Linear Inversion in Layered Viscoacoustic Media Using a Time Domain Method

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Abstract

Real Earth media attenuate and disperse propagating waves. If these effects are neglected, inversion can produce erroneous results. Viscoelasticity provides an appropriate and powerful tool to model these anelastic effects, so forms a plausible basis for inversion algorithms. The τ-p (intercept time-slowness) domain permits economical modeling and inversion of 3-D wave propagation in layered media. The adjoint state method leads to an efficient algorithm for linear least squares inversion of short wavelength model components.

Numerical experiments show that accurate inversion requires conditioning of the data by a time varying gain, hence also of the simulator output, to "equalize" the influence of earlier strong events and later attenuated events.

1 Introduction

Viscoelasticity provides a good model for representing real earth media [3]. Attenuation of propagating waves is often measured by the quality factor Q. Sedimentary rocks have been shown to have nearly constant Q over the seismic frequency range. A simple viscoelastic model using standard linear solids in parallel can closely approximate this behavior, [2], [3], [5]. In many applications a single standard linear solid suffices,[2],[3]. We have developed a reliable simulator which solves the viscoelastic wave equation using the finite difference method. The forward simulator has already been used in 2-D and 3-D scattering studies and been subject to several convergence tests [4], [11]. The inversion algorithm derived here is similar to an algorithm by Martinez and McMechan, [8]. Both algorithms use plane wave data and a layered media model, but Martinez and McMechan's algorithm is in contrast based in the frequency domain and uses propagator matrices to model the wave field.

We will start by briefly discussing the simulator and its modification for plane wave layered media simulations. We will then use this simulator to derive a least squares inversion algorithm for plane wave data and layered media, through the adjoint state technique. Finally, numerical experiments show that accurate inversion requires conditioning of the data by a time varying gain, hence also of the simulator output, to "equalize" the influence of earlier strong events and later attenuated events.

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2 Forward Modeling

The three dimensional viscoacoustic wave equation for one standard linear solid can be written [3], [10],

$$
\text{(2.1)} \qquad \begin{cases} p_{,t} = -K(1+\tau)\nabla\cdot\vec{v} - r + f(\vec{x}_s,t) \\[2ex] r_{,t} = -\dfrac{1}{\tau_\sigma}(r + K\tau\nabla\cdot\vec{v}) \\[2ex] \vec{v}_{,t} = -\dfrac{1}{\rho}\nabla p \end{cases},
$$

which is an initial value problem, i.e., $p = r = 0$, $\vec{v} = \vec{0}$, $t \ll 0$. K is the bulk modulus, ρ is the density and τ is roughly proportional to the inverse of Q and a measure of attenuation, [2]. τ_σ is a relaxation time and determines the frequency range where there is most attenuation. p is the pressure (stress) and $\vec{v} = (v_x, v_y, v_z)$ is the particle velocity. r is called a memory variable, which reflects the 'memory' of the viscoelastic medium. Waves are initiated by sources, $f(\vec{x}_s, t)$, which are added to the right hand side of equation (2.1). The field p is sampled at receiver locations, $\vec{x}_r$, to produce a pressure seismogram, $p(\vec{x}_s, \vec{x}_r, t)$.

Assuming a layered model, $K = K(z)$, $\rho = \rho(z)$, $\tau = \tau(z)$, it is possible to Radon transform equation (2.1) to the τ-p_s domain. The result is a 1-D wave equation with a modified bulk modulus.

$$
\text{(2.2)} \qquad \begin{cases} p_{,t} = -K_{mod}(1+\tau)v_{,z} - r + f(z_s,t) \\[2ex] r_{,t} = -\dfrac{1}{\tau_\sigma}(r + K_{mod}\,\tau v_{,z}) \\[2ex] v_{,t} = -\dfrac{1}{\rho}p_{,z} \end{cases},
$$

where

$$
\text{(2.3)} \qquad K_{mod} = \frac{K}{1 - p_s^2 K/\rho},
$$

$v = v_z$, and p_s is the horizontal slowness. The solutions to this transformed wave equation are plane waves with normals (Poynting-vectors) in the x-z plane. The simulated seismogram is defined as $F(\vec{\mu}, t, p_s) = p(z_r, t, p_s)$.

The formula for scaling of the bulk modulus was derived using a non-viscous wave equation, which results in a small approximation for the viscoelastic case. Equation (2.2) was solved with the finite difference method. The finite difference solver uses a staggered scheme, which is second order accurate in time and fourth order accurate in space. p and v are updated in time by the leap-frog scheme and r by Crank-Nicolson, [10].

3 Linearized Inversion

The object of seismic inversion is to find an earth model, $\vec{\mu} = (K, \rho, \tau)$, which yields the best fit between the simulated seismogram, $F(\vec{\mu})$, and the measured seismogram, F_0^{obs}. Measuring the fit with the L^2 norm leads to minimizing the following functional J,

$$J = \frac{1}{2}(F(\vec{\mu}) - F_0^{obs}, F(\vec{\mu}) - F_0^{obs}) \equiv \frac{1}{2}\int\int (F(\vec{\mu}) - F_0^{obs})^2 \, dt dp_s. \tag{3.1}$$

$F(\vec{\mu})$ is clearly a nonlinear function of the parameters $\vec{\mu}$. Assuming there exists a reasonably good estimate, $\vec{\mu}_0$, it is justified to linearize $F(\vec{\mu})$ to get a linearized map $DF(\delta\vec{\mu})$ and use it instead of F in the functional J, $\vec{\mu} = \vec{\mu}_0 + \delta\vec{\mu}$. It is of course also necessary to change F_0^{obs} to $F^{obs} = F_0^{obs} - F(\vec{\mu}_0)$. This yields,

$$J = \frac{1}{2}(DF[\delta\vec{\mu}] - F^{obs}, DF[\delta\vec{\mu}] - F^{obs}), \tag{3.2}$$

where square brackets are used to emphasize that $DF[\delta\vec{\mu}]$ is linear in $\delta\vec{\mu}$. Since J is a positive semi-definite form it will have a minimum when a perfect fit occurs. The gradient of J with respect to $\delta\vec{\mu}$ will therefore vanish at this point. A small variation of J yields,

$$dJ = (DF[d\delta\vec{\mu}], DF[\delta\vec{\mu}] - F^{obs}), \tag{3.3}$$

since DF is linear. $d\delta\vec{\mu}$ is a variation of $\delta\vec{\mu}$. By defining the adjoint of DF as DF^*, it is possible to rewrite equation (3.3) as,

$$dJ = (d\delta\vec{\mu}, DF^*[DF[\delta\vec{\mu}] - F^{obs}]), \tag{3.4}$$

Now it is possible to identify the gradient of J with respect to $\delta\vec{\mu}$ as,

$$Grad\, J = DF^*[DF[\delta\vec{\mu}] - F^{obs}]. \tag{3.5}$$

Setting this gradient to zero yields the normal equations,

$$DF^*[DF[\delta\vec{\mu}]] = DF^*[F^{obs}]. \tag{3.6}$$

Solving equation (3.6) yields the $\delta\vec{\mu}$ that minimizes J.

To utilize the formula in equation (3.6) we first have to derive the linearized forward map DF, by perturbing the parameters determining the media (K, ρ, τ) and the fields describing the viscoelastic waves (p, v, r). The linear terms in the perturbations are kept to produce the linearized forward map. Due to the scaling of K in equation (2.3) the linearized bulk modulus is,

$$\delta K_{mod} = \delta K \left(1 + \frac{K_{mod}}{\rho} p_s^2\right)^2 - \delta\rho \left(\frac{K_{mod}^2}{\rho^2} p_s^2\right). \tag{3.7}$$

This linearized forward map is identical to a Born approximation.

The adjoint DF^* of DF is found through the adjoint state technique, [6]. This involves solving a final value equation system. The new equation system is the adjoint of the linearized equation.

$$
\begin{cases}
u_{,t} = -\left(\frac{1}{\rho}q\right)_{,z} - q0 \\
s_{,t} = \frac{1}{\tau_\sigma}(s+u) \\
q_{,t} = -\left(K_{mod}(1+\tau)u - K_{mod}\tau s\right)_{,z}
\end{cases}
\tag{3.8}
$$

$u = q = s = 0$, $t \gg 0$. q, u, and s are the adjoint fields. The source, $q0(z_r, t, p_s)$, in equation (3.8) is either a residual seismogram, i.e., F^{obs}, or output from the linearized forward map, $F[\vec{\mu}]$. As a matter of fact it can be any "seismogram-like" object. The adjoint fields are used to find the adjoint for the modified parameters by cross correlating the adjoint fields in equation (3.8) with the background fields, equation (2.1),

$$
\begin{cases}
\delta K_{mod-est} = \int_0^T -(1+\tau)v_{,z}u - \tau v_{,z}s \; dt \\
\delta\tau_{mod-est} = \int_0^T -K_{mod}v_{,z}u - K_{mod}v_{,z}s \; dt \\
\delta\rho_{mod-est} = \int_0^T \frac{1}{\rho^2}p_{,z}q \; dt
\end{cases}
\tag{3.9}
$$

As a final step to find the full adjoint DF^* it is necessary to find the adjoint of the operator modifying the linearized bulk modulus in equation (3.7). This adjoint modifies the preliminary estimated density and bulk modulus;

$$
\begin{cases}
\delta K_{est} = \delta K_{mod-est}\left(1 + \frac{K_{mod}}{\rho}p_s^2\right)^2 \\
\delta\tau_{est} = \delta\tau_{mod-est} \\
\delta\rho_{est} = \delta\rho_{mod-est} - \delta K_{mod-est}\frac{K_{mod}^2}{\rho^2}p_s^2
\end{cases}
.
\tag{3.10}
$$

Equation (3.10) is the final adjoint output. It is necessary to derive the adjoint DF^* using the discretized system to find the correct discrete adjoint DF^*. No additional numerical analysis of the adjoint equation [equation (3.8)] is necessary, since it has the same numerical properties as

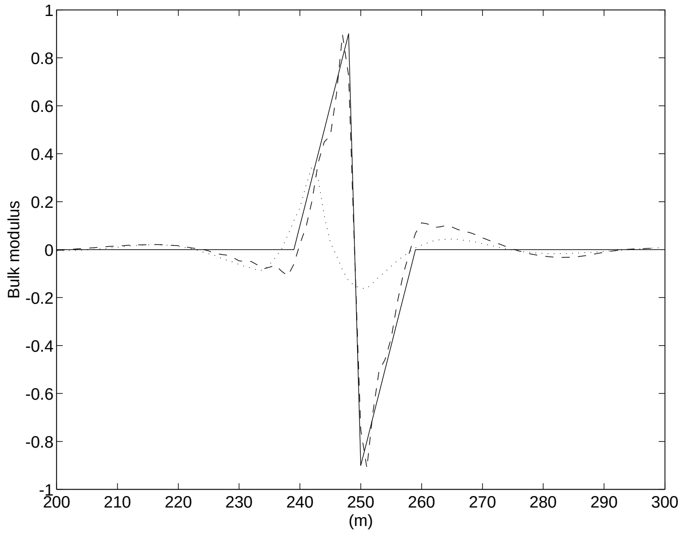

Figure 4.1: The true reflector with two inversion estimates. Solid - True reflector, Dashed - Viscoelastic estimate, Dotted - Elastic estimate.

the forward equation [equation (2.2)], which in turn has been thoroughly analysed [10]. It is clear from the above derivation that the linearized forward map DF and its adjoint DF^* do not exist explicitly. Therefore, it is necessary to use an iterative method, for instance the conjugate gradient method, to find the solution of the inverse problem $\delta\vec{\mu}$, i.e., to solve the normal equation (3.6).

4 Results (pure 1D)

We produced a synthetic 1-D seismogram by using a single value for the slowness, p_s. The synthetic model consisted of one distinct reflector (perturbation in bulk-modulus, the velocity $2.0km/s$, the density $2.6g/cm^3$, and Q was approximately 40). We used this seismogram as input to our inversion routine and performed viscoelastic and elastic inversion. The result from the two inversions is shown in Figure 4.1, where the original reflector is compared to reflectors predicted by elastic and viscoelastic inversion. It is clear that the result from the elastic inversion is quite poor. The wavenumber spectra of the reflectors yield more insight to the differences. The wavenumber spectra of the three reflectors are shown in Figure 4.2. We can see an almost perfect match in the range where the reflector is illuminated by the source wavelet for the viscoelastic case but poor reconstruction in the elastic case.

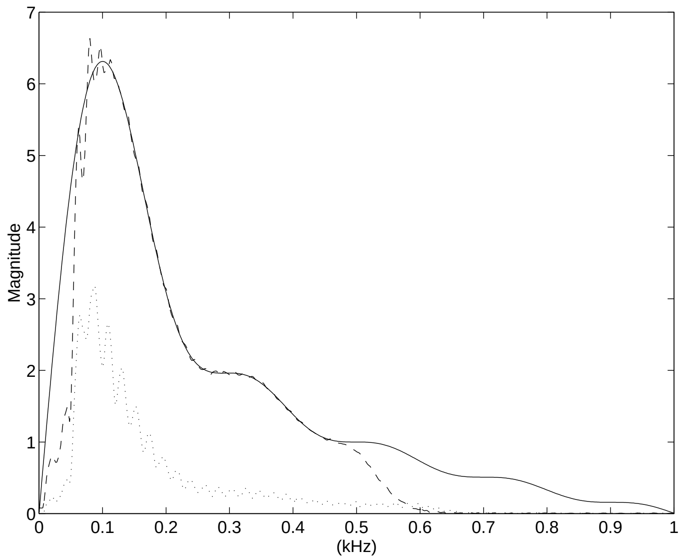

Figure 4.2: Frequency content of the true reflector and the two estimates. Solid - True reflector, Dashed - Viscoelastic estimate, Dotted - Elastic estimate.

5 Analysis of the 1D estimation results

Both modeling and inversion were done with a constant reference velocity in 1-D. This enables us to easily switch between wavenumber and frequency domain, which makes it possible to compare the wavenumber response of the different estimated reflectors (Figure 4.2) with propagation effects, due to the viscoelastic medium, derived in the frequency domain. These propagation effects are the likely cause for the incorrect estimate found from the elastic inversion using the viscoelastic seismogram.

For 1-D constant velocity/attenuation medium it is possible to derive a simple expression for the effect on different frequency components in the seismogram. The different components in the frequency spectra decays in time according to

$$e^{-\omega t \frac{\tau \omega \tau_\sigma}{1+\omega^2 \tau_\sigma^2}}, \tag{5.1}$$

for a viscoelastic medium with a constitutive relation corresponding to a standard linear solid. Since there is only one reflector it is possible to approximate the time t in equation (5.1) by dividing the distance from the receiver to the reflector by the velocity in the medium.

The above effect should be possible to track in the inverted results if an incorrect value for the attenuation in the media was used, e.g., elastic inversion for viscoelastic (realistic) seismogram, and should in other words cause erroneous estimation of reflectors. Dividing the wavenumber spectra for an elastically estimated reflector with the wavenumber spectra for a viscoelastically estimated reflector should yield a rough approximation to the inverse of the function in equation (5.1). This

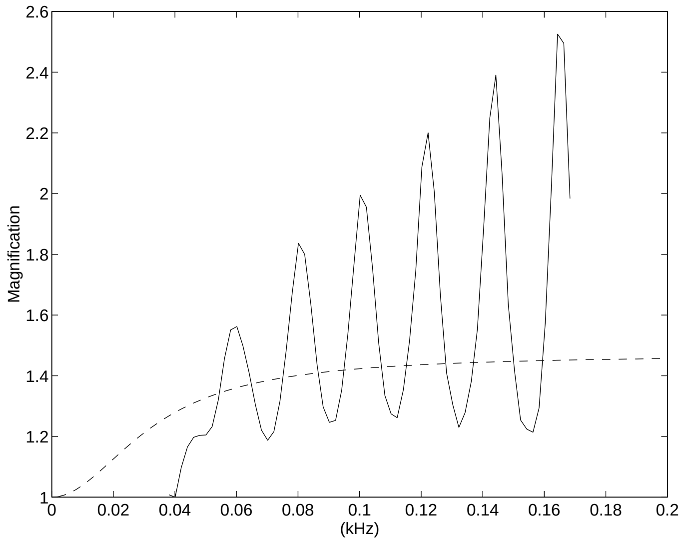

Figure 5.1: Ratio of the wavenumber spectra of two estimates and the predicted effect from attenuation in medium, equation (5.1). Solid - Ratio of the wavenumber spectra of two estimates, Dashed - The inverse of the predicted effect for decay.

is shown in Figure 5.1. The ratio between the two wave number spectra has only been calculated on the interval where the spectra have a good support. The ratio of the wavenumber spectra is oscillating around the predicted curve which suggests that the erroneous result is due to neglect of the propagation effects (attenuation and dispersion) for the elastic inversion.

6 Results for Layered Media Inversion

We produced a synthetic seismogram by viscoelastic modeling, Figure 6.1. The synthetic model consisted of four distinct reflectors which were perturbations in bulk modulus δK, see Figure 6.2. $\delta\rho = \delta\tau = 0$, $\rho = 2.6\ g/cm^3$, and $\tau = 0.05$ were constant throughout the model. The τ-value corresponds to a minimum Q of approximately 40. K was set so that the velocity in the model was constant $2.0\,km/s$ down to $250\,m$ and then linearly incresing by $(2.7\,m/s)/m$. The source was a Ricker wavelet, with a center frequency of $240\,Hz$, initiated at a depth of $100\,m$. The seismogram was used as input for the inversion. The result from the inversion is shown in Figure 6.2, where the original reflectors are compared to reflectors predicted by inversion. The deeper reflectors are less well resolved. Figure 6.3 shows a close-up of the deepest reflector. The predicted reflector is clearly underestimated. The side-lobes in the estimates are due to the finite frequency content of the source wavelet, [1].

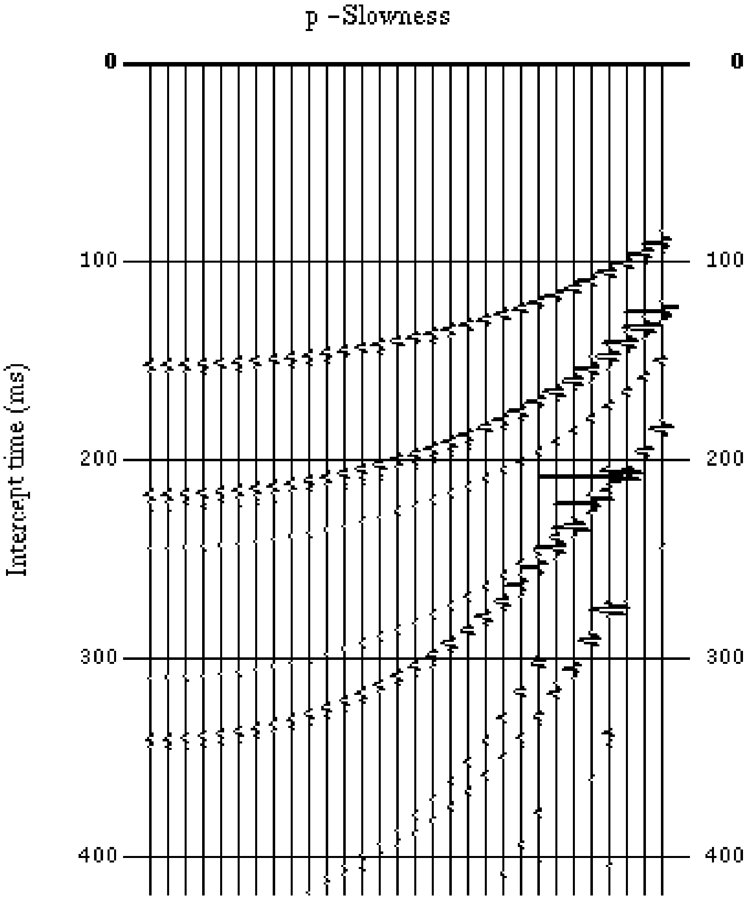

Figure 6.1: τ-p_s seismogram used in the inversion routine. Note that the seismogram contains surface multiples.

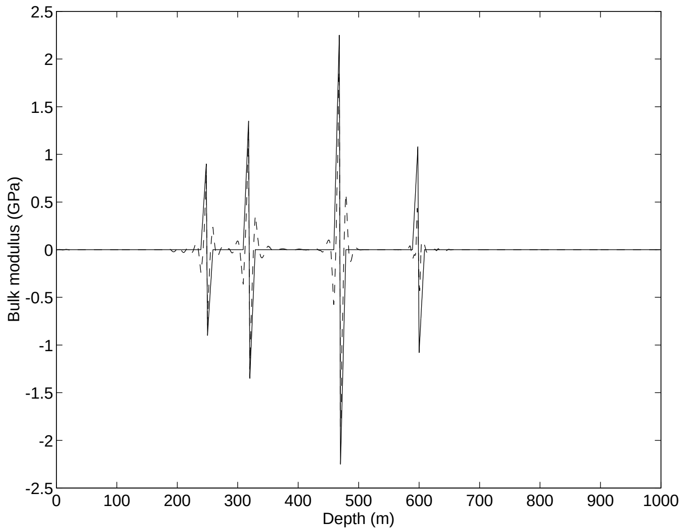

Figure 6.2: True and estimated reflectors without post conditioning. Solid - True reflectors, Dashed - Estimated reflectors.

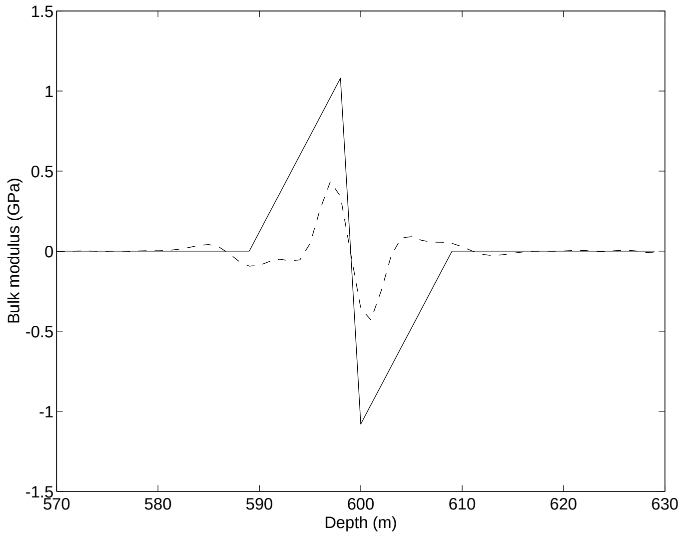

Figure 6.3: True and estimated reflector without post conditioning. Solid - True reflector, Dashed - Estimated reflector.

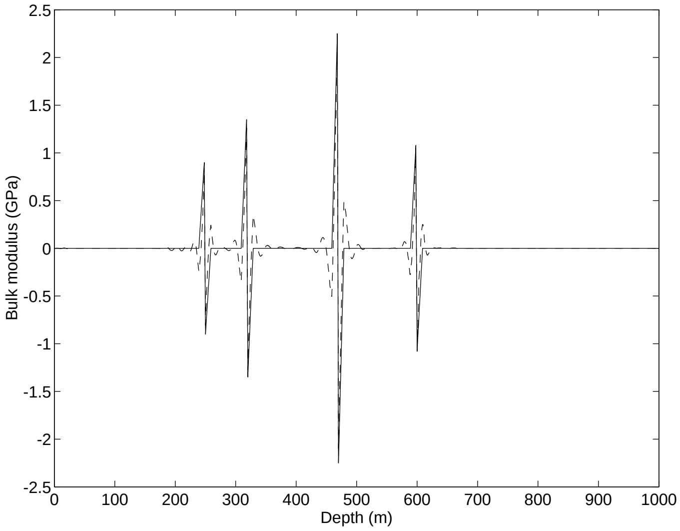

Figure 7.1: True and estimated reflectors with post conditioning. Solid - True reflectors, Dashed - Estimated reflectors.

7 Analysis and Improvement of Estimation Results (Layered Media)

The inversion tries to fit the simulated seismogram as well as possible to the measured (or observed) seismogram. Since viscoelastic (realistic) media attenuate propagating waves, later arrivals (corresponding to deeper reflectors) yield smaller amplitudes than earlier arrivals, see Figure 6.1. Hence later arrivals have less effect on the seismogram as well as on the residual. Deeper reflectors correspond, losely speaking, to smaller eigenvalues of the forward operator. Inversion routines that rely on Krylov-space based iterative methods will favor reflectors corresponding to large eigenvalues i.e., estimate these correctly at an earlier stage in the iterative method. The common solution to this problem is to use a pre-conditioner, i.e. a change of variables in model space. We use instead a change of variables in data space: we pointwise multiply the seismogram by an increasing function of time to counter the attenuation. This is more accurately called post-conditioning. A crude approximation to the viscoelastic attenuation as a function of time is,

$$e^{-t\frac{\tau}{2\,\tau_\sigma}}. \tag{7.1}$$

Hence, the seismogram should be multiplied by the reciprocal of (7.1) before inversion. The output of the linearized forward map and the input of the adjoint have also to be multiplied by the same gain function. Figure 7.1 shows true and predicted reflectors, resulting from such inversion. The same number of iterations were performed as for the result shown in Figure 6.2 . The close-up of the deepest reflector, Figure 7.2, shows a much better agreement than before (Figure 6.3).

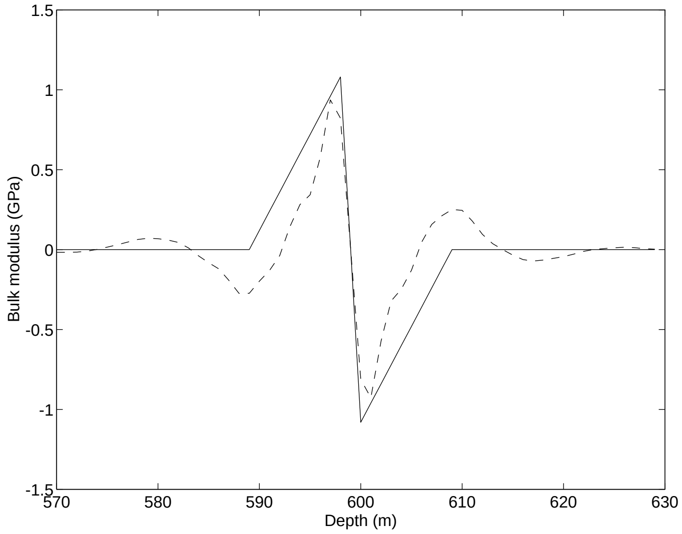

Figure 7.2: True and estimated reflector with post conditioning. Solid - True reflector, Dashed - Estimated reflector.

8 Conclusions

We have described a viscoelastic linearized inversion algorithm for layered media. The error in estimates of reflectors can be severe if significant anelastic effects are neglected. It seems necessary to post-condition seismograms to correctly estimate deep reflectors. Immediate areas of research are effect on AVO [Amplitude Versus Offset analysis, [7]] and inversion for smoothly varying Q.

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