Kinematics of Reverse Time S-G Migration

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Agenda: explore *prestack focussing properties* of RT S-G migration, proper definition of *image volume*, using ray theory.

- "Standard" PSDM (CO, CS, CSA) exhibits *kinematic artifacts* in complex structure (TRIP,...): image gathers *not flat* when velocity is correct.

- Stolk-deHoop '01: no artifacts in prestack S-G migration (perfect focussing of offset image panels at zero offset, even in complex velocity structures). Limitations: reflector dip subhorizontal, rays do not turn ("DSR assumption")

- RT formulation permits arbitrary reflector orientation, propagation. Image volume combining horizontal, vertical offsets focusses near zero offset.

Outline, Part I:

- Born approximation, extended models, common offset, angle PSDM
- kinematic image artifacts: why image gathers may not be flat at correct velocity
- double reflector model, double reflector PSDM,
- relation to S-G migration via DSR equation.
- reverse time adjoint computation

Outline, Part II:

- kinematics of double reflector model, horizontal offsets and focussing property under DSR assumption
- why horizontal offset is insufficient; combining horizontal and vertical offset: filtered coordinate image volumes
- derivation of focussing property, limitation to small offset corridor
- some implementation details: how to make RT S-G as fast as standard RT

Born approximation = linearized seismic inverse problem, acoustic version: given smooth *velocity* $v(x, y, z) = v(\mathbf{x})$, seismic data $d(\mathbf{x}_r, t; \mathbf{x}_s)$, find oscillatory *reflectivity* $r(\mathbf{x}) \equiv \delta v(\mathbf{x})/v(\mathbf{x})$ to fit the data:

$$F[v]r \simeq d$$

Definition of *Born modeling = acoustic forward* operator $F[v]$, via PDEs: acoustic Green's function G and its perturbation δG solve

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G = \delta(t)\delta(\mathbf{x} - \mathbf{x}_s), \;\; \left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta u = \frac{2r}{v^2} G$$

plus suitable bdry and initial conditions. Then

$$F[v]r(\mathbf{x}_r, t; \mathbf{x}_s) = \delta G(\mathbf{x}_r, t; \mathbf{x}_s)$$

[Note: lots of things ignored - source, P-S conversion, anelasticity,...]

Integral representation of Born modeling:

$$\delta G(\mathbf{x}_r,t;\mathbf{x}_s) = \frac{\partial^2}{\partial t^2}\int dy \int d\tau \,\frac{2r(\mathbf{y})}{v^2(\mathbf{y})} G(\mathbf{y},t-\tau;\mathbf{x}_r)G(\mathbf{y},\tau;\mathbf{x}_s) \quad (1)$$

Insert asymptotic repn. of Green's function:

$$G(\mathbf{x},t;\mathbf{x}_s) = A(\mathbf{x};\mathbf{x}_s)\delta(t-\tau(\mathbf{x},\mathbf{x}_s))$$

gives *Kirchhoff* or *Ray-Born* or *GRT* approximation to linearized fwd map:

$$\delta G(\mathbf{x}_r,t;\mathbf{x}_s) = \frac{\partial^2}{\partial t^2}\int dy \,\frac{2r(\mathbf{y})}{v^2(\mathbf{y})} A(\mathbf{y},\mathbf{x}_r)A(\mathbf{y},\mathbf{x}_s)\delta(t-\tau(\mathbf{y},\mathbf{x}_r)-\tau(\mathbf{y},\mathbf{x}_s))$$

BUT this is only valid for r concentrated near sources and receivers (no caustics!) - so we won't use it.

"Prestack" or extended modeling, common offset: in integral representation of δG, *permit* r to depend on (half) offset $\mathbf{h} = (\mathbf{x}_r - \mathbf{x}_s)/2$, call it $R(\mathbf{x},\mathbf{h})$:

$$\tilde{F}[v]R(\mathbf{x}_s + 2\mathbf{h}, t; \mathbf{x}_s) \equiv$$

$$= \frac{\partial^2}{\partial t^2} \int dy \int d\tau \, \frac{2R(\mathbf{y},\mathbf{h})}{v^2(\mathbf{y})} G(\mathbf{y}, t-\tau; \mathbf{x}_s + 2\mathbf{h}) G(\mathbf{y},\tau;\mathbf{x}_s)$$

Two things worth noting:

- Each offset bin is modeled *independently*.
- If $R(\mathbf{x},\mathbf{h})$ is *independent of* $\mathbf{h}$, i.e. $R(\mathbf{x},\mathbf{h}) = r(\mathbf{x})$, then this is simply Born modeling - $\tilde{F}[v]R = F[v]r$ - so $R(\mathbf{x},\mathbf{h})$ is an *extended model*.

Properties of common offset extended (or prestack) Born modeling operator $\tilde{F}[v]$:

- invertible in mild structure, i.e. absent caustics - asymptotic inverse is also GRT [Beylkin, Rakesh, Bleistein, Burridge, Miller, DeHoop, Spencer, tenKroode, Smit,...]

- asymptotic inverse is a *true amplitude migration* operator: reflectors (local oscillatory plane wave components of R) correctly positioned with correct amplitude, with correct amplitude, but low-frequency trends are not recovered.

- adjoint operator $\tilde{F}[v]^*$ is also a migration operator, i.e. positions reflectors correctly but with possibly incorrect amplitudes (again, absent caustics!):

$$\tilde{F}[v]^* d(\mathbf{x},\mathbf{h}) =$$

$$\frac{2}{v^2(\mathbf{y})}\int dx_s \int d\tau\, G(\mathbf{y}, t-\tau; \mathbf{x}_s + 2\mathbf{h}) G(\mathbf{y}, \tau; \mathbf{x}_s) \frac{\partial^2}{\partial t^2} d(\mathbf{x}_s + 2\mathbf{h}, t; \mathbf{x}_s)$$

$+$ asymptotic Green's function $=$ Kirchhoff common offset depth migration (with particular choice of amplitude).

The basis of velocity analysis:

If velocity is correct, then *image volume* $\tilde{F}[v]^* d$ has *same reflectors* as true reflectivity

$\Rightarrow$ independent of $\mathbf{h}$ $\Rightarrow$ image *gathers* are *flat*

Bad News: If structure is sufficiently complex, so that caustics and multipathing occur, the Kirchhoff PSDM operator $\tilde{F}[v]^*$ is *not* an inverse (in the kinematic sense - except for amplitudes) of $\tilde{F}$: apparent reflectors with nonzero dip appear - *kinematic image artifacts*.

$\Rightarrow$ common image gathers are not flat even when velocity is correct.

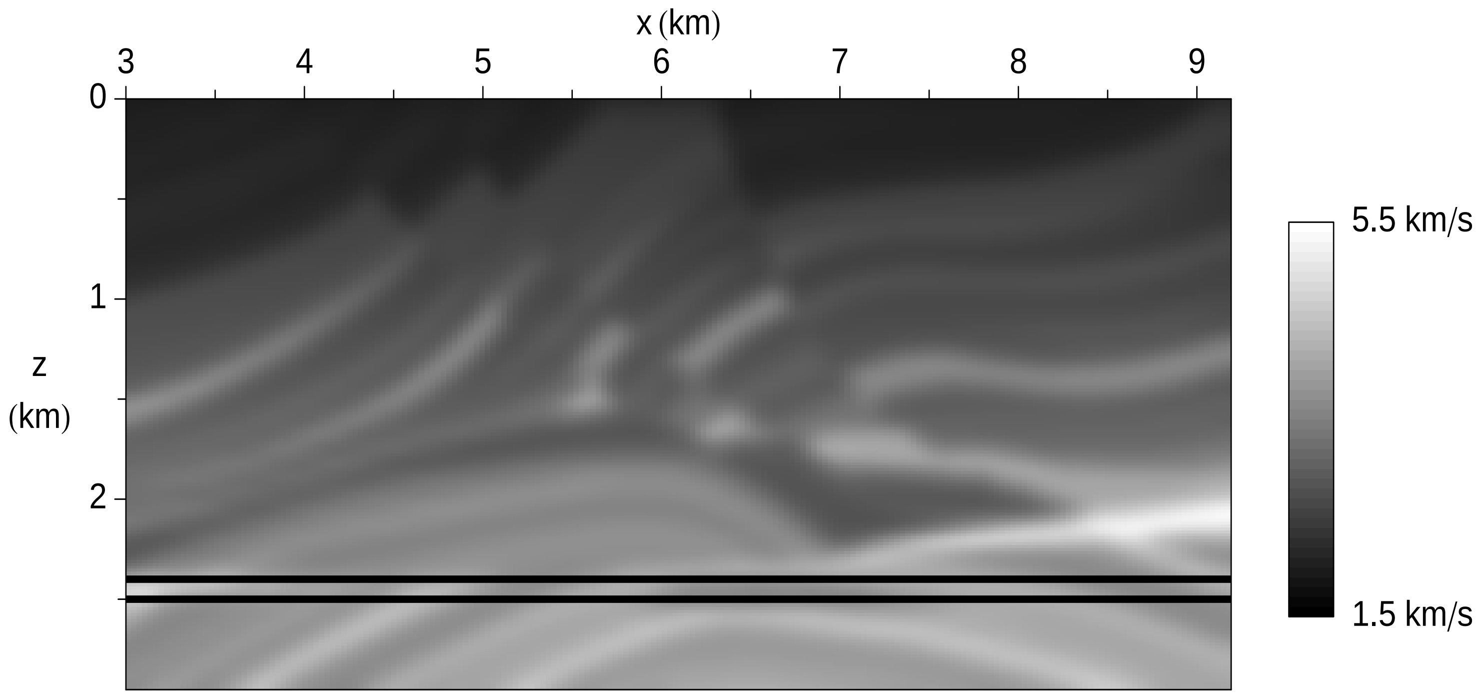
Common shot migration: Nolan, 1996 SEG; common offset, scattering angle (Kirchhoff) migration: Stolk & S. 2002 SEG.

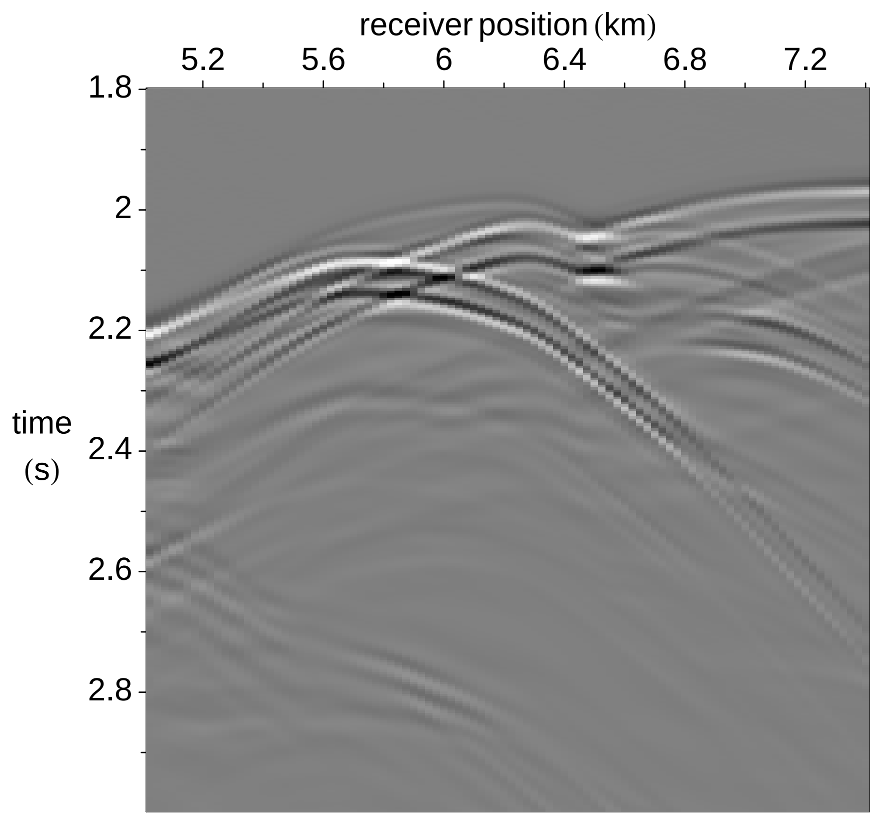
Example (data courtesy of G. Lambaré; migration computations by C. Stolk):

"Canonical test case" (Xu et al, 2001): $v =$ Marmousi model smoothed by Gaussian, half-power radius = 150 m; $r =$ two flat reflectors, depths 2400, 2500 m.

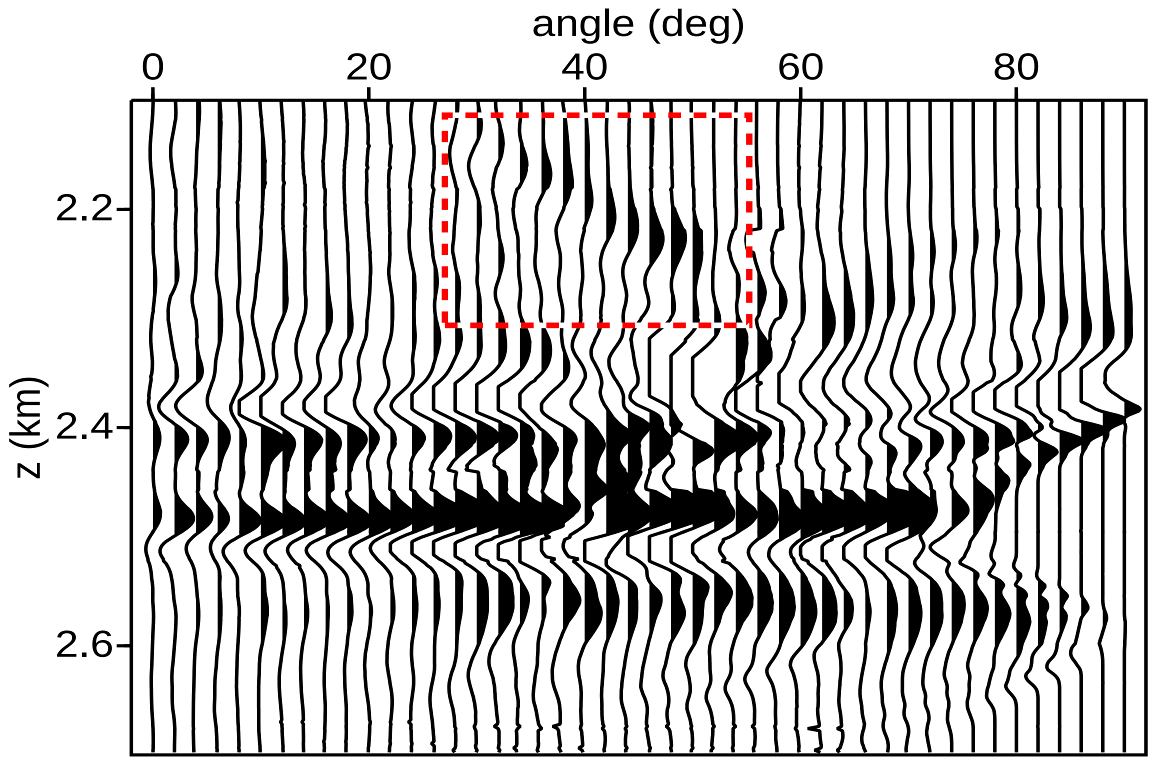
Migration operator = multiarrival Kirchhoff, i.e. proper asymptotic approximation of integral representation of $\tilde{F}[v]^*$, implemented via dynamic ray tracing.

to appear in *Geophysics*

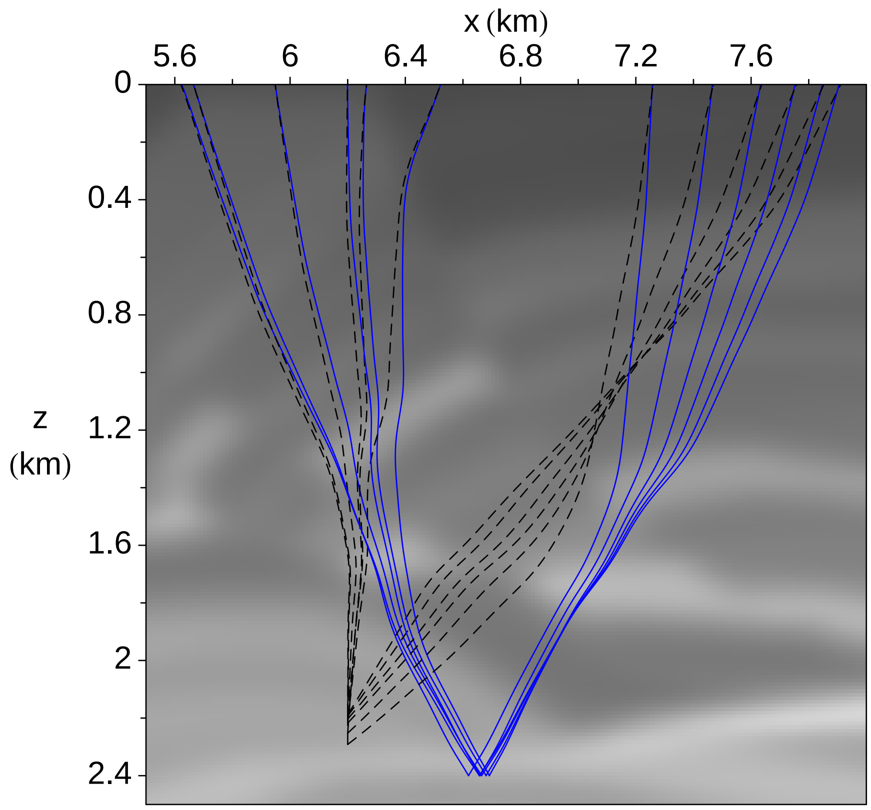

x (km)
3
4
5
6
7
8
9
0
1
2
z
(km)
5.5 km/s
1.5 km/s



Typical shot gather: much evidence of multipathing, caustic formation.



Typical common scattering angle image gather: note nonflat event in box. This is kinematic, not a signal processing artifact: it results from data event migrating along different ray pair than that which produced it.



Blue rays = energy path producing data event. Black rays: energy path for migration, resulting in displaced, angle-dependent image artifact.

S-G modeling - begin with a *different extension* of Born modeling:

$$\bar{F}[v]R(\mathbf{x}_r, t; \mathbf{x}_s) =$$

$$\frac{\partial^2}{\partial t^2} \int dy \int dh \int d\tau \, \frac{2R(\mathbf{y},\mathbf{h})}{v^2(\mathbf{y})} G(\mathbf{y}+\mathbf{h}, t-\tau; \mathbf{x}_r) G(\mathbf{y}-\mathbf{h}, \tau; \mathbf{x}_s)$$

Looks similar to common offset extension, BUT:

- "offset" parameter $\mathbf{h}$ is *not* same as surface offset $(\mathbf{x}_r - \mathbf{x}_s)/2$ - *two* reflection points $\mathbf{y} \pm \mathbf{h}$ - *double reflector model*

- each output point $(\mathbf{x}_r, t; \mathbf{x}_s)$ depends on all model points $(\mathbf{x}, \mathbf{h})$

- Same as Born modeling - $\bar{F}[v]R = F[v]r$ - when $R(\mathbf{x}, \mathbf{h}) = r(x)\delta(\mathbf{h})$, i.e. (double) reflectivity *focussed* at offset zero (rather than flat as in common offset extension).

An immediate difficulty - too many parameters.

- for common offset extension, $\mathbf{h}$ = surface offset, therefore (essentially) horizontal, prestack model $R(\mathbf{x}, \mathbf{h})$ has same number of parameters as data - this is needed for invertibility.

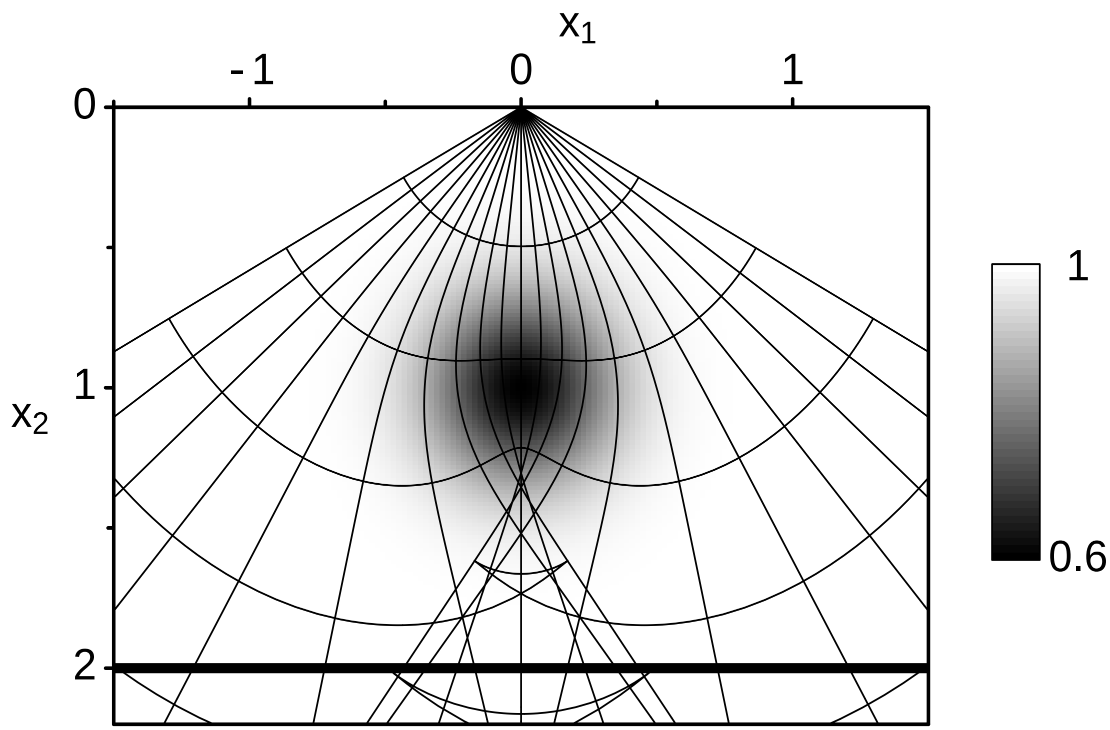
- for double reflector model, $\mathbf{h}$ is not surface offset, therefore not constrained by geometry of sources and receivers - can be essentially arbitrary! Too many parameters, $\bar{F}[v]$ cannot be invertible as defined.

An obvious solution: *mandate* that $\mathbf{h}$ is horizontal, i.e. $h_z \equiv 0$.

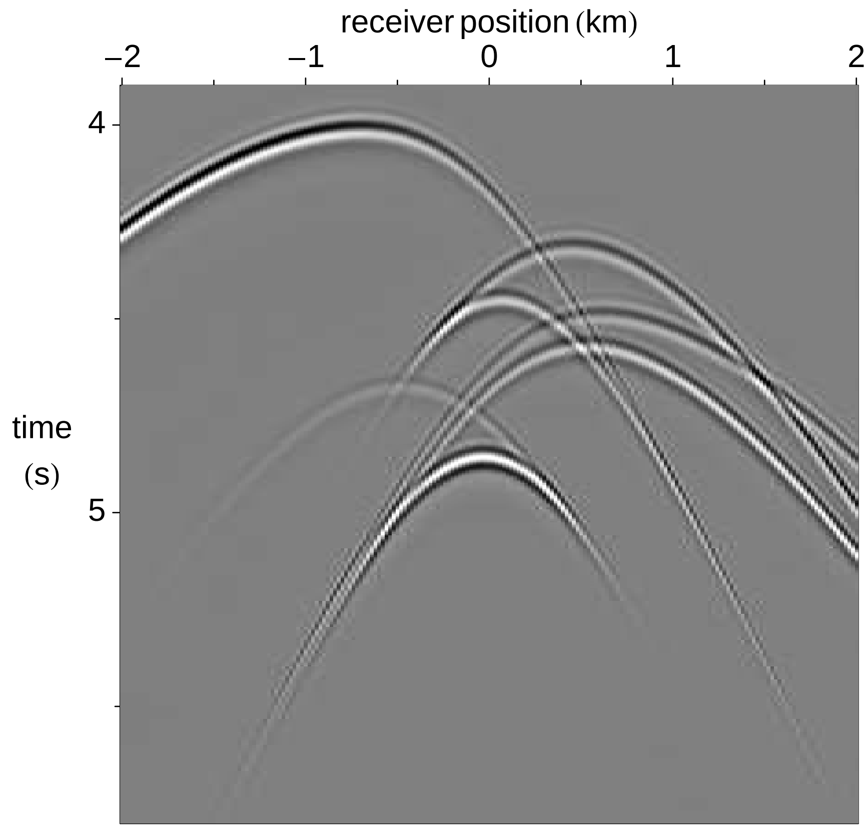
Good News: Suppose also that rays carrying significant energy do not turn ("DSR assumption"), and that all shot, receiver locations are present in data ("true 3D", "complete coverage"). Then (i) $\bar{F}[v]$ is *always* invertible, regardless of multipathing; (ii) the adjoint $\bar{F}[v]^*$ focusses energy in image gathers at zero offset; (iii) $\bar{F}[v]^*$ is kinematically (except for amplitudes) identical to Claerbout's S-G DSR migration operator (Stolk-DeHoop 2001).

The rest of this seminar:

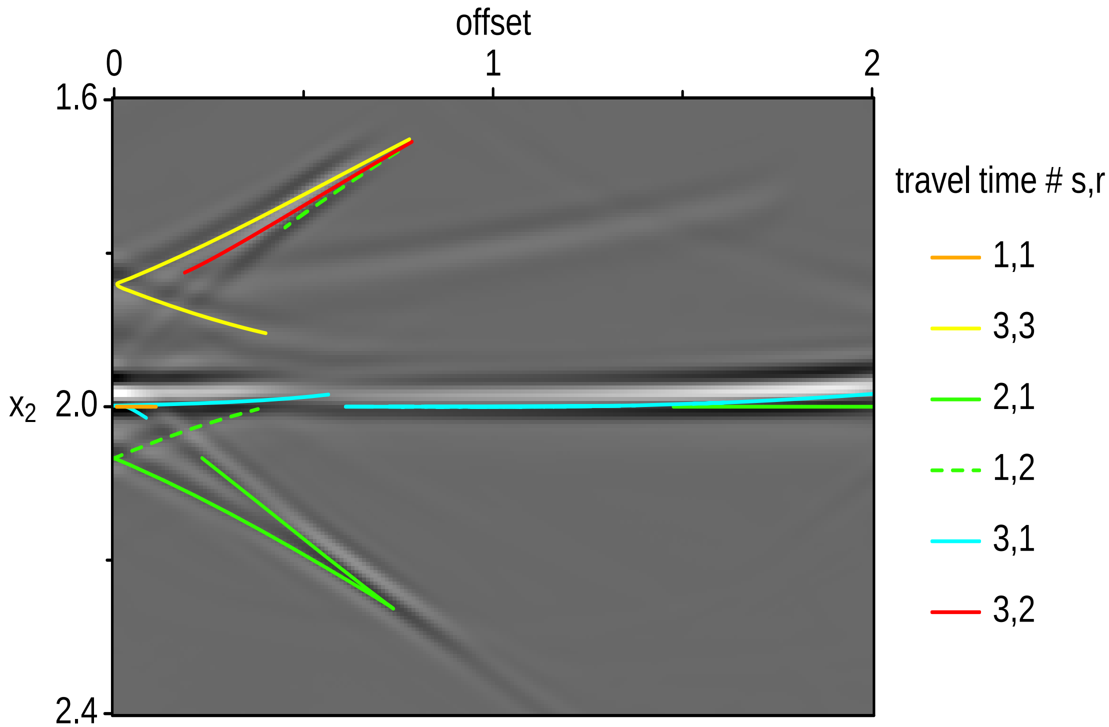
- examples of S-G focussing in the presence of multipathing
- ray-theoretic analysis of focussing
- how and why to get rid of the DSR / horizontal offset assumptions.



Gaussian lens velocity model, flat reflector at depth 2 km, overlain with rays and wavefronts (Stolk & S. 2002 SEG).

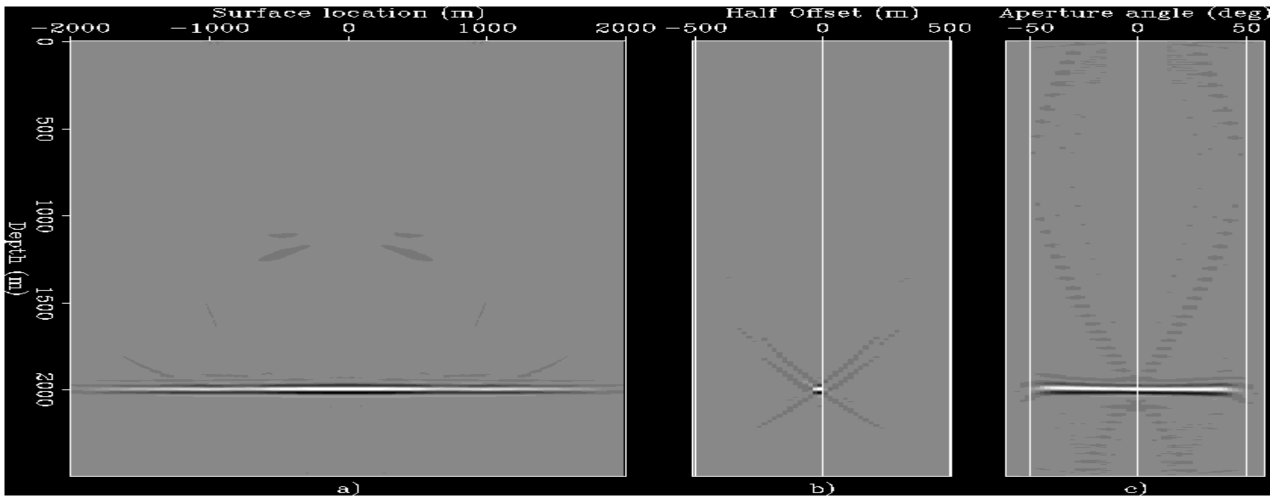


Typical shot gather - lots of arrivals



Offset common image gather (slice of $\tilde{F}[v]^*d$), with kinematically predicted reflector images overlain.

Middle panel: S-G common image gather, horizontal offset (slice of $\bar{F}[v]^* d$), courtesy B. Biondi.



Relation with S-G migration via depth extrapolation (Claerbout IEI 1985): begin by introducing *source-receiver parametrization* $\bar{R}(\mathbf{y}_r, \mathbf{y}_s) = R(\mathbf{y}, \mathbf{h})$ where $\mathbf{y} = (\mathbf{y}_r + \mathbf{y}_s)/2$, $\mathbf{h} = (\mathbf{y}_r - \mathbf{y}_s)/2$ (''sunken'' midpoint, offset). Rewrite

$$\bar{F}[v]\bar{R}(\mathbf{x}_r, t; \mathbf{x}_s) =$$

$$\frac{\partial^2}{\partial t^2} \int dy_r \int dy_s \int d\tau \, \frac{2\bar{R}(\mathbf{y}_r, \mathbf{y}_s)}{v^2(\mathbf{y})} G(\mathbf{y}_r, t-\tau; \mathbf{x}_r) G(\mathbf{y}_s, \tau; \mathbf{x}_s)$$

The LHS is the value at $\mathbf{x} = \mathbf{x}_r$ of a field $u(\mathbf{x}, t; \mathbf{x}_s)$ which satisfies

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) u(\mathbf{x}, t; \mathbf{x}_s) = \int dy_r \, R(\mathbf{x}, \mathbf{y}_s) G(\mathbf{y}_s, t; \mathbf{x}_s)$$

Define the RHS of the last equation to be the field

$$\equiv w_s(\mathbf{x}, t; \mathbf{x}_s)$$

so that

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) u(\mathbf{x}, t; \mathbf{x}_s) = w_s(\mathbf{x}, t; \mathbf{x}_s)$$

("upward continue the receivers"), and note that $w_s(\mathbf{x}, t; \mathbf{x}_s)$ is the value at $\mathbf{y} = \mathbf{x}_s$ of a field which satisfies

$$\left(\frac{1}{v(\mathbf{y})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{y}}\right) w_s(\mathbf{x}, t; \mathbf{y}) = R(\mathbf{x}, \mathbf{y})\delta(t)$$

("upward continue the sources") (w_r defined similarly).

Depth extrapolation begins with *approximate factorization* of the wave operator. To keep notation under control, stick with 2D: $\mathbf{x} = (x, z_r), \mathbf{y} = (y, z_s)$,

$$\frac{\partial^2}{\partial z_r^2} - \left(\frac{1}{v(x)^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) \simeq \left(\frac{\partial}{\partial z_r} - B_r\right)\left(\frac{\partial}{\partial z_r} + B_r\right)$$

similarly for wave operator in z_s, y, t. That is

$$B_r \simeq \sqrt{\frac{1}{v(x)^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}},\ B_s \simeq \sqrt{\frac{1}{v(y)^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}}$$

Good questions:

- when is this possible - A: when the DSR assumption holds (rays carrying significant energy do not turn);
- what does $\simeq$ mean - A: this is an asymptotic result;
- what sort of operator is B - A: see CIME notes on TRIP web site;
- how do you compute B - A: see the depth extrapolation industry.

Upcoming projections:

$$\tilde{u} = \left(\frac{\partial}{\partial z_s} + B_s\right)\left(\frac{\partial}{\partial z_r} + B_r\right) u$$

$$\tilde{w}_s = \left(\frac{\partial}{\partial z_s} + B_s\right) w_s,$$

(similar definition for $\tilde{w}_r$). Since z will be singled out (and eventually $R(\mathbf{x}, \mathbf{y})$ will have a factor of $\delta(\mathbf{x}, \mathbf{y})$), impose the constraint that

$$\bar{R}(x, z, y, z_s) = \tilde{R}(x, y, z)\delta(z - z_s)$$

Then

$$\left(\frac{\partial}{\partial z_r} - B_r\right) \tilde{u}(x, z_r, t; y, z_s) = \tilde{w}_s(x, z_r, t; y, z_s)$$

$$\left(\frac{\partial}{\partial z_s} - B_s\right) \tilde{w}_s(x, z_r, t; y, z_s) = \tilde{R}(x, y, z_r)\delta(z_r - z_s)\delta(t)$$

Simultaneous upward continuation:

$$\frac{\partial}{\partial z}\tilde{u}(x,z,t;y,z) = \frac{\partial}{\partial z_r}\tilde{u}(x,z_r,t;y,z)|_{z=z_r} + \frac{\partial}{\partial z_r}\tilde{u}(x,z,t;y,z_s)|_{z=z_s}$$

$$= [B_r\tilde{u} + \tilde{w}_s + B_s\tilde{u} + \tilde{w}_r]_{z_r=z_s=z}$$

Since $\tilde{w}_s(y,z,t;x,z) = \tilde{w}_r(x,z,t;y,z) = \tilde{R}(x,y,z)\delta(t)$, $\tilde{u}$ is seen to satisfy the **DSR modeling equation**:

$$\left(\frac{\partial}{\partial z} - B_r - B_s\right)\tilde{u}(x,z,t;y,z) = 2\tilde{R}(x,y,z)\delta(t)$$

Since the $\tilde{u}$, the upcoming projection of u, has same events as u (wrong amplitudes, dip-dep. filter), replace forward map $\bar{F}[v]$ by

$$\widehat{F}[v]\tilde{R}(x_r, t; x_s) = \tilde{u}(x_r, 0, t; x_s, 0)$$

retaining same kinematics. Now easy task ("adjoint state method", see CIME notes) to show that adjoint of $\widehat{F}[v]$ given by Claerbout's **DSR migration equation:** solve

$$\left(\frac{\partial}{\partial z} - B_r - B_s\right)\tilde{q}(x, y, z, t) = 0$$

in *increasing* z with initial condition at $z = 0$:

$$\tilde{q}(x_r, x_s, 0, t) = d(x_r, x_s, t)$$

("downward continue sources and receivers"). Then $\widehat{F}[v]^* d(x, y, z) = \tilde{q}(x, y, z, 0)$ ("prestack image extracted at $t = 0$").

Back to 3D. Reverse time computation of adjoint:

$$\bar{F}^*[v]d(\mathbf{x},\mathbf{h}) = -\int dx_s \int_0^T dt \, \frac{\partial q}{\partial t}(\mathbf{x}+\mathbf{h}.t;\mathbf{x}_s)\nabla^2 G(\mathbf{x}-\mathbf{h},t;\mathbf{x}_s)$$

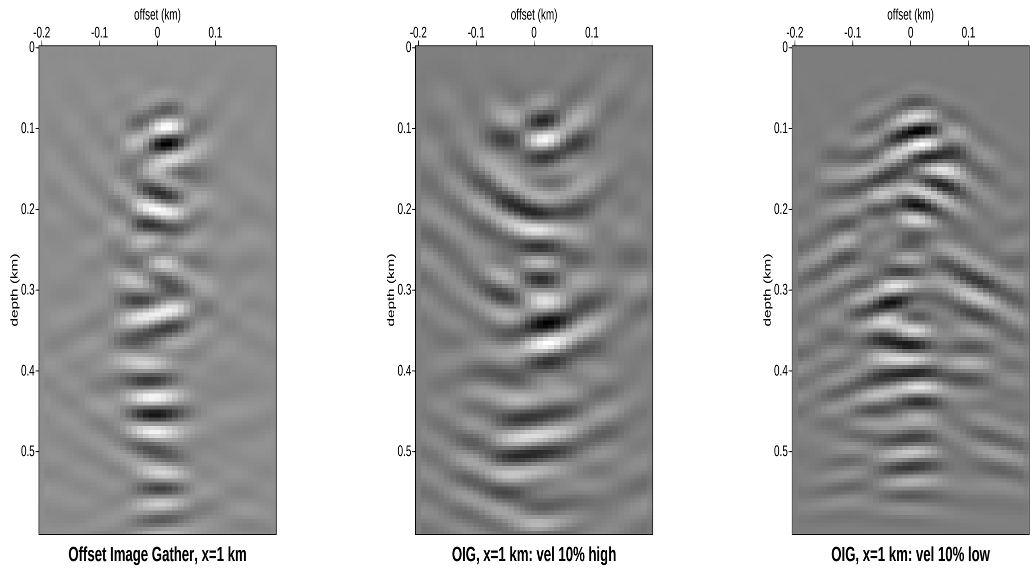
where the *adjoint state* or backpropagated field $q(\mathbf{x},t;\mathbf{x}_s)$ satisfies $q \equiv 0$, $t \geq T$ and

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) q(\mathbf{x},t;\mathbf{x}_s) = \int dx_r \, d(\mathbf{x}_r,t;\mathbf{x}_s)\delta(\mathbf{x}-\mathbf{x}_r)$$

(" use data traces as time-reversed sources, add up")

Points to note:

- The Green's function G and the adjoint field q are *exactly* the same as the usual fields appearing in RT imaging (Whitmore, Lailly, McMechan,...).
- The only difference is the displacement of the correlation point by the offset $\mathbf{h}$.
- The loop over $\mathbf{h}$ can be restricted to various subsets of 3D offset space (eg. to be horizontal) - this only affects the imaging (crosscorrelation) loop, all other computations remaining the same.
- Implementation can use any accurate discretization of the wave equation.
- Derivation: another instance of the adjoint state method, see eg. CIME notes.



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2D RT horizontal offset S-G image gathers of data from random reflectivity, constant velocity. Computation uses (4,2) FD scheme. Synthetic FD data: 40 shots, fixed split spread. From left to right: correct velocity, 10% high, 10% low. Note *focussing* at $h = 0$ for correct velocity, as predicted by theory.