3. Wave equation migration

(i) Reverse time

(ii) Reverse depth

Reverse time computation of adjoint $F[v]^*$:

Start with the zero-offset case - easier, but only if you replace it with the exploding reflector model, which replaces $F[v]$ by

$$\tilde{F}[v]r(\mathbf{x}_s,t) = w(\mathbf{x}_s,t),\ \mathbf{x}_s \in X_s, 0 \le t \le T$$

$$\left(\frac{4}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) w = \delta(t)\frac{2r}{v^2},\ w \equiv 0,\ t < 0$$

To compute the adjoint, start with its definition: choose $d \in \mathcal{E}(X_s \times (0,T))$, so that

$$< \tilde{F}[v]^* d, r > = < d, \tilde{F}[v] r >$$

$$= \int_{X_s} dx_s \int_0^T dt\, d(\mathbf{x}_s,t) w(\mathbf{x}_s,t)$$

The only thing you know about w is that it solves a wave equation with r on the RHS. To get this fact into play, (i) rewrite the integral as a space-time integral:

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \int_{X_s} dx_s \, d(\mathbf{x}_s, t)\delta(\mathbf{x} - \mathbf{x}_s) w(\mathbf{x}, t)$$

(ii) write the other factor in the integrand as the image of a field q under the (adjoint of the) wave operator (it's self-adjoint), that is,

$$\left(\frac{4}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) q(\mathbf{x}, t) = \int_{X_s} dx_s \, d(\mathbf{x}_s, t)\delta(\mathbf{x} - \mathbf{x}_s)$$

so

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \left[\left(\frac{4}{v^2(\mathbf{x})}\frac{\partial^2}{\partial t^2} - \nabla^2\right) q(\mathbf{x}, t)\right] w(\mathbf{x}, t)$$

(iii) integrate by parts

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \left[\left(\frac{4}{v^2(\mathbf{x})} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) w(\mathbf{x}, t) \right] q(\mathbf{x}, t)$$

which works if $q \equiv 0$, $t > T$ (*final value condition*); (iv) use the wave equation for w

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \, \frac{2}{v(\mathbf{x})^2} r(\mathbf{x}) \delta(t) q(\mathbf{x}, t)$$

(v) observe that you have computed the adjoint:

$$= \int_{\mathbf{R}^3} dx \, r(\mathbf{x}) \left[\frac{2}{v(\mathbf{x})^2} q(\mathbf{x}, 0) \right] = < r, \tilde{F}[v]^* d >$$

i.e.

$$\tilde{F}[v]^* d = \frac{2}{v(\mathbf{x})^2} q(\mathbf{x}, 0)$$

Summary of the computation, with the usual description:

- Use that data as sources, backpropagate in time - i.e. solve the final value ("reverse time") problem

$$\left(\frac{4}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) q(\mathbf{x},t) = \int_{X_s} dx_s\, d(\mathbf{x}_s,t)\delta(\mathbf{x}-\mathbf{x}_s),\ q \equiv 0,\ t > T$$

- read out the "image" (= adjoint output) at $t = 0$:

$$\tilde{F}[v]^* d = \frac{2}{v(\mathbf{x})^2} q(\mathbf{x},0)$$

Note: The adjoint (time-reversed) field q is *not* the physical field (δu) run backwards in time, contrary to some imputations in the literature.

Known as "two way reverse time finite difference migration" in geophysical literature (Whitmore, 1982) - uses full (two way) wave equation, propagates adjoint field backwards in time, generally implemented using finite difference discretization. Same as "adjoint state method", Lions 1968, Chavent 1974 for control and inverse problems for PDEs - much earlier for control of ODEs - Lailly, Tarantola '80s.

A slightly messier computation computes the adjoint of $F[v]$ (i.e. multioffset or *prestack* migration):

$$F[v]^*d(\mathbf{x}) = -\frac{2}{v(\mathbf{x})}\int dx_s \int_0^T dt \left(\frac{\partial q}{\partial t}\nabla^2 u\right)(\mathbf{x},t;\mathbf{x}_s)$$

where *adjoint field* q satisfies $q \equiv 0$, $t \geq T$ and

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) q(\mathbf{x},t;\mathbf{x}_s) = \int dx_r\, d(\mathbf{x}_r,t;\mathbf{x}_s)\delta(\mathbf{x}-\mathbf{x}_r)$$

Proof:

$$< F[v]^* d, r > = < d, F[v] r >$$

$$= \int \int dx_s \, dx_r \int_0^T dt \, d(\mathbf{x}_r, t; \mathbf{x}_s) \frac{\partial \delta u}{\partial t}(\mathbf{x}_r, t; \mathbf{x}_s)$$

$$= \int dx_s \int dx \int_0^T dt \left\{ \int dx_r \, d(\mathbf{x}_r, t; \mathbf{x}_s) \delta(\mathbf{x} - \mathbf{x}_r) \right\} \frac{\partial \delta u}{\partial t}(\mathbf{x}, t; \mathbf{x}_s)$$

$$= \int dx_s \int dx \int_0^T dt \left[\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) q \right] \frac{\partial \delta u}{\partial t}(\mathbf{x}, t; \mathbf{x}_s)$$

$$= - \int dx_s \int dx \int_0^T dt \left[\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \delta u \right] \frac{\partial q}{\partial t}(\mathbf{x}, t; \mathbf{x}_s)$$

(boundary terms in integration by parts vanish because (i) $\delta u \equiv 0$, $t << 0$; (ii) $q \equiv 0$, $t >> 0$; (iii) both vanish for large $\mathbf{x}$, at each t)

$$= -\int dx_s \int dx \int_0^T dt \left(\frac{2r}{v^2} \frac{\partial^2 u}{\partial t^2} \frac{\partial q}{\partial t} \right) (\mathbf{x}, t; \mathbf{x}_s)$$

$$= -\int dx_s \int dx \, r(\mathbf{x}) \frac{2}{v^2(\mathbf{x})} \int_0^T dt \left(\frac{\partial^2 u}{\partial t^2} \frac{\partial q}{\partial t} \right) (\mathbf{x}, t; \mathbf{x}_s)$$

$$= < r, F[v]^* d >$$

q.e.d.

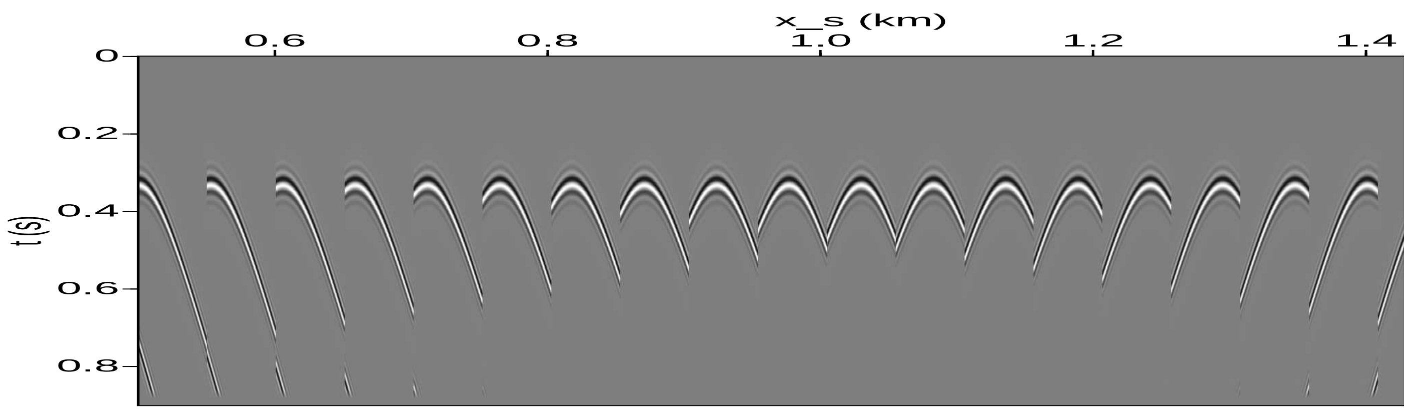
Algorithm: finite difference or finite element discretization in $\mathbf{x}$, finite difference time stepping.

- For each $\mathbf{x}_s$, solve wave equation for u forward in t, record final (t=T) Cauchy data, also (for example) Dirichlet boundary data.
- Step u and q backwards in time together; at each time step, data serves as source for q ("backpropagate data")
- During backwards time stepping, accumulate (approximations to)

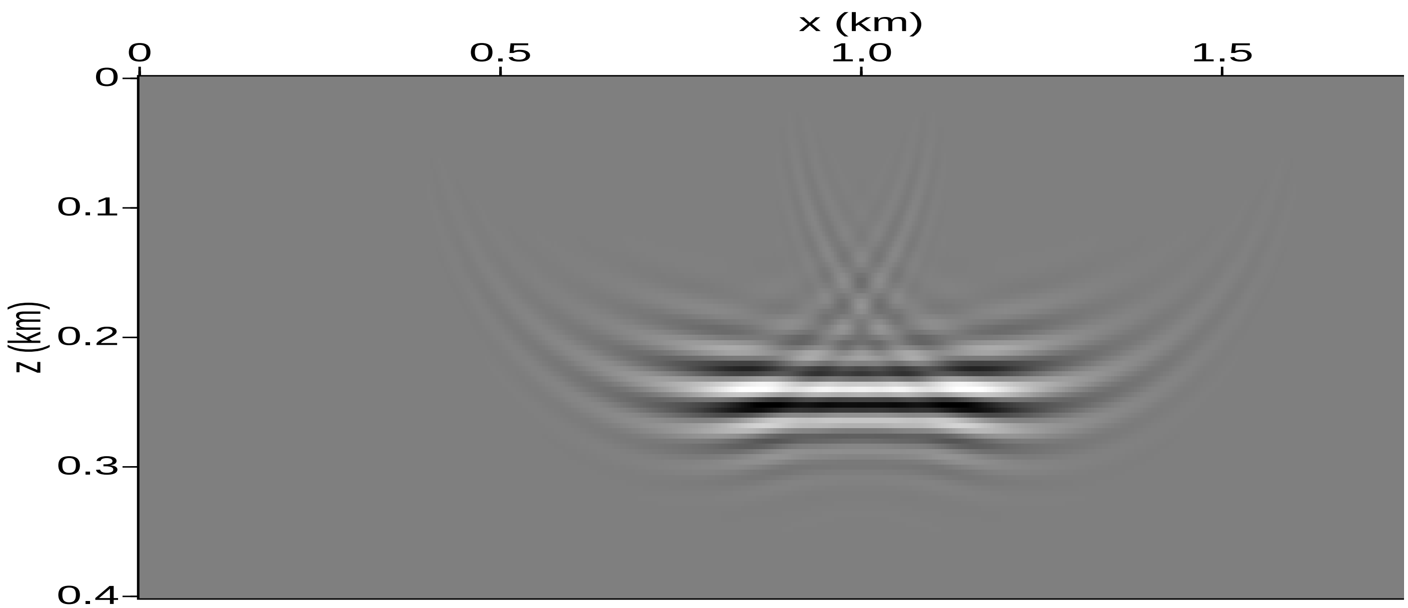
$$Q(\mathbf{x}) += \frac{2}{v^2(\mathbf{x})} \int_0^T dt \left(\frac{\partial^2 u}{\partial t^2} \frac{\partial q}{\partial t} \right) (\mathbf{x}, t; \mathbf{x}_s)$$

  ("crosscorrelate reference and backpropagated field").
- next $\mathbf{x}_s$ - after last $\mathbf{x}_s$, $F[v]^* d = Q$.

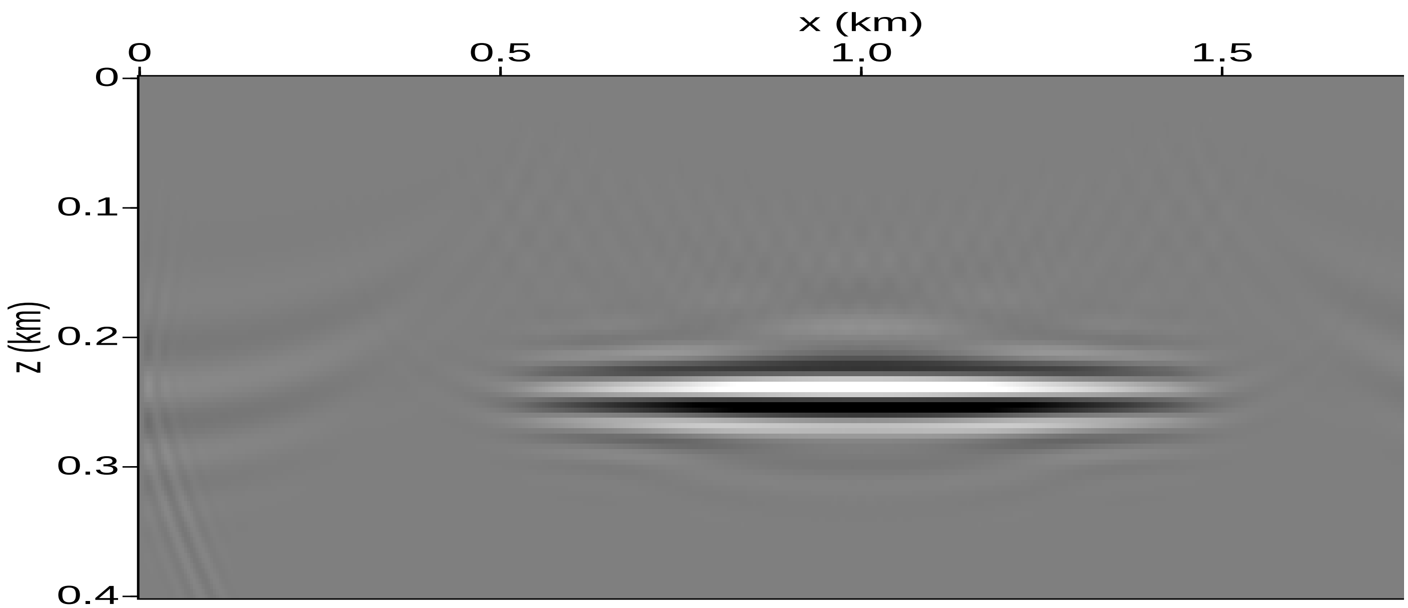
Example: constant (reference) velocity $v \equiv 1.5$ km/s, flat reflector at $z = 0.25$ km ($r(z) = 0, z < 0.25$ km, $= 0.15$ km/s else).



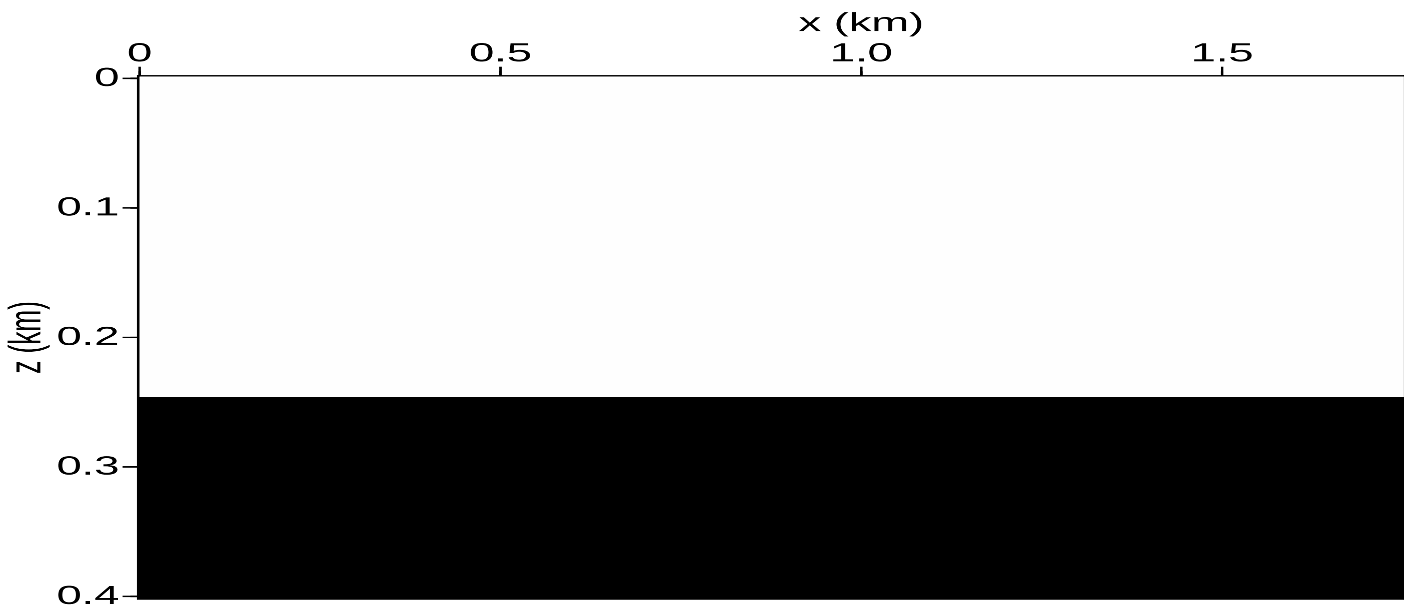
21 CSGs, flat reflector, const vel.



CSG RT mig, 1 shot (x_s=1.0 km)



CSG RT mig, 21 shots (x_s = 0.5-1.5 km



Reflectivity model, flat reflector

Reverse depth computation of $F[v]^*$

- Claerbout, early 70's
- zero offset version: Claerbout IEI ("swimming pool equation").
- multioffset version: "survey sinking", double-square-root ("DSR") equation, BEI.

Start with zero-offset. Again, assume exploding reflector model:

$$\tilde{F}[v]r(\mathbf{x}_s, t) = w(\mathbf{x}_s, t),\ \mathbf{x}_s \in X_s, 0 \le t \le T$$

$$\left(\frac{4}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) w = \delta(t)\frac{2r}{v^2},\ w \equiv 0,\ t < 0$$

Basic idea: 2nd order wave equation permits waves to move in all directions, but waves carrying reflected energy are (mostly) moving *up*. Should satisfy a 1st order equation for wave motion in one direction.

For the moment use 2D notation $\mathbf{x} = (x,z)$ etc. Write wave equation as evolution equation in z:

$$\frac{\partial^2 w}{\partial z^2} - \left(\frac{4}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) w = -\delta(t)\frac{2r}{v^2}$$

Suppose that you could take the square root of the operator in parentheses - call it B. Then the LHS of the wave equation becomes

$$\left(\frac{\partial}{\partial z} - B\right)\left(\frac{\partial}{\partial z} + B\right) w = -\delta(t)\frac{2r}{v^2}$$

so setting

$$\tilde{w} = \left(\frac{\partial}{\partial z} + B\right) w$$

you get

$$\left(\frac{\partial}{\partial z} - B\right)\tilde{w} = -\delta(t)\frac{2r}{v^2}$$

which might be the required equation for upcoming waves.

Two major problems: (i) how the h–l do you take the square root of a PDO? (ii) what guarantees that the equation just written governs upcoming waves?

Calculus of pseudodifferential operators: recall that products of ΨDOs are ΨDOs. Computations simple for *subclass* of ΨDOs with symbols given by asymptotic expansions:

$$p(\mathbf{x},\xi) \sim \sum_{j\le m} p_j(\mathbf{x},\xi),\ |\xi| \to \infty$$

in which p_j is *homogeneous in* ξ *of degree* j:

$$p_j(\mathbf{x},\tau\xi) = \tau^j p_j(\mathbf{x},\tau\xi),\ \tau, |\xi| \ge 1$$

The *principal symbol* is the homogeneous term of highest degree, i.e. p_m above.

Product rule for ΨDOs: if

$$p^1(\mathbf{x},\xi) = \sum_{j\le m^1} p^1_j(\mathbf{x},\xi),\ p^2(\mathbf{x},\xi) = \sum_{j\le m^2} p^2_j(\mathbf{x},\xi)$$

then principal symbol of $p^1(\mathbf{x},D)p^2(\mathbf{x},D)$ is $p^1_{m^1}(\mathbf{x},\xi)p^2_{m^2}(\mathbf{x},\xi)$, and there is an algorithm for computing the rest of the expansion.

In an open neighborhood $X\times\Xi$ of $(\mathbf{x}_0,\xi_0)$, symbol of $p^1(\mathbf{x},D)p^2(\mathbf{x},D)$ depends only on symbols of p^1, p^2 in $X\times\Xi$.

Consequence: if $a(\mathbf{x}, D)$ has an asymptotic expansion and is of order $m \in \mathbf{R}$, and $a_m(\mathbf{x}_0, \xi_0) > 0$ in $\mathcal{P} \subset \mathbf{R}^n \times \mathbf{R}^n - 0$, then there exists $b(\mathbf{x}, D)$ of order $m/2$ with asymptotic expansion for which

$$(a(\mathbf{x}, D) - b(\mathbf{x}, D)b(\mathbf{x}, D))u \in \mathcal{E}(\mathbf{R}^n)$$

for any $u \in \mathcal{E}'(\mathbf{R}^n)$ with $WF(u) \subset \mathcal{P}$.

Moreover, $b_{m/2}(\mathbf{x}, \xi) = \sqrt{a_m(\mathbf{x}, \xi)}$, $(\mathbf{x}, \xi) \in \mathcal{P}$. Will call b a *microlocal square root* of a.

Similar construction: if $a(\mathbf{x}, \xi) \neq 0$ in $\mathcal{P}$, then there is $c(\mathbf{x}, D)$ of order $-m$ so that

$$c(\mathbf{x}, D)a(\mathbf{x}, D)u - u,\ a(\mathbf{x}, D)c(\mathbf{x}, D)u - u \in \mathcal{E}(\mathbf{R}^n)$$

for any $u \in \mathcal{E}'(\mathbf{R}^n)$ with $WF(u) \subset \mathcal{P}$.

Moreover, $c_{-m}(\mathbf{x}, \xi) = 1/a_m(\mathbf{x}, \xi)$, $(\mathbf{x}, \xi) \in \mathcal{P}$. Will call b a *microlocal inverse* of a.

Application: symbol of

$$a(x,z,D_t,D_x) = \frac{\partial^2}{\partial x^2} - \frac{4}{v(x,z)^2}\frac{\partial^2}{\partial t^2} = \frac{4}{v(x,z)^2}D_t^2 - D_x^2$$

is

$$a(x,z,\tau,\xi) = \frac{4}{v(x,z)^2}\tau^2 - \xi^2$$

For $\delta > 0$, set

$$\mathcal{P}_\delta(z) = \left\{(x,t,\xi,\tau) : \frac{4}{v(x,z)^2}\tau^2 > (1+\delta)\xi^2\right\}$$

Then according to the last slide, there is an order 1 ΨDO-valued function of z, $b(x,z,D_t,D_x)$, with principal symbol

$$b_1(x,z,\tau,\xi) = \sqrt{\frac{4}{v(x,z)^2}\tau^2 - \xi^2} = \tau\sqrt{\frac{4}{v(x,z)^2} - \frac{\xi^2}{\tau^2}},\ (x,t,\xi,\tau) \in \mathcal{P}_\delta(z)$$

for which $a(x, z, D_t, D_x)u \simeq b(x, z, D_t, D_x)b(x, z, D_t, D_x)u$ if $WF(u) \subset \mathcal{P}_\delta(z)$.

b is the world-famous **single square root** ("SSR") operator - see Claerbout, BEI.

To what extent has this construction factored the wave operator:

$$\left(\frac{\partial}{\partial z} - ib(x, z, D_x, D_t)\right)\left(\frac{\partial}{\partial z} + ib(x, z, D_x, D_t)\right)$$

$$= \frac{\partial^2}{\partial z^2} + b(x, z, D_x, D_t)b(x, z, D_x, D_t) + \frac{\partial b}{\partial z}(x, z, D_x, D_t)$$

SSR Assumption: For some $\delta > 0$, the wavefield w satisfies

$$(x, z, t, \xi, \zeta, \tau) \in WF(w) \Rightarrow (x, t, \xi, \tau) \in \mathcal{P}_\delta(z) \text{ and } \zeta\tau > 0$$

This statement has a ray-theoretic interpretation (which will eventually make sense): rays carrying significant energy are nowhere horizontal. Along any such ray, z decreases as t increases - *coming up!*

$$\tilde{w}(x,z,t) = \left(\frac{\partial}{\partial z} + ib(x,z,D_x,D_t)\right) w(x,z,t)$$

$$b(x,z,D_x,D_t)b(x,z,D_x,D_t)w \simeq \left(\frac{4}{v(x,z)^2}D_t^2 - D_x^2\right) w$$

with a smooth error, so

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t)\right) \tilde{w}(x,z,t) = -\frac{2r(x,z)}{v(x,z)^2}\delta(t)$$

$$+i\left(\frac{\partial}{\partial z} b(x,z,D_x,D_t)\right) w(x,z,t)$$

(since b depends on z, the z deriv. does not commute with b).
So $\tilde{w} = \tilde{w}_0 + \tilde{w}_1$, where

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t)\right)\tilde{w}_0(x,z,t) = -\frac{2r(x,z)}{v(x,z)^2}\delta(t)$$

(this is the **SSR modeling equation**)

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t)\right)\tilde{w}_1(x,z,t) = i\left(\frac{\partial}{\partial z}b(x,z,D_x,D_t)\right)w(x,z,t)$$

Claim: $WF(\tilde{w}_1) \subset WF(w)$.

Granted this $\Rightarrow WF(\tilde{w}_0) \subset WF(w)$ also.

Upshot: SSR modeling

$$\tilde{F}_0[v]r(x_s, z_s, t) = \tilde{w}_0(x_s, z_s, t)$$

produces the same singularities (i.e. the same waves) as exploding reflector modeling, so is as good a basis for migration.

SSR migration: assume that sources all lie on $z_s = 0$.

$$< \tilde{F}_0[v]^* d, r > = < d, \tilde{F}_0[v] r >$$

$$= \int dx_s \int dt\, d(x_s, t) \tilde{w}_0(x_s, 0, t)$$

$$= \int dx_s \int dt \int dz\, d(\bar{x}_s, t)\delta(z)\tilde{w}_0(x_s, z, t)$$

Define the adjoint field q by

$$\left(\frac{\partial}{\partial z} - b(x, z, D_x, D_t)\right) q(x, z, t) = d(x, t)\delta(z),\;\; q(x, z, t) \equiv 0,\; z < 0$$

which is equivalent to solving the initial value problem

$$\left(\frac{\partial}{\partial z} - ib(x, z, D_x, D_t)\right) q(x, z, t) = 0,\; z > 0;\,,\;\; q(x, 0, t) = d(x, t)$$

Insert in expression for inner product, integrate by parts, use self-adjointness of b, get

$$< d, \tilde{F}_0[v]r >= \int dx \int dz\, \frac{2r(x, z)}{v(x, z)^2} q(x, z, 0)$$

whence

$$\tilde{F}_0[v]^* d(x,z) = \frac{2}{v(x,z)^2} q(x,z,0)$$

Standard description of this algorithm:

- downward continue data (i.e. solve for q)
- image at $t = 0$.

The art of SSR migration: computable approximations to $b(x, z, D_x, D_t)$
- swimming pool operator, many successors.

Unfinished business: proof of claim

Depends on celebrated **Propagation of Singularities** theorem of Hörmander (1970).

Given symbol $p(\mathbf{x},\xi)$, order m, with asymptotic expansion, define *bicharateristics* as solutions $(\mathbf{x}(t),\xi(t))$ of Hamiltonian system

$$\frac{d\mathbf{x}}{dt} = \frac{\partial p}{\partial \xi}(\mathbf{x},\xi),\ \frac{d\xi}{dt} = -\frac{\partial p}{\partial \mathbf{x}}(\mathbf{x},\xi)$$

with $p(\mathbf{x}(t),\xi(t)) \equiv 0$.

Theorem: Suppose $p(\mathbf{x},D)u = f$, and suppose that for $t_0 \le t \le t_1$, $(\mathbf{x}(t),\xi(t)) \notin WF(f)$. Then either $\{(\mathbf{x}(t),\xi(t)) : t_0 \le t \le t_1\} \subset WF(u)$ or $\{(\mathbf{x}(t),\xi(t)) : t_0 \le t \le t_1\} \subset T^*(\mathbf{R}^n) - WF(u)$.

At least two distinct proofs:

- Nirenberg, 1972
- Hörmander, 1970 (in Taylor, 1981)

Proof of claim: check that bicharacteristics for SSR operator are just upcoming rays of geom. optics for wave equation. These pass into $t < 0$ where RHS is smooth, also initial condn at large z is smooth - so each ray has one "end" outside of $WF(\tilde{w}_1)$. If ray carries singularity, must pass of WF of w, but then it's entirely contained by P of S applied to w. **q. e. d.**

Nonzero offset ("prestack"): starting point is integral representation of the scattered field

$$F[v]r(\mathbf{x}_r,t;\mathbf{x}_s) = \frac{\partial^2}{\partial t^2}\int dx\,\frac{2r(\mathbf{x})}{v(\mathbf{x})^2}\int ds\,G(\mathbf{x}_r,t-s;\mathbf{x})G(\mathbf{x}_s,s;\mathbf{x})$$

By analogy with zero offset case, would like to view this as "exploding reflectors in both directions": reflectors propagate energy upward to sources and to receivers. However can't do this because reflection location is *same* for both.

Bold stroke: introduce a new space variable $\mathbf{y}$, define

$$\tilde{F}[v]R(\mathbf{x}_r,t;\mathbf{x}_s) = \frac{\partial^2}{\partial t^2}\int\int dx\,dy\,R(\mathbf{x},\mathbf{y})\int ds\,G(\mathbf{x}_r,t-s;\mathbf{x})G(\mathbf{x}_s,s;\mathbf{y})$$

and note that $\tilde{F}[v]R = F[v]r$ if

$$R(\mathbf{x},\mathbf{y}) = \frac{2r}{v^2}\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\delta(\mathbf{x}-\mathbf{y})$$

This trick decomposes $F[v]$ into two "exploding reflectors":

$$\tilde{F}[v]R(\mathbf{x}_r,t;\mathbf{x}_s) = u(\mathbf{x},t;\mathbf{x}_s)|_{\mathbf{x}=\mathbf{x}_r}$$

where

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) u(\mathbf{x},t;\mathbf{x}_s) = \int dy\, R(\mathbf{x},\mathbf{y})G(\mathbf{x}_s,t;\mathbf{y})$$

$$\equiv w_s(\mathbf{x}_s,t;\mathbf{x})$$

("upward continue the receivers"),

$$\left(\frac{1}{v(\mathbf{y})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{y}}\right) w_s(\mathbf{y},t;\mathbf{x}) = R(\mathbf{x},\mathbf{y})\delta(t)$$

("upward continue the sources").

This factorization of $F[v]$ ($r \mapsto R \mapsto \tilde{F}[v]R$) leads to a reverse time computation of adjoint with - will discuss on Friday.

It's equally possible to continue the receivers first, then the sources, which leads to

$$\left(\frac{1}{v(\mathbf{y})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{y}}\right) u(\mathbf{x}_r, t; \mathbf{y}) = \int dx\, R(\mathbf{x},\mathbf{y}) G(\mathbf{x}_r, t; \mathbf{x})$$

$$\equiv w_r(\mathbf{x}_r, t; \mathbf{y})$$

("upward continue the sources"),

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) w_r(\mathbf{x}, t; \mathbf{y}) = R(\mathbf{x},\mathbf{y})\delta(t)$$

("upward continue the receivers").

Apply reverse depth concept: as before, go 2D temporarily, $\mathbf{x} = (x, z_r), \mathbf{y} = (y, z_s)$, all sources and receivers on $z = 0$.

Double Square Root ("DSR") assumption: For some $\delta > 0$, the wavefield u satisfies

$$(x, z_r, t, y, z_s, \xi, \zeta_s, \tau, \eta, \zeta_r) \in WF(u) \Rightarrow$$

$$(x, t, \xi, \tau) \in \mathcal{P}_\delta(z_r), \ (y, t, \eta, \tau) \in \mathcal{P}_\delta(z_s), \text{ and } \zeta_r\tau > 0, \ \zeta_s\tau > 0,$$

As for SSR, there is a ray-theoretic interpretation: rays from source and receiver to scattering point stay away from the vertical and decrease in z for increasing t, i.e. they are all upcoming.

Since z will be singled out (and eventually $R(\mathbf{x},\mathbf{y})$ will have a factor of $\delta(\mathbf{x},\mathbf{y})$), impose the constraint that

$$R(x,z,x,z_s) = \tilde{R}(x,y,z)\delta(z-z_s)$$

Define upcoming projections as for SSR:

$$\tilde{w}_s = \left(\frac{\partial}{\partial z_s} + ib(y,z_s,D_y,D_t)\right) w_s,$$

$$\tilde{w}_r = \left(\frac{\partial}{\partial z_r} + ib(x,z_r,D_x,D_t)\right) w_r,$$

$$\tilde{u} = \left(\frac{\partial}{\partial z_s} + ib(y,z_s,D_y,D_t)\right)\left(\frac{\partial}{\partial z_r} + ib(x,z_r,D_x,D_t)\right) u$$

Except for lower order commutators which we justify throwing away as before,

$$\left(\frac{\partial}{\partial z_s} - ib(y, z_s, D_y, D_t)\right)\tilde{w}_s = \tilde{R}\delta(z_r - z_s)\delta(t),$$

$$\left(\frac{\partial}{\partial z_r} - ib(x, z_r, D_x, D_t)\right)\tilde{w}_r = \tilde{R}\delta(z_r - z_s)\delta(t),$$

$$\left(\frac{\partial}{\partial z_r} - ib(x, z_r, D_x, D_t)\right)\tilde{u} = \tilde{w}_s$$

$$\left(\frac{\partial}{\partial z_s} - ib(y, z_s, D_y, D_t)\right)\tilde{u} = \tilde{w}_r$$

Initial (final) conditions are that $\tilde{w}_r, \tilde{w}_s$, and $\tilde{u}$ all vanish for large z - the equations are to be solve in decreasing z ("upward continuation").

Simultaneous upward continuation:

$$\frac{\partial}{\partial z}\tilde{u}(x,z,t;y,z) = \frac{\partial}{\partial z_r}\tilde{u}(x,z_r,t;y,z)|_{z=z_r} + \frac{\partial}{\partial z_r}\tilde{u}(x,z,t;y,z_s)|_{z=z_s}$$

$$= [ib(x,z_r,D_x,D_t)\tilde{u} + \tilde{w}_s + ib(y,z_s,D_y,D_t)\tilde{u} + \tilde{w}_r]_{z_r=z_s=z}$$

Since $\tilde{w}_s(y,z,t;x,z) = \tilde{w}_r(x,z,t;y,z) = \tilde{R}(x,y,z)\delta(t)$, $\tilde{u}$ is seen to satisfy the **DSR modeling equation**:

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t) - ib(y,z,D_y,D_t)\right)\tilde{u}(x,z,t;y,z) = 2\tilde{R}(x,y,z)\delta(t)$$

$$\tilde{F}[v]\tilde{R}(x_r,t;x_s) = \tilde{u}(x_r,0,t;x_s,0)$$

Computation of adjoint follows same pattern as for SSR, and leads to

DSR migration equation: solve

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t) - ib(y,z,D_y,D_t)\right)\tilde{q}(x,y,z,t) = 0$$

in *increasing* z with initial condition at $z=0$:

$$\tilde{q}(x_r,x_s,0,t) = d(x_r,x_s,t)$$

Then $\tilde{F}[v]^*d(x,y,z) = \tilde{q}(x,y,z,0)$

The physical DSR model has $\tilde{R}(x,y,z) = r(x,z)\delta(x-y)$, so final step in DSR computation of $F[v]^*$ is adjoint of $r \mapsto \tilde{R}$:

$$F[v]^*d(x,z) = \tilde{q}(x,x,z,0)$$

Standard description of DSR migration (Claerbout, IEI):

- downward continue sources and receivers (solve DSR migration equation)
- image at $t = 0$ and zero offset ($x = y$)

Another moniker: “survey sinking”: DSR field $\tilde{q}$ is (related to) the field that you would get by conducting the survey with sources and receivers at depth z. At any given depth, the zero-offset, time-zero part of the field is the instantaneous response to scatterers on which source = receiver is sitting, therefore constitutes an image.

As for SSR, the art of DSR migration is in the approximation of the DSR operator.

4. Velocity Analysis

Partially linearized seismic inverse problem: given observed seismic data d^{obs}, find smooth *velocity* $v \in \mathcal{E}(X), X \subset \mathbf{R}^3$ oscillatory *reflectivity* $r \in \mathcal{E}'(X)$ so that

$$D\mathcal{F}[v](vr) = F[v]r \simeq d^{\text{obs}}$$

where the acoustic potential field u and its perturbation δu solve

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u = f(t)\delta(\mathbf{x} - \mathbf{x}_s),$$

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta u = 2r\nabla^2 u$$

plus suitable bdry and initial conditions.

$$F[v]r = \left.\frac{\partial \delta u}{\partial t}\right|_Y$$

data acquisition manifold $Y = \{(\mathbf{x}_r, t; \mathbf{x}_s)\} \subset \mathbf{R}^7$, dimn $Y \leq 5$ (many idealizations here!).

$F[v] : \mathcal{E}'(X) \to \mathcal{D}'(Y)$ is a linear map but dependence on v is quite nonlinear, so this inverse problem is nonlinear.

Direct approach, eg. via output least squares - hopeless! (Gauthier et al., 1986; Kolb et al., 1986; Bunks et al., 1995)

Velocity analysis = clever indirect approach to this inverse problem, based on concept of extended model or extension of $F[v]$.

Extension of $F[v]$ (aka *extended model*): manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \rightarrow \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \rightarrow \mathcal{D}'(Y)$ so that

$$\begin{array}{ccccc} & & \bar{F}[v] & & \\ & \mathcal{E}'(\bar{X}) & \rightarrow & \mathcal{D}'(Y) & \\ \chi & \uparrow & & \uparrow & \text{id} \\ & \mathcal{E}'(X) & \rightarrow & \mathcal{D}'(Y) & \\ & & F[v] & & \end{array}$$

commutes, i.e.

$$\bar{F}[v]\chi r = F[v]r$$

(Familiar) Example: the *Convolutional Model*

- *Approximation* of P. L. model, accurate when v, r functions of z only

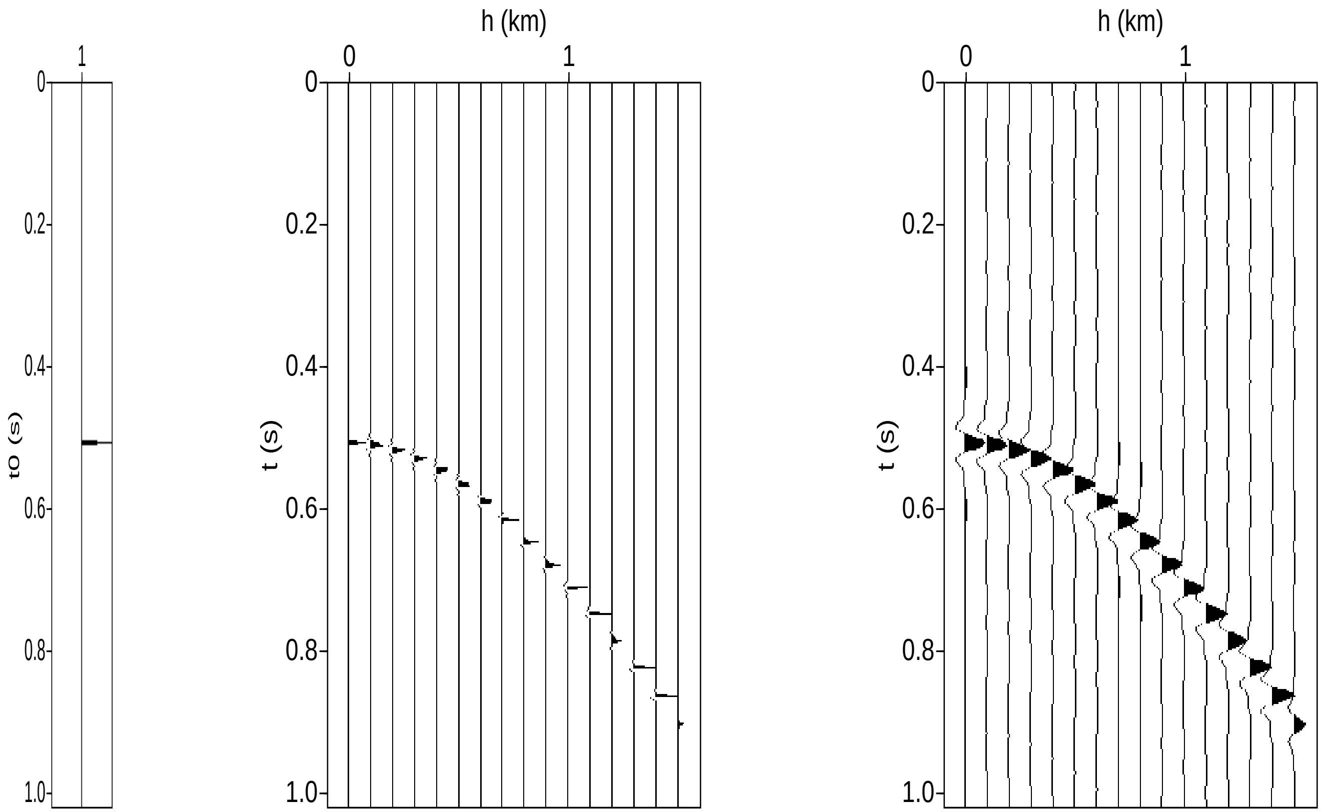
- data function of t, $h = (x_r - x_s)/2$ *half-offset*

- two-way traveltime $\tau(z,h)$, inverse $\zeta(t,h)$

- if $v =$const. then $\tau(z,h) = 2\sqrt{z^2 + h^2}/v$

$$F[v]r(t,h) = \int dt' \, f(t-t')r(\zeta(t',h))$$

("inverse NMO, convolve with source")



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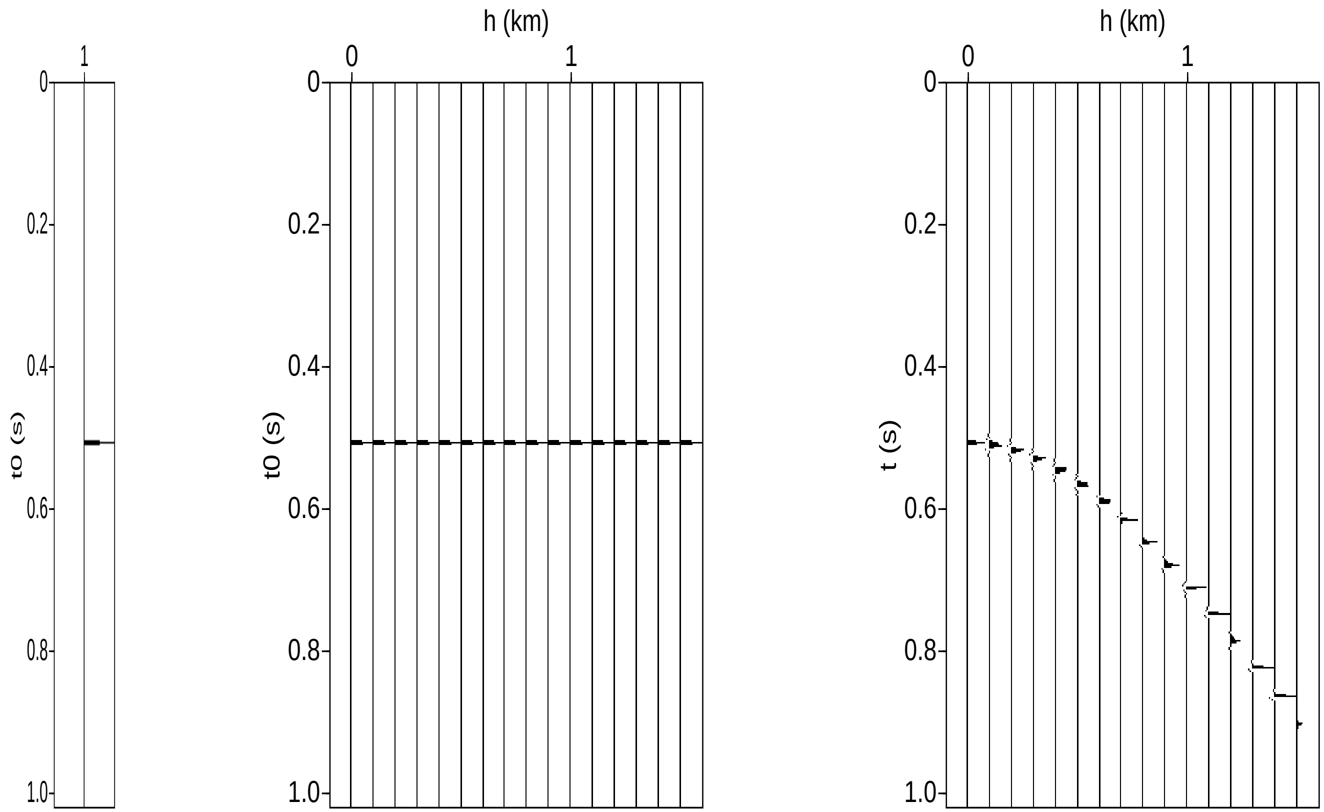
Left: $r(t_0)$. Middle: $r(\zeta(t,h))$. Right: $F[v]r(t,h)$

Factor convolutional model through extension: *replicate* r for each h

$$\chi : r(z) \mapsto \bar{r}(z,h) = r(z)$$

then apply inverse NMO and convolve with source, independently for each h

$$\bar{F}[v] : \bar{r}(z,h) \mapsto f * \bar{r}(\zeta(t,h),h)$$



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Left: $r(t_0)$. Middle: $\bar{r}(t_0,h) = \chi r$. Right: $\bar{r}(\zeta(t,h),h)$.

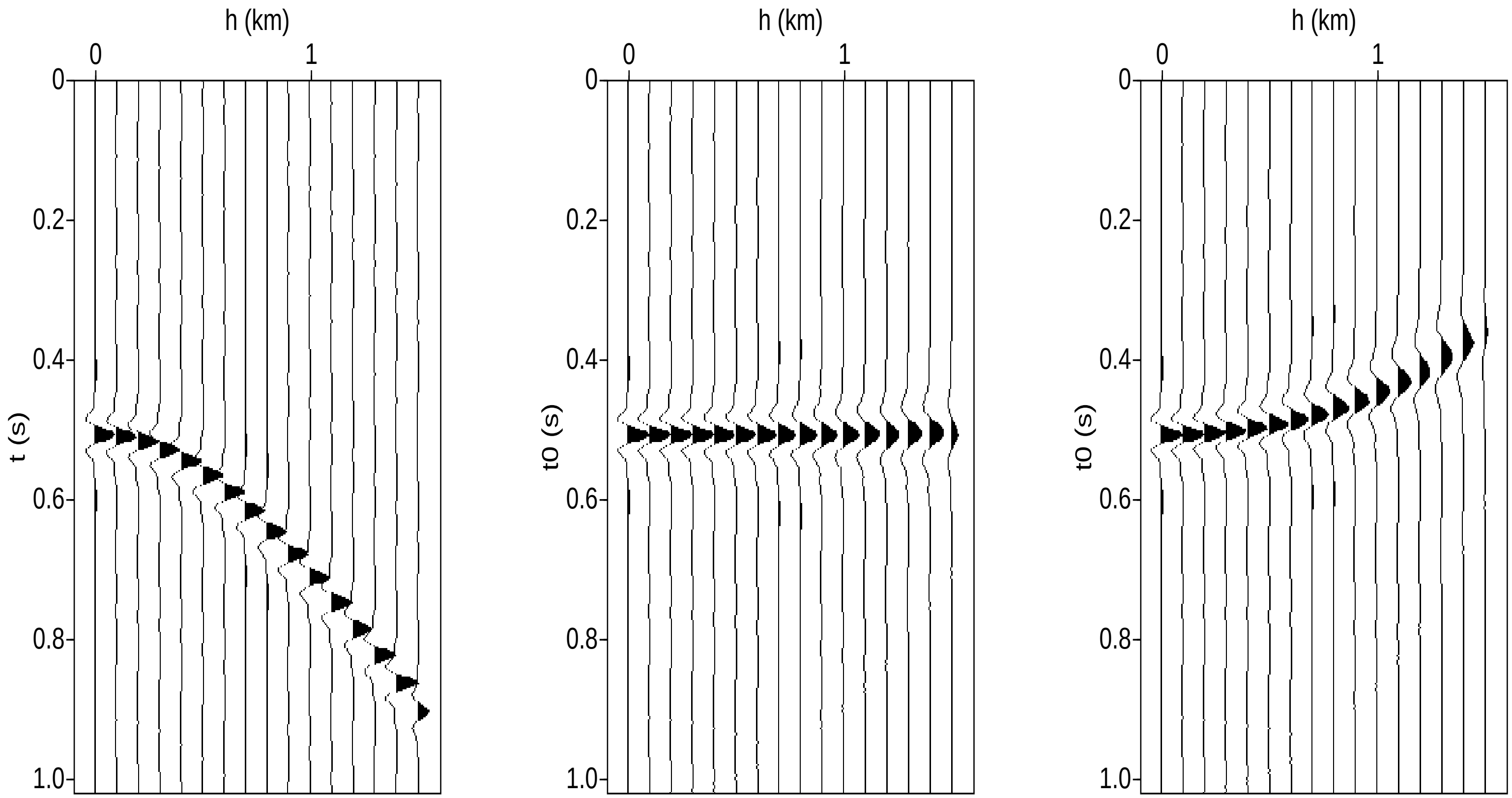
Invertible extension: $\bar{F}[v]$ has a *right parametrix* $\bar{G}[v]$, i.e.

$$I - \bar{F}[v]\bar{G}[v]$$

is smoothing.

Example: for the convolutional model, $\bar{G}[v]$ is signature decon followed by NMO, applied trace-by-trace.

NB: The trivial extension - $\bar{X} = X, \bar{F} = F$ - is virtually never invertible.



Left: $d(t,h) = F[v']r(t,h)$. Middle: $\bar{G}[v']d(t_0,h)$. Right: $\bar{G}[v_1]d(t_0,h), v_1 \neq v$

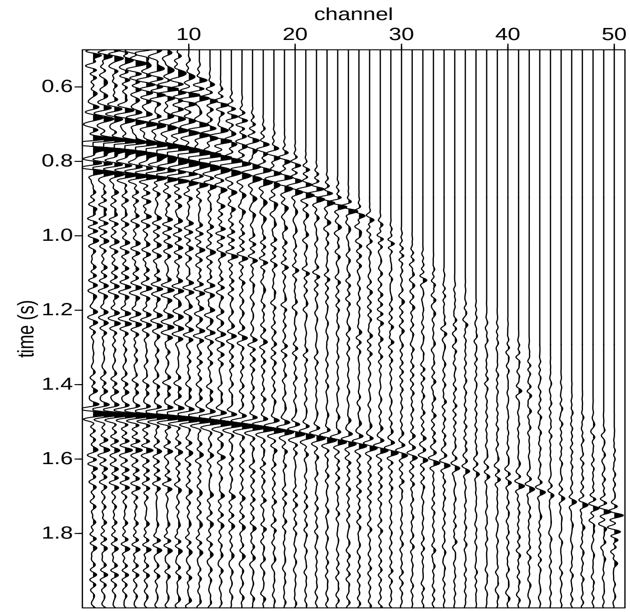
Reformulation of inverse problem: given d^{obs}, find v so that $\bar{G}[v]d^{\text{obs}} \in$ the range of χ.

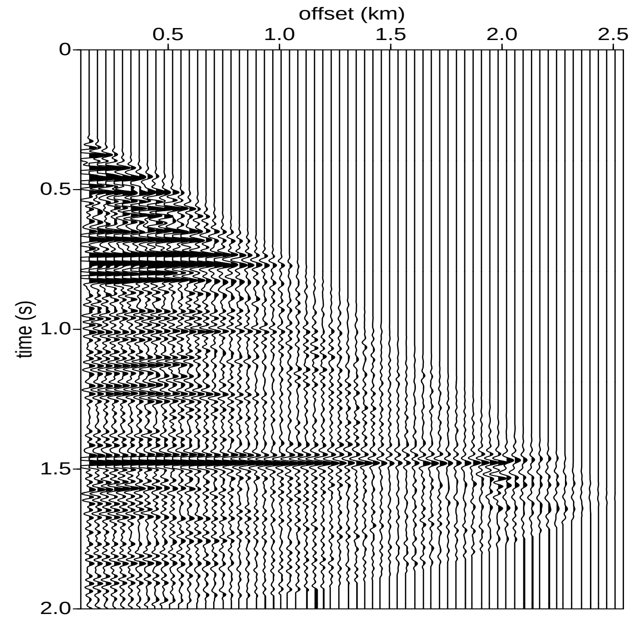
Proof: that is, $\bar{G}[v]d^{\text{obs}} = \chi r$ for some r, so $d^{\text{obs}} \simeq \bar{F}[v]\bar{G}[v]d^{\text{obs}} = \bar{F}[v]\chi r = F[v]r$ Q. E. D.

This is velocity analysis!

Example: Standard VA. Apply convolutional model to each midpoint in CMP-binned data. Range of $\chi = \bar{r}(z,h)$ *independent of h*, i.e. flat. SO: twiddle v so that $\bar{G}[v]d^{\text{obs}}$ shows flat events.

Caveats: in practice, be happy when $\bar{G}[v]d^{\text{obs}}$ is in range of χ except for wrong amplitudes, finite frequency effects, and obvious (!) noise.





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Left: part of survey (d^{obs}) from North Sea (thanks: Shell Research), lightly preprocessed.
Right: restriction of $\bar{G}[v]d^{\text{obs}}$ to $\mathbf{x}_m =$ const (function of depth, offset): shows rel. sm'ness in h (offset) for properly chosen v.

The usual extended model behind Migration Velocity Analysis:

- v, r functions of all space variables

- $\chi r(\mathbf{x}, \mathbf{h}) = r(\mathbf{x})$ (so $\bar{r} \in$ range of $\chi \Leftrightarrow$ plots of $\bar{r}(\cdot, \cdot, z, h)$ appear *flat*)

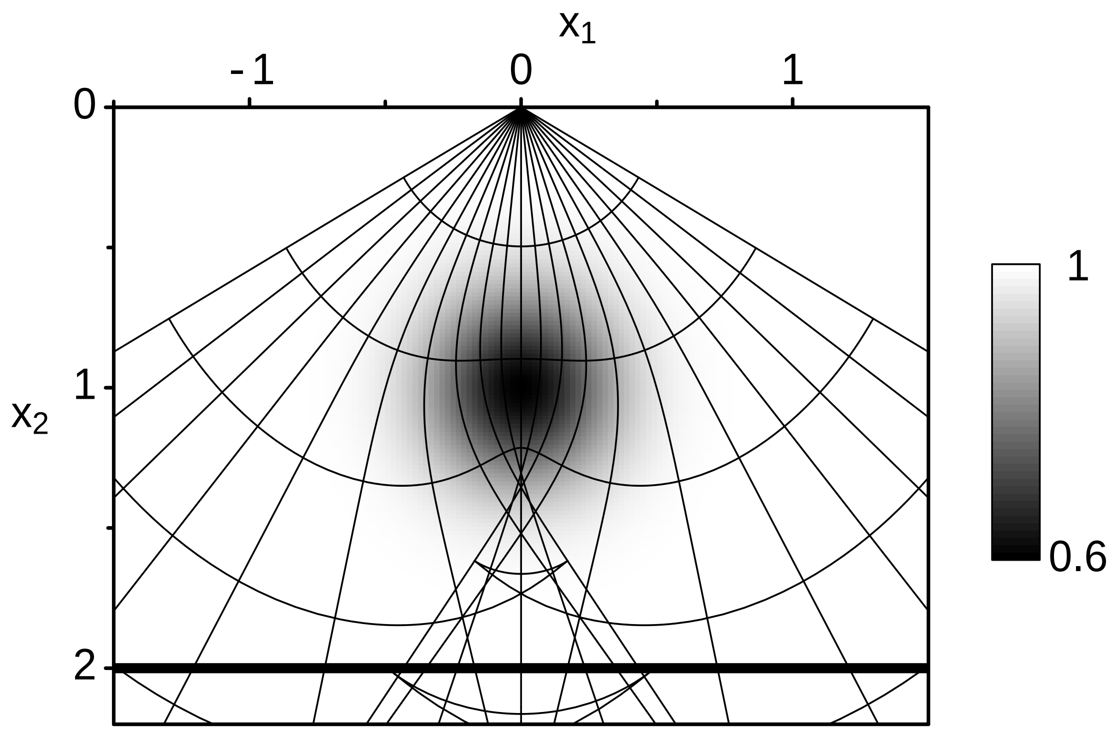
- $$\bar{F}[v]\bar{r}(\mathbf{x}_r, \mathbf{x}_s, t) = \frac{\partial^2}{\partial t^2} \int dx\, \bar{r}(\mathbf{x}, \mathbf{h}) \int ds\, G(\mathbf{x}_r, t - s; \mathbf{x}) u(\mathbf{x}_s, s; \mathbf{x})$$
  (recall $\mathbf{h} = (\mathbf{x}_r - \mathbf{x}_s)/2$)

NB: $\bar{F}$ is "block diagonal" - family of operators (FIOs) parametrized by $\mathbf{h}$.

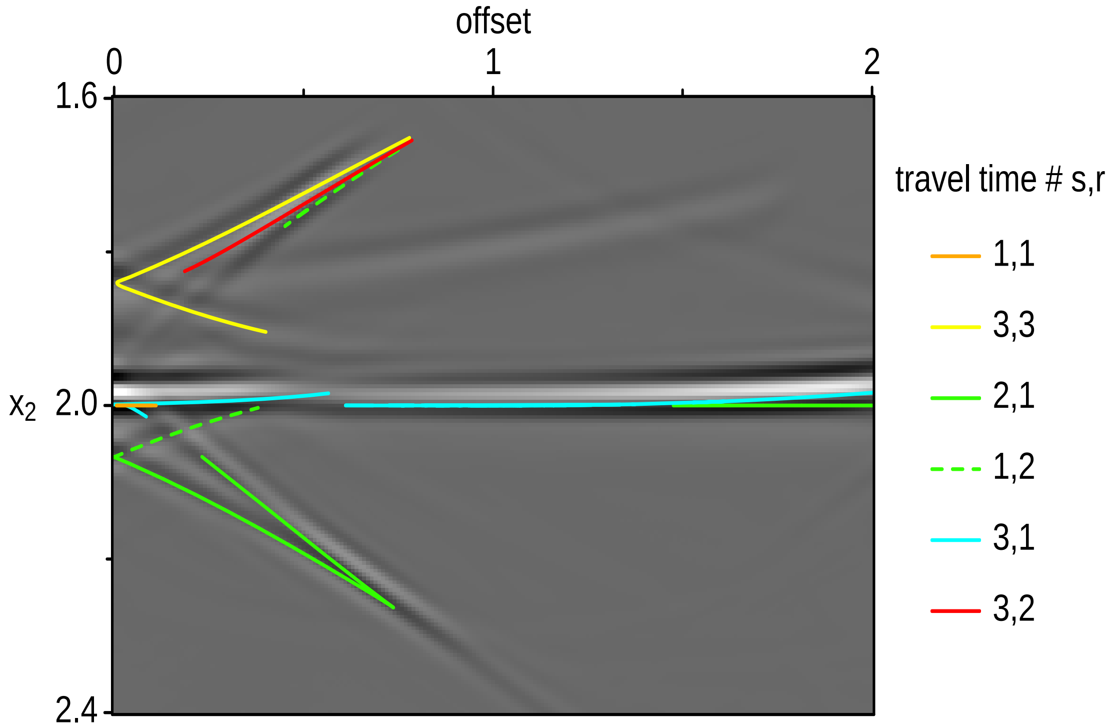
- Beylkin (1985), Rakesh (1988): if $\|\nabla^2 v\|_{C^0}$ "not too big", then the usual extension is invertible.

- $\bar{G}$ = *common offset* migration-inversion aka ray-Born inversion aka true-amplitude migration etc. etc. Usually implemented as generalized Radon transform = "weighted diffraction stack" (Beylkin, Bleistein, DeHoop,...)

- Nolan, Stolk, WWS: if $\|\nabla^2 v\|_{C^0}$ is too big, *usual extension is not invertible!*



Example: Gaussian lens over flat reflector at depth 2 km ($r(\mathbf{x}) = \delta(z-2)$).



Common Image Gather (const. x, y slice) of $\bar{G}[v]d^{\text{obs}}$: not flat, i.e. not in range of χ even allowing for amplitude, finite frequency errors, and *even though velocity is correct!* [Stolk, WWS 2002]

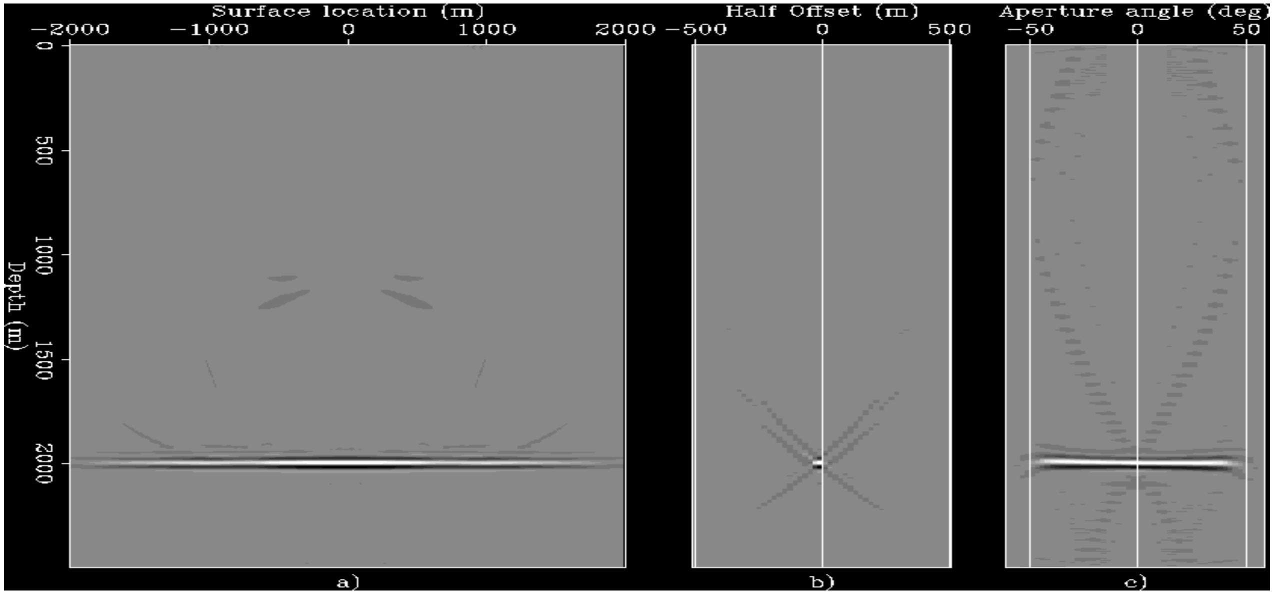
Claerbout's extended model = basis of *survey sinking* or *shot-geophone* migration:

- $\chi r(\mathbf{x},\mathbf{h}) = r(\mathbf{x})\delta(\mathbf{h})$, so $\bar{r} \in$ range of $\chi \Leftrightarrow$ plots of $\bar{r}(\cdot,\cdot,z,h)$ appear *focussed* at $\mathbf{h} = 0$

$$\bar{F}[v]\bar{r}(\mathbf{x}_r,\mathbf{x}_s,t)$$

$$= \frac{\partial^2}{\partial t^2}\int dx \int dh\, \bar{r}(\mathbf{x},\mathbf{h}) \int ds\, G(\mathbf{x}_r, t-s; \mathbf{x}+\mathbf{h}) u(\mathbf{x}_s, s; \mathbf{x}-\mathbf{h})$$

- This extension is invertible, assuming (i) $\mathbf{h}_3 = 0$ (horizontal offset only) and (ii) "DSR hypothesis": rays do not turn. Then adjoint map is equivalent modulo elliptic ΨDO factor to shot-geophone migration via DSR equation [Stolk-DeHoop 2001]



Lens data, shot-geophone migration [B. Biondi, 2002]
Left: Image via DSR. Middle: $\bar{G}[v]d$ - well-focused (at $\mathbf{h} = 0$),
i.e. in range of χ to extent possible. Right: Angle CIG.

Alternate expression for extended S-G model:

$$\bar{F}[v]\bar{r}(\mathbf{x}_r, t; \mathbf{x}_s) = \frac{\partial}{\partial t}\delta\bar{u}(\mathbf{x}, t; \mathbf{x}_s)|_{\mathbf{x}=\mathbf{x}_r}$$

where

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right)\delta\bar{u}(\mathbf{x}, t; \mathbf{x}_s) = \int_{\mathbf{x}+2\Sigma_d} dy\,\bar{r}(\mathbf{x}, \mathbf{y})u(\mathbf{y}, t; \mathbf{x}_s)$$

Computing $\bar{G}[v]$: instead of parametrix, be satisfied with adjoint - the two differ by a ΨDO factor, which will not affect smoothness of CIGs.

Computing the adjoint: use the *adjoint state method* (WWS, Biondi & Shan, SEG 2002).

Define *adjoint state* w:

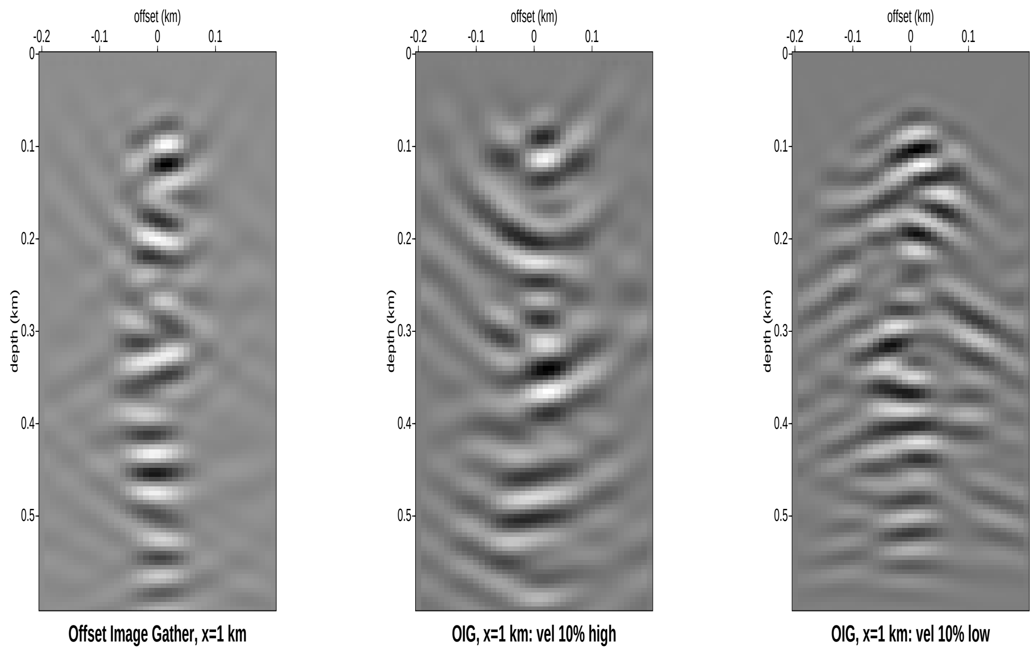
$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla_{\mathbf{x}}^2\right) w(\mathbf{x},t;\mathbf{x}_s) = \int dx_r\, d(\mathbf{x}_r,t;\mathbf{x}_s)\delta(\mathbf{x}-\mathbf{x}_r)$$

with $w(\mathbf{x},t;\mathbf{x}_s) = 0, t >> 0$. [This is exactly the backpropagated field of standard reverse time prestack migration, cf. Lines talk.]

Then

$$\bar{F}[v]^* d(\mathbf{x},\mathbf{h}) = \int dx_s \int dt\, u(\mathbf{x}+2\mathbf{h},t;\mathbf{x}_s) w(\mathbf{x},t;\mathbf{x}_s)$$

[This is exactly the same computation as for standard reverse time prestack, except that crosscorrelation occurs at an offset $2\mathbf{h}$].



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Two way reverse time horizontal offset S-G image gathers of data from random reflectivity, constant velocity. From left to right: correct velocity, 10% high, 10% low.

Some Loose Ends

- invertibility of S-G extended model only known under DSR assumption with horizontal offsets [Stolk-DeHoop, 2001]. Vertical offsets are good when DSR breaks down, eg. to image overhanging reflectors [Biondi, WWS 2002]. Current best result: data focusses only at offset = 0 within a limited range off offsets; focussing at large offsets not ruled out [WWS, 2002]. What actually happens?

- S-G extension amounts to construction of annihilators [cf. DeHoop]. How can one characterize globally invertible annihilator representations?

- quantification of non-membership in range of χ (DSO) - which ones yield good optimization problems locally [Stolk-WWS, IP 2003] or globally?

Conclusions

- Most of contemporary SDP related *partially linearized* seismic inverse problem
- Velocity analysis = approach to solution of PL seismic inverse problem via invertible extended models
- Usual extended models (common offset, common shot, common angle,...) are not invertible when the velocity structure is complex, due to multipathing
- The extended model of shot-geophone migration is invertible even in the presence of multipathing
- Shot-geophone migration has a reverse-time implementation