Mathematics of Seismic Imaging Part 4: Global Asymptotics

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4. Global Asymptotics

Why Beylkin isn't enough

Geometry of Reflection

Development of Global Asymptotics

Kinematic Artifacts in Migration

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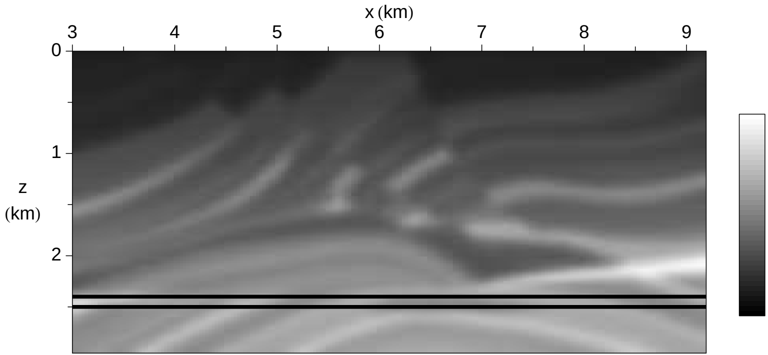
The theory developed by Beylkin and others cannot be the end of the story:

B. White, "The Stochastic Caustic" (1982): For "random but smooth" $v(\mathbf{x})$ with variance σ, points at distance $O(\sigma^{-2/3})$ from source have more than one ray connecting to source, with probability 1 - *multipathing* associated with formation of *caustics* = ray envelopes.

Example: Marmousi, lightly smoothed



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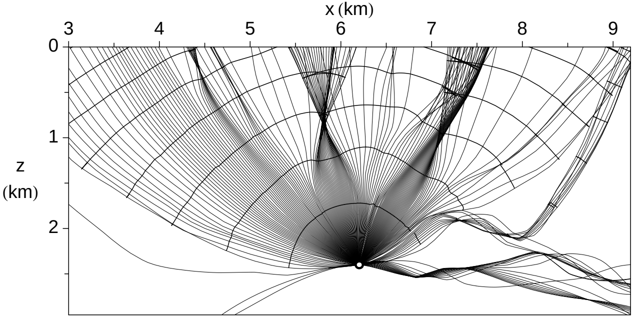
Example: Marmousi, lightly smoothed

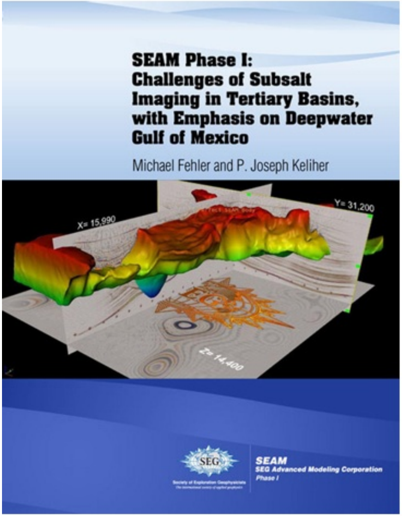


Multipathing, formation of caustics invalidates asymptotic analysis on which GRT representation is based:

multiple rays $\Rightarrow$ traveltime no longer *function* of position

Why it matters

Strong refraction: salt (4-5 km/s) structures embedded in sedimentary rock (2-3 km/s) (eg. Gulf of Mexico), also chalk in North Sea, gas seeps - some of the most promising petroleum provinces!



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Escape from simplicity - the Canonical Relation

How do we get away from "simple geometric optics", SSR, DSR,... - all violated in sufficiently complex (and realistic) models?

Rakesh *Comm. PDE* 1988, Nolan *Comm. PDE* 1997: global description of $F_\delta[v]$ as mapping reflectors $\mapsto$ reflections.

Escape from simplicity - the Canonical Relation

$Y = \{\mathbf{x}_s, t, \mathbf{x}_r\}$ (time $\times$ set of source-receiver pairs) submfd of $\mathbf{R}^7$ of dim. ≤ 5, $\Pi : T^*(\mathbf{R}^7) \rightarrow T^*Y$ the natural projection

$\mathrm{supp}\, r \subset X \subset \mathbf{R}^3$

Escape from simplicity - the Canonical Relation

Canonical relation
$C_{F_\delta[v]} \subset T^*(X) \setminus \{\mathbf{0}\} \times T^*(Y) \setminus \{\mathbf{0}\}$ describes singularity mapping properties of F:

$$(\mathbf{x}, \xi, \mathbf{y}, \eta) \in C_{F_\delta[v]} \Leftrightarrow \text{for some } u \in \mathcal{E}'(X),$$

$$(\mathbf{x}, \xi) \in WF(u) \text{and } (\mathbf{y}, \eta) \in WF(Fu)$$

Rakesh's Construction

recall defn of *rays*: solutions of Hamiltonian system

$$\frac{d\mathbf{X}}{dt} = \nabla_{\mathbf{\Xi}} H(\mathbf{X}, \mathbf{\Xi}), \;\; \frac{d\mathbf{\Xi}}{dt} = -\nabla_{\mathbf{X}} H(\mathbf{X}, \mathbf{\Xi})$$

with

$$H(\mathbf{X}, \mathbf{\Xi}) = \frac{1}{2}(1 - v^2(\mathbf{X})|\mathbf{\Xi}|^2) = 0$$

Rakesh's Construction

Characterization of C_F:

$$((\mathbf{x},\xi),(\mathbf{x}_s,t,\mathbf{x}_r,\xi_{\mathbf{s}},\tau,\xi_{\mathbf{r}})) \in C_{F_\delta[v]}$$

$$\subset T^*(X)-\{\mathbf{0}\}\times T^*(Y)-\{\mathbf{0}\}$$

$\Leftrightarrow$ there are rays $(\mathbf{X}_s,\mathbf{\Xi}_s)$, $(\mathbf{X}_r,\mathbf{\Xi}_r)$, times t_s,t_r so that

$$\Pi(\mathbf{X}_s(0),t,\mathbf{X}_r(t),\mathbf{\Xi}_s(0),\tau,\mathbf{\Xi}_r(t))=(\mathbf{x}_s,t,\mathbf{x}_r,\xi_s,\tau,\xi_r),$$

$$\mathbf{X}_s(t_s)=\mathbf{X}_r(t-t_r)=\mathbf{x},\ t_s+t_r=t,\ \mathbf{\Xi}_s(t_s)-\mathbf{\Xi}_r(t-t_r)||\xi$$

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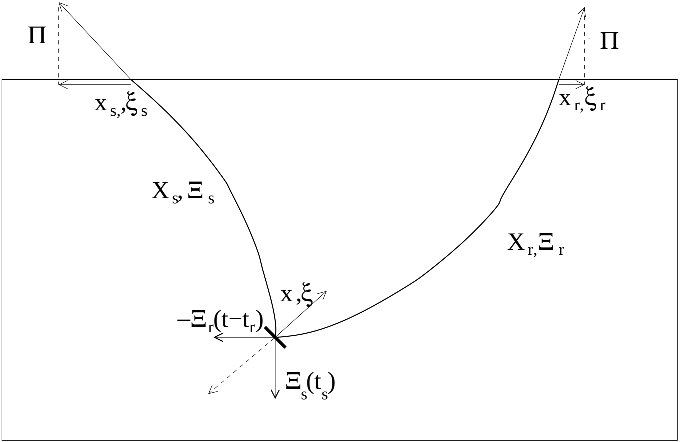
Rakesh's Construction

$$|\mathbf{\Xi}_s(t_s)| = |\mathbf{\Xi}_r(t - t_r)| \Rightarrow$$

sum $=$ bisector $\Rightarrow$

velocity vectors of incident ray from source and reflected ray from receiver (traced backwards in time) make equal angles with reflector at $\mathbf{x}$ *with normal* ξ.

Rakesh's Construction



Rakesh's Construction

Upshot: canonical relation of $F_\delta[v]$ simply enforces the equal-angles law of reflection.

Further, *rays carry high-frequency energy*, in exactly the fashion that seismologists imagine.

Finally, *Rakesh's characterization of C_F is global:* no assumptions about ray geometry, other than no forward scattering and no grazing incidence on the acquisition surface Y, are needed.

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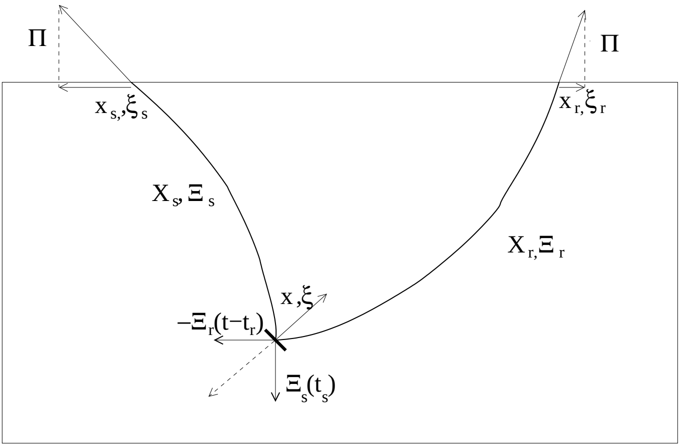
Rakesh's Construction

Inversion aperture Γ:

$(\mathbf{x}, \boldsymbol{\xi}) \in \Gamma \Leftrightarrow$

there is $(\mathbf{x}_s, t, \mathbf{x}_r) \in Y$ and rays connecting $\mathbf{x}_s$ and $\mathbf{x}_r$ with $\mathbf{x}$ so that $\boldsymbol{\xi}$ bisects ray velocity vectors, and total time along the two rays is $= t$

Rakesh's Construction



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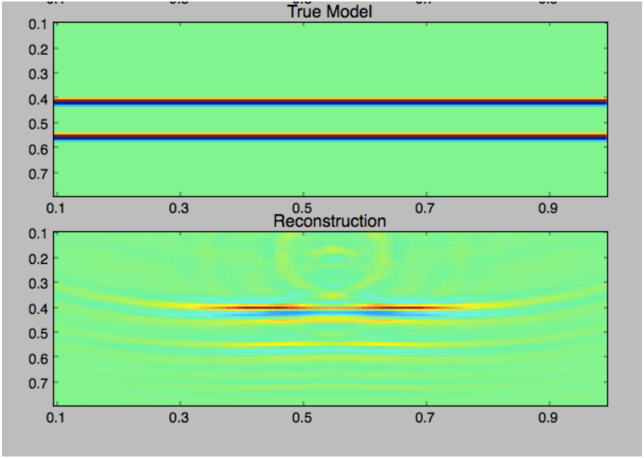
Rakesh's Construction

Exercise based on `HorizontalReflector2D.py`



Explain why inverted layers do not extend to boundary - explain where they end

Proof: Plan of attack

Recall that

$$F[v]r(\mathbf{x}_r, t; \mathbf{x}_s) = \frac{\partial \delta u}{\partial t}(\mathbf{x}_r, t; \mathbf{x}_s)$$

where

$$\frac{1}{v^2}\frac{\partial^2 \delta u}{\partial t^2} - \nabla^2 \delta u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} r$$

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = \delta(t)\delta(\mathbf{x} - \mathbf{x}_s)$$

and $u, \delta u \equiv 0,\ t < 0$.

Proof: Plan of attack

Need to understand (1) $WF(u)$, (2) relation $WF(r) \leftrightarrow WF(ru)$, (3) WF of soln of WE in terms of WF of RHS (this also gives (1)!).

Singularities of the Acoustic Potential Field

Main tool: **Propagation of Singularities** theorem of Hörmander (1970).

Given symbol $p(\mathbf{x}, \boldsymbol{\xi})$ w. asymptotic expansion, order m, define *null bicharateristics* (= rays) as solutions $(\mathbf{x}(t), \boldsymbol{\xi}(t))$ of Hamiltonian system

$$\frac{d\mathbf{x}}{dt} = \frac{\partial p_m}{\partial \boldsymbol{\xi}}(\mathbf{x}, \boldsymbol{\xi}), \quad \frac{d\boldsymbol{\xi}}{dt} = -\frac{\partial p_m}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\xi})$$

with $p_m(\mathbf{x}(t), \boldsymbol{\xi}(t)) \equiv 0$.

Singularities of the Acoustic Potential Field

Theorem: Suppose $p(\mathbf{x}, D)u = f$, $t \mapsto (\mathbf{x}(t), \boldsymbol{\xi}(t))$ is a null bicharacteristic, and for $t_0 \le t \le t_1$, $(\mathbf{x}(t), \boldsymbol{\xi}(t)) \notin WF(f)$. Then either

- $\{(\mathbf{x}(t), \boldsymbol{\xi}(t)) : t_0 \le t \le t_1\} \subset WF(u)$

or

- $\{(\mathbf{x}(t), \boldsymbol{\xi}(t)) : t_0 \le t \le t_1\} \subset T^*(\mathbf{R}^n) - WF(u)$

Source to Field

RHS of wave equation for $u = \delta$ function in $\mathbf{x}, t$. WF set $= \{(\mathbf{x}, t, \boldsymbol{\xi}, \tau) : \mathbf{x} = \mathbf{x}_s, t = 0\}$ - i.e. no restriction on covector part.

$\Rightarrow (\mathbf{x}, t, \boldsymbol{\xi}, \tau) \in WF(u)$ iff it lies on a null bichar passing over $(\mathbf{x}_s, 0)$

$\Rightarrow (\mathbf{x}, t)$ lies on the "light cone" with vertex at $(\mathbf{x}_s, 0)$.

[Principal symbol for wave op is $p_2(\mathbf{x}, t, \boldsymbol{\xi}, \tau) = \frac{1}{2}(\tau^2 - v^2(\mathbf{x})|\boldsymbol{\xi}|^2)$]

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Source to Field

Hamilton's equations for null bicharacteristics (using t for parameter) are

$$\frac{dt}{dt} = 1 = \tau, \ \frac{d\tau}{dt} = 0$$

$$\frac{d\mathbf{X}}{dt} = -v^2(\mathbf{X})\mathbf{\Xi}, \ \frac{d\mathbf{\Xi}}{dt} = \nabla \log v(\mathbf{X})$$

Thus ξ is proportional to velocity vector of ray.

[Exercise: show (ξ, τ) *normal* to light cone]

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Singularities of Products

To compute $WF(ru)$ from $WF(r)$ and $WF(u)$, use *Gabor calculus* (Duistermaat, Ch. 1)

r is really $(r \circ \pi)$, where $\pi(\mathbf{x}, t) = \mathbf{x}$. Choose bump function ϕ localized near $(\mathbf{x}, t)$

$$[\phi(r \circ \pi) u]^{\wedge}(\boldsymbol{\xi}, \tau) = \int d\boldsymbol{\xi}' \, d\tau' \widehat{\phi r}(\boldsymbol{\xi}') \delta(\tau') \widehat{u}(\boldsymbol{\xi} - \boldsymbol{\xi}', \tau - \tau')$$

$$= \int d\boldsymbol{\xi}' \widehat{\phi r}(\boldsymbol{\xi}') \widehat{u}(\boldsymbol{\xi} - \boldsymbol{\xi}', \tau)$$

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decays rapidly as $|(\boldsymbol{\xi},\tau)| \to \infty$ unless (i) exist $(\mathbf{x}',\boldsymbol{\xi}') \in WF(r)$ so that $\mathbf{x},\mathbf{x}' \in \pi(\mathrm{supp}\phi)$, $(\boldsymbol{\xi}-\boldsymbol{\xi}',\cdot) \in WF(u)$, i.e. $(\boldsymbol{\xi},\cdot) \in WF(r\circ\pi) + WF(u)$, or (ii) $\boldsymbol{\xi} \in WF(r)$ or $(\boldsymbol{\xi},\cdot) \in WF(u)$.

Possibility (ii) will not contribute, so effectively

$$WF((r\circ\pi)u) = \{(\mathbf{x},t_s,\boldsymbol{\xi}+\boldsymbol{\Xi}_s(t_s),\cdot) :$$
$$(\mathbf{x},\boldsymbol{\xi}) \in WF(r),\ \mathbf{x} = \mathbf{X}_s(t_s)\}$$

for a ray $(\mathbf{X}_s,\boldsymbol{\Xi}_s)$ with $\mathbf{X}_s(0) = x_s$, some τ.

Wavefront set of Scattered Field

Propagation of singularities:
$(\mathbf{x}_r, t, \boldsymbol{\xi}_r, \tau_r) \in WF(\delta u) \Leftrightarrow$ on ray $(\mathbf{X}_r, \boldsymbol{\Xi}_r)$ passing through $WF(ru)$. Can argue that time of intersection is $t - t_r < t$ (Exercise: do it!)

That is,

$$\mathbf{X}_r(t) = \mathbf{x}_r, \mathbf{X}_r(t - t_r) = \mathbf{X}_s(t_s) = x,$$

$t = t_r + t_s$, and

$$\boldsymbol{\Xi}_r(t_s) = \boldsymbol{\xi} + \boldsymbol{\Xi}_s(t_s)$$

for some $\boldsymbol{\xi} \in WF(r)$. **Q. E. D.**

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Rakesh's Thesis

Rakesh (1986):

- $F[v]$ is *Fourier Integral Operator* = class of oscillatory integral operators, introduced by Hörmander and others in the '70s to describe the solutions of nonelliptic PDEs (ΨDOs are special FIOs.)
- Adjoint of FIO = FIO with inverse canonical relation
- Composition of FIOs $\neq$ FIO in general - not an algebra (unlike ΨDOs)

Rakesh's Thesis

- Beylkin: $F[v]^*F[v]$ is FIO (ΨDO, actually), given simple ray geometry hypothesis - but this is only sufficient
- Rakesh: follows from general results of Hörmander: *simple ray geometry* $\Leftrightarrow$ *canonical relation is graph of ext. deriv. of phase function*.

The Shell Guys and TIC

Inversion aperture $\neq T^*X \Rightarrow F[v]^*F[v]$ cannot be boundedly invertible

[Exercise: Why? Hint: revisit relation between symbol and operator, recall that inversion aperture is not all of T^*X]

The Shell Guys and TIC

A *microlocal parametrix* for a ΨDO P in a conic set Γ is an operator Q for which

$$u - QPu \in C^\infty$$

if $WF(u) \subset \Gamma$

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The Shell Guys and TIC

Smit, tenKroode and Verdel (1998): provided that

- source, receiver positions $(\mathbf{x}_s, \mathbf{x}_r)$ form an *open* 4D manifold ("complete coverage" - all source, receiver positions at least locally), and
- the *Traveltime Injectivity Condition* ("TIC") holds: $C_{F[v]}^{-1} \subset T^*Y \setminus \{0\} \times T^*X \setminus \{0\}$ is a *function* - that is, initial data for source and receiver rays *projected into* T^*Y and total travel time together determine ray pair uniquely....

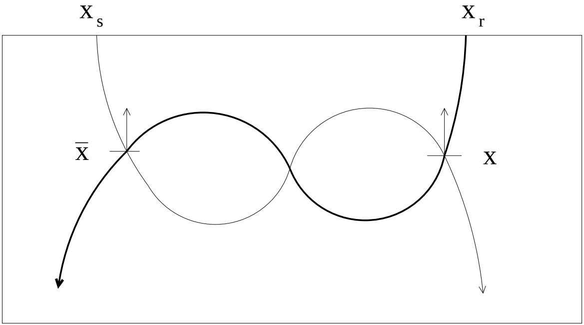
The Shell Guys and TIC

then $F[v]^*F[v]$ is ΨDO

$\Rightarrow$ application of $F[v]^*$ produces image

and $F[v]^*F[v]$ has microlocal parametrix ("asymptotic inverse") in inversion aperture

TIC is a nontrivial constraint!



Symmetric waveguide: time $(\mathbf{x}_s \rightarrow \bar{\mathbf{x}} \rightarrow \mathbf{x}_r)$ same as time $(\mathbf{x}_s \rightarrow \mathbf{x} \rightarrow \mathbf{x}_r)$, so TIC fails.

Stolk's Thesis

Stolk (2000): for dim=2, under "complete coverage" hypothesis, v for which $F[v]^*F[v] =$ [ΨDO + rel. smoothing op] open, dense set in $C^\infty(\mathbf{R}^2)$ (without assuming TIC!). Conjecture: same for dim=3.

Also, for any dim, v for which $F[v]^*F[v]$ is FIO open, dense in $C^\infty(\mathbf{R}^2)$.

Operto's Thesis

Application of $F[v]^*$ involves accounting for *all* rays connecting source and receiver with reflectors.

Standard practice at time attempted to simplify integral kernel with single choice of ray pair (shortest time, max energy,...).

Operto et al (2000): nice illustration that all rays must be included in general to obtain good image.

Nolan's Thesis

Limitation of Smit-tenKroode-Verdel: most idealized data acquisition geometries violate "complete coverage": for example, idealized marine streamer geometry (src-recvr submfd is 3D)

Nolan (1997): result remains true without "complete coverage" condition: requires only TIC plus addl condition so that projection $C_{F[v]} \to T^*Y$ is embedding - but examples violating TIC are much easier to construct when source-receiver submfd has positive codim.

Nolan's Thesis

Sinister Implication: When data is just a single gather - common shot, common offset - image may contain *artifacts*, i.e. spurious reflectors not present in model.

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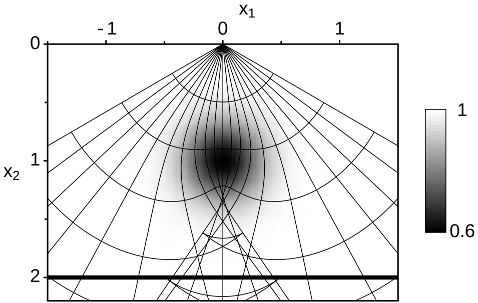
Horrible Example

Synthetic 2D Example (see Stolk and WWS, *Geophysics* 2004 for this and other horrible expls)

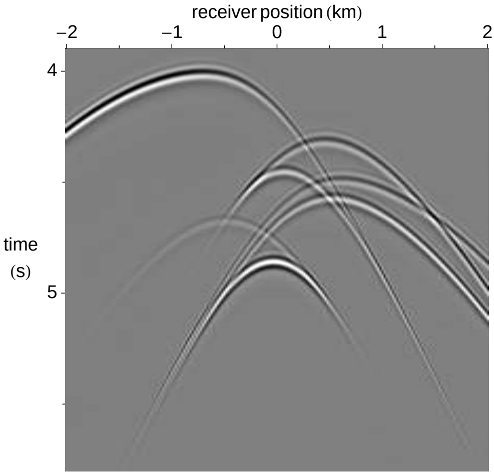
Strongly refracting acoustic lens (v) over horizontal reflector (r), $S^{\rm obs} = F[v]r$.

(i) for open source-receiver set, $F[v]^* S^{\rm obs}$ = good image of reflector - within limits of finite frequency implied by numerical method, $F[v]^* F[v]$ acts like ΨDO;

(ii) for *common offset* submfd (codim 1), TIC is violated and $WF(F[v]^* S^{\rm obs})$ is larger than $WF(r)$.



Gaussian lens velocity model, flat reflector at depth 2 km, overlain with rays and wavefronts (Stolk & S. 2002 SEG).



Typical shot gather - lots of arrivals

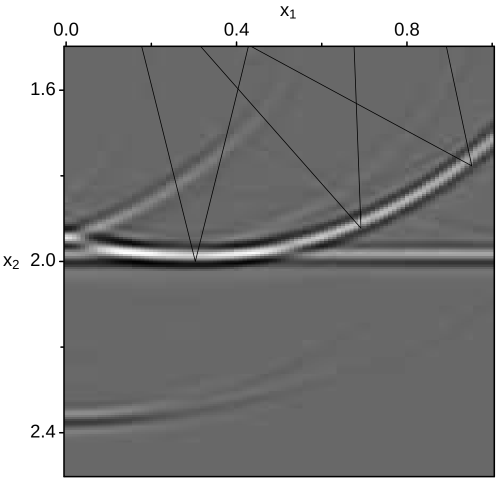


Image from common offset gather, $h = 0.3$ - TIC fails (3 ray pairs with same data), image has “artifact” WF

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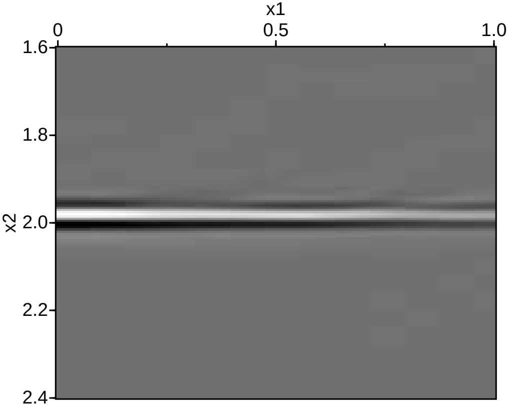

Image from all offsets - TIC holds, “WF” recovered

What it all means

Note that a gather scheme makes the scattering operator block-diagonal: for example with data sorted into common offset gathers $h = (x_r - x_s)/2$,

$$F[v] = [F_{h_1}[v], ..., F_{h_N}[v]]^T, \; d = [d_{h_1}, ..., d_{h_N}]^T$$

Thus $F[v]^* d = \sum_i F_{h_i}[v]^* d_{h_i}$. Otherwise put: to form image, **migrate** *i*th gather (apply migration operator $F_{h_i}[v]^*$, then **stack** individual migrated images.

What it all means

Horrible Examples show that individual offset gather images may contain nonphysical apparent reflectors (artifacts).

Smit-tenKroode-Verdel, Nolan, Stolk: if TIC holds, then these artifacts are not stationary with respect to the gather parameter, hence *stack out* (interfere destructively) in final image.