAT-JANNAUD, E. BECACHE, and E. CHOVET. Reflection tomography: how to cope with multiple arrivals? Extended Abstracts, SEG 62th annual international meeting and exposition, New Orleans, 1992. 741-744.
10. S.LANG. Real and Functional Analysis. Third Edition. Springer-Verlag,1993.
11. H. SONG and W. W. SYMES. Inversion of Crosswell Seismograms via Differential Semblence Optimization. *TRIP Annual Report*, 1994.
12. A.TARANTOLA. Inversion of seismic reflection data in the acoustic approximation. *Geophysics*, 51:1893-1903,1986.

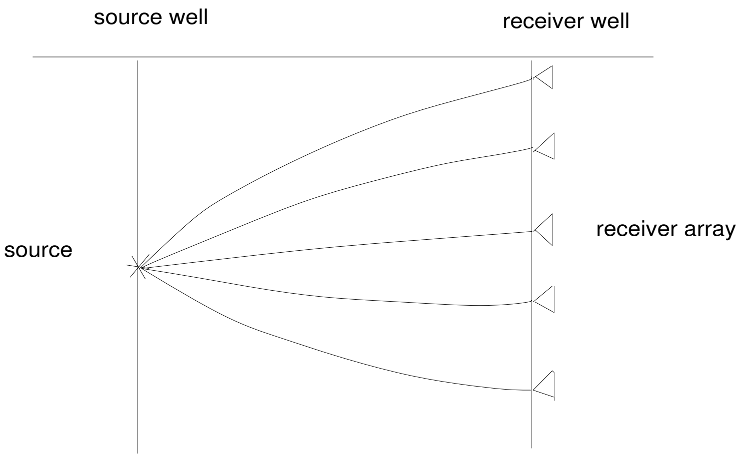

FIGURE 2.1: Experiment visualization: one source.

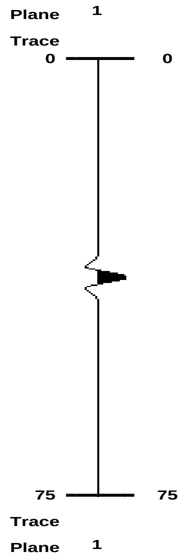

FIGURE 2.2: Ricker source wavelet.

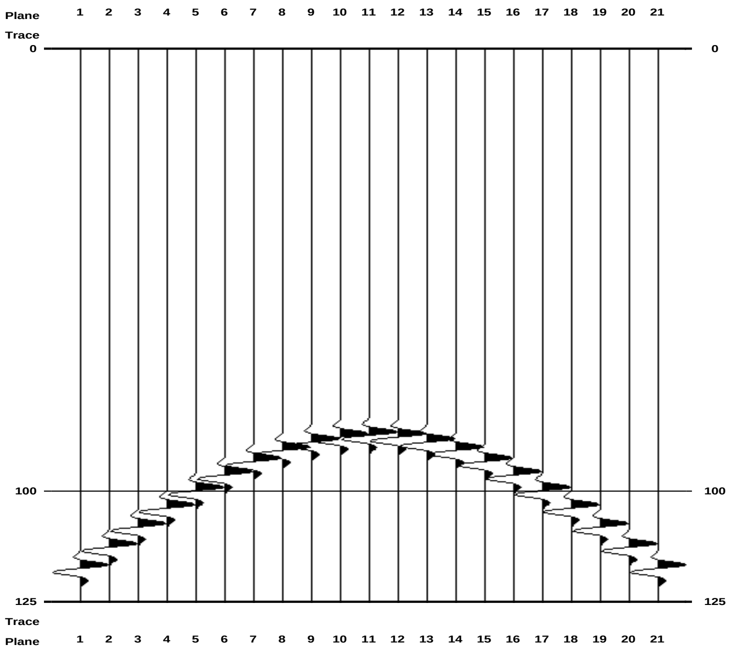

Figure 2.3: Seismogram.

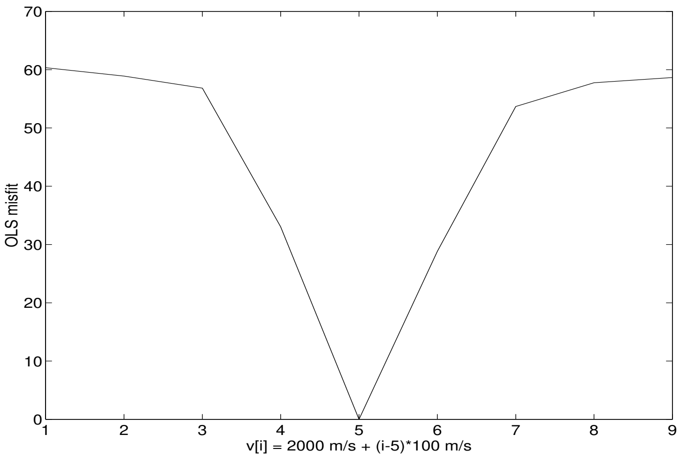

Figure 2.4: Output least squares objective function.

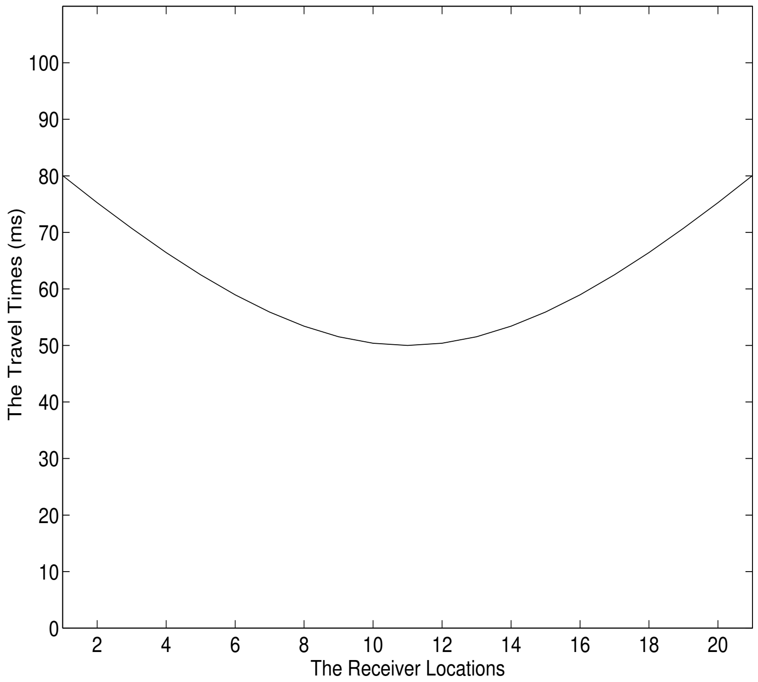

FIGURE 2.5: Traveltime data.

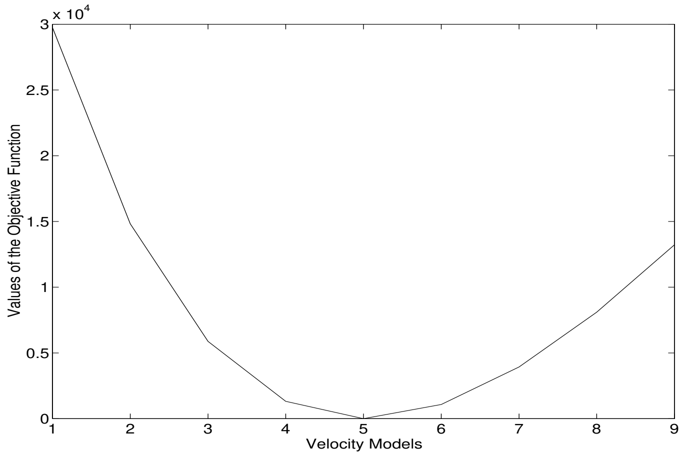

FIGURE 2.6: Traveltime tomography objective function.

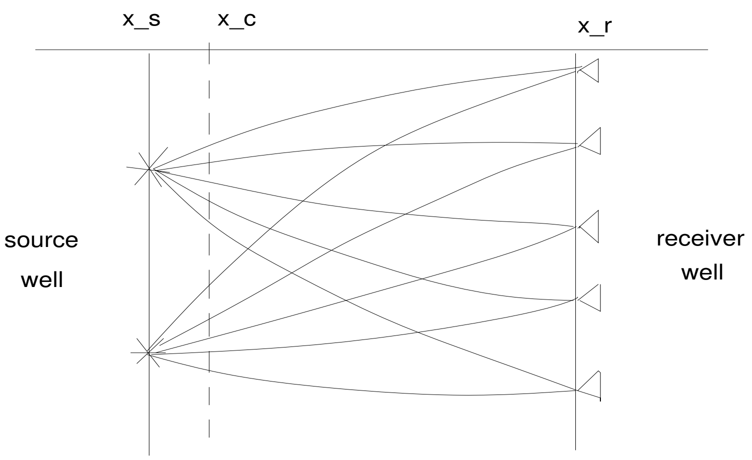

Figure 2.7: Multiple sources.

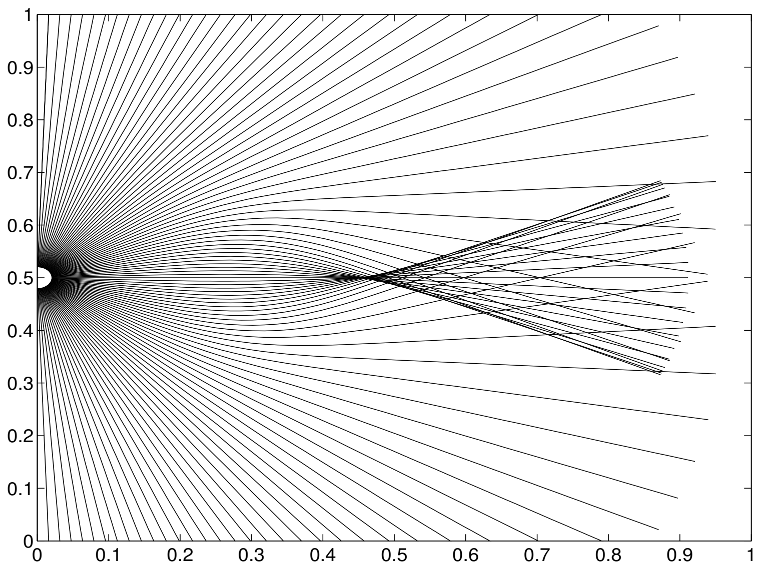

Figure 3.1: Intersecting rays caused by a low velocity zone.

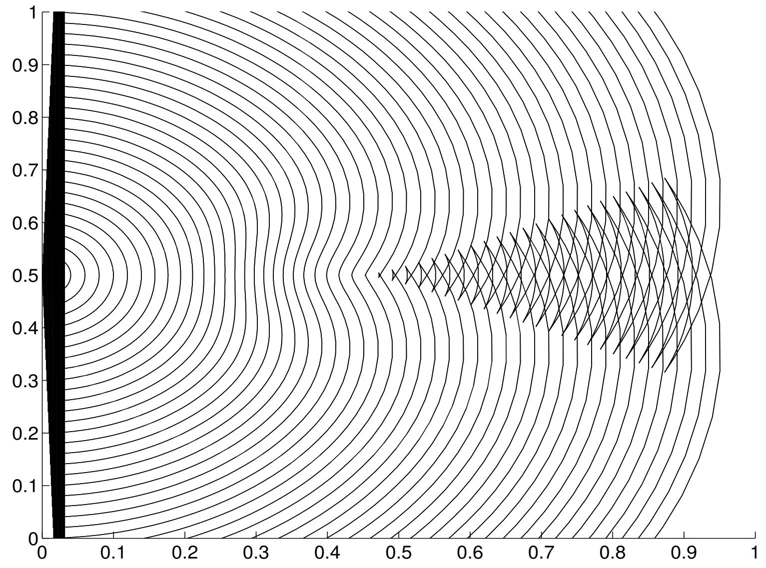

FIGURE 3.2: Self-folding wave fronts.

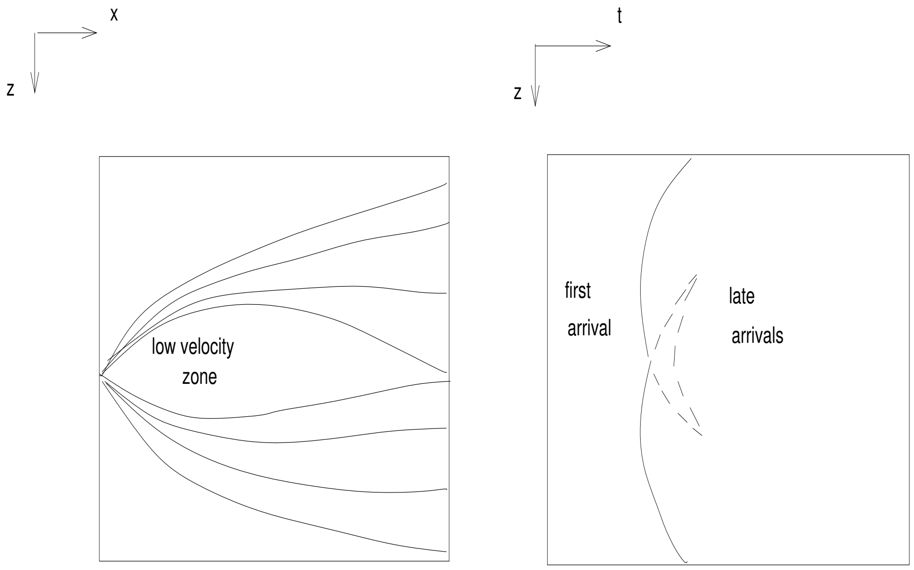

FIGURE 3.3: First arrival does not constrain the low velocity zone.

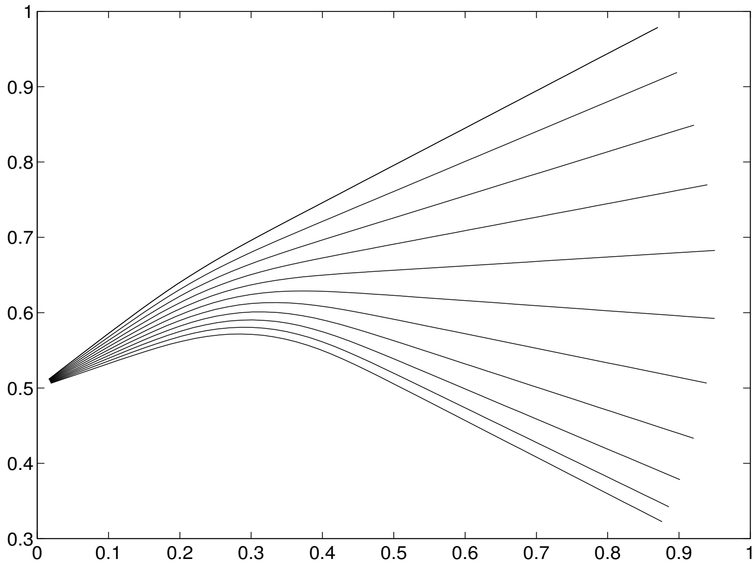

FIGURE 3.4: Upper (first) group.

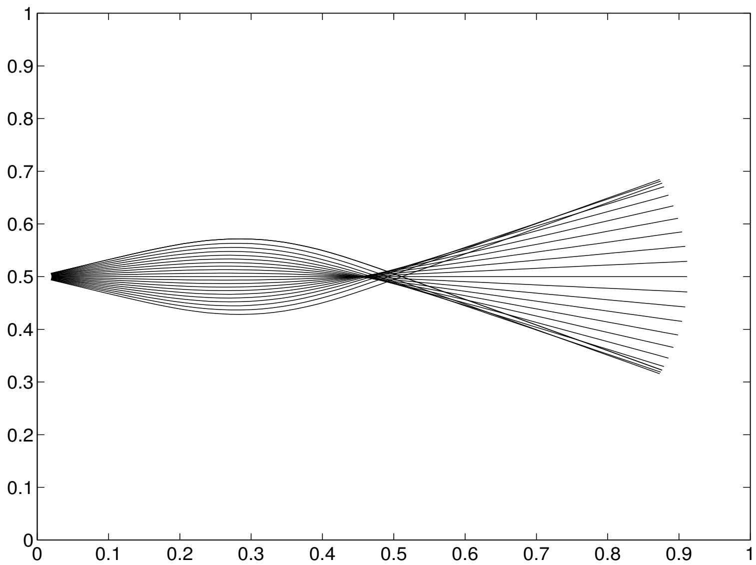

FIGURE 3.5: Center (second) group.

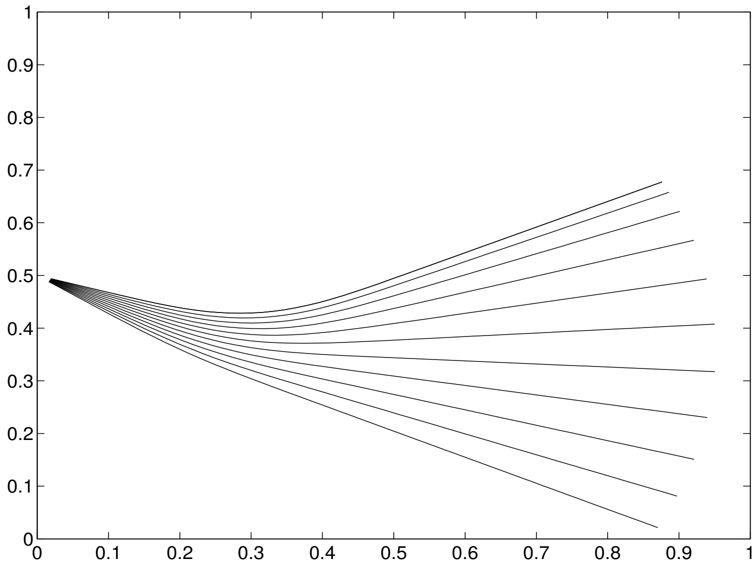

FIGURE 3.6: Lower (third) group.

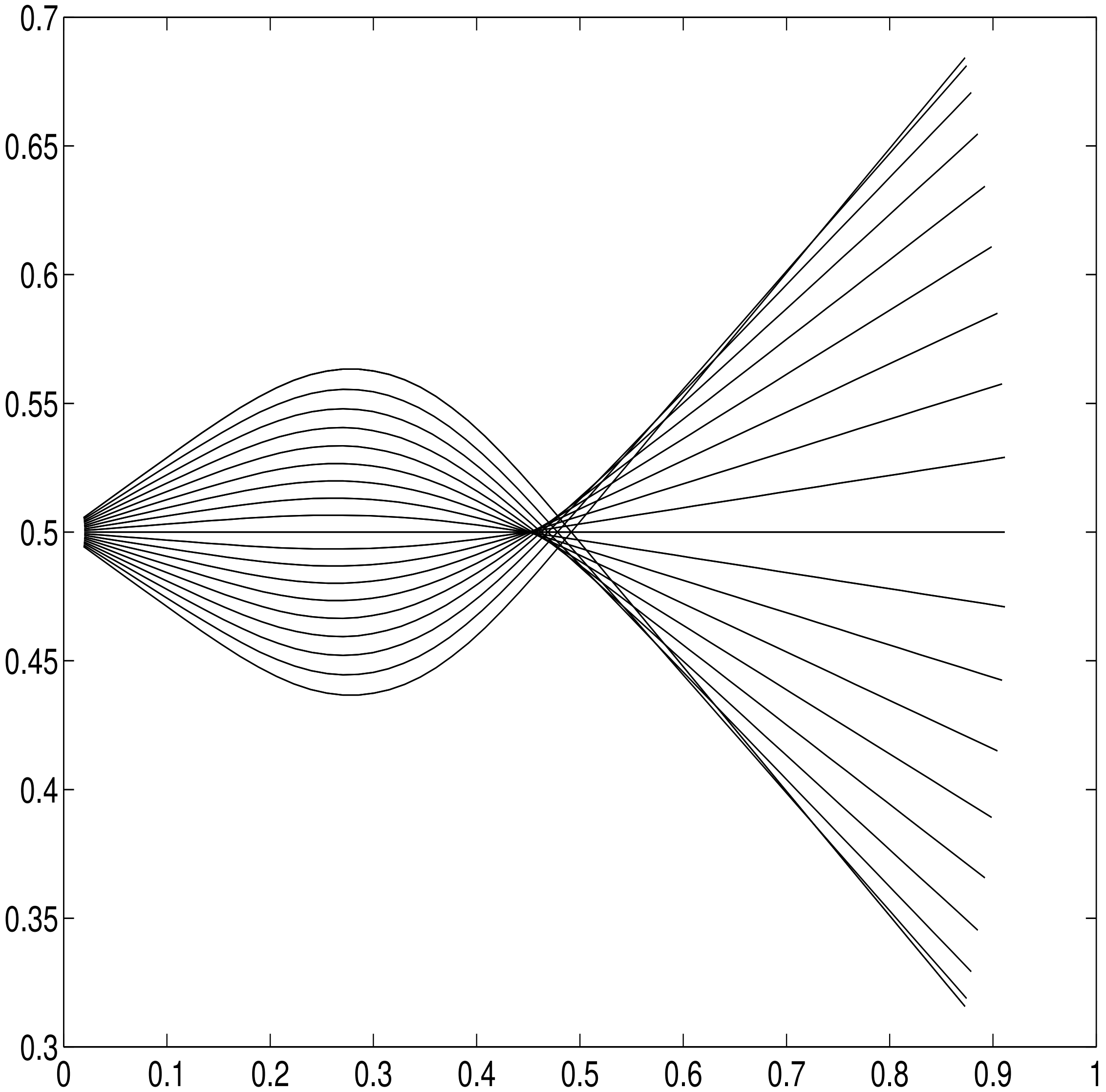

Figure 3.7: The change of variable from z_r to z is multi-valued, and singular on the caustics if x is to the right of the cusp.