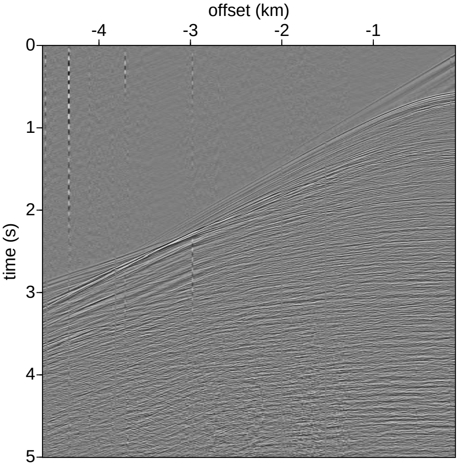
Mathematics of Seismic Imaging Part 1: Modeling and Inverse Problems

William W. Symes

Rice University

0. Introduction

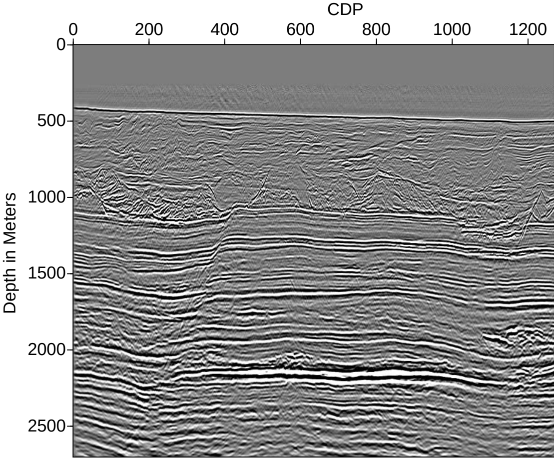
How do you turn lots of this...



(field seismogram from the Gulf of Mexico - thanks: Exxon Production Co.)

0. Introduction

into this - an *image* of subsurface structure



0. Introduction

resembling actual subsurface structure

0. Introduction

Also: what does imaging have to do with

inversion

$=$ construction of a physical model that explains data

?

0. Introduction

Main goal of these lectures: coherent mathematical view of reflection seismic imaging, as practiced in petroleum industry, and its relation to seismic inversion

- imaging = approximate solution of inverse problem for wave equation
- most practical imaging methods based on linearization ("perturbation theory")
- high frequency asymptotics ("microlocal analysis") key to understanding
- beyond linearization, asymptotics - many open problems

0. Introduction

Lots of mathematics - much yet to be created - with practical implications!

1. Modeling & Inverse Problems

Reflection seismology

aka active source seismology, seismic sounding/profiling

uses seismic (elastic) waves to probe the Earth's sedimentary crust

main exploration tool of oil & gas industry, also used in environmental and civil engineering (hazard detection, bedrock profiling) and academic geophysics (structure of crust and mantle)

Reflection seismology

highest resolution imaging technology for deep Earth exploration, in comparison with static (gravimetry, resistivity) or diffusive (passive, active source EM) techniques - works because

waves transfer space-time resolved information from one place to another with (relatively) little loss

wavelengths at easily accessible frequencies $\sim$ *scale of important structural features*

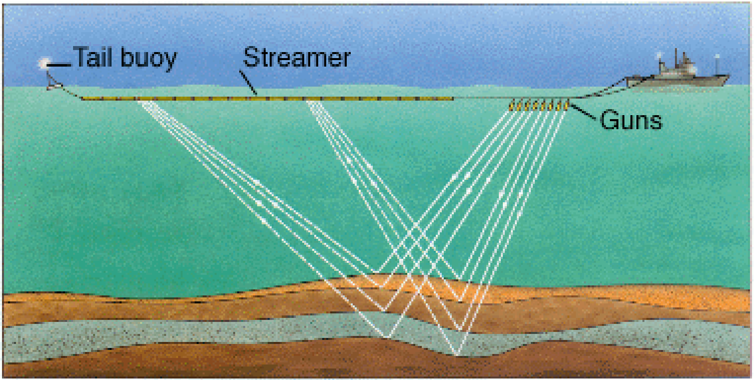
RICE

Reflection seismology

Three components:

- energy/sound source - creates wave traveling into subsurface
- receivers - record waves (echoes) reflected from subsurace
- recording and signal processing instrumentation

Reflection seismology



http://www.glossary.oilfield.slb.com/en/Terms/s/streamer.aspx, 12.07.2013

Reflection seismology

Marine reflection seismology:

- typical energy source: *airgun array* - releases (array of) supersonically expanding bubbles of compressed air, generates sound pulse in water
- typical receivers: hydrophones (waterproof microphones) in one or more 5-10 km flexible streamer(s) - wired together 500 - 30000 *groups* (each group produces a single channel / time series)
- survey ships - lots of recording, processing capacity

Reflection seismology

Survey consists of many experiments = shots = source positions $\mathbf{x}_s$

Simultaneous recording of reflections at many localized receivers, positions $\mathbf{x}_r$, time interval = $0 - O(10)$s after initiation of source.

Data acquired on land and at sea ("marine") - vast bulk (90%+) of data acquired each year is marine.

Reflection seismology

Marine seismic data parameters:

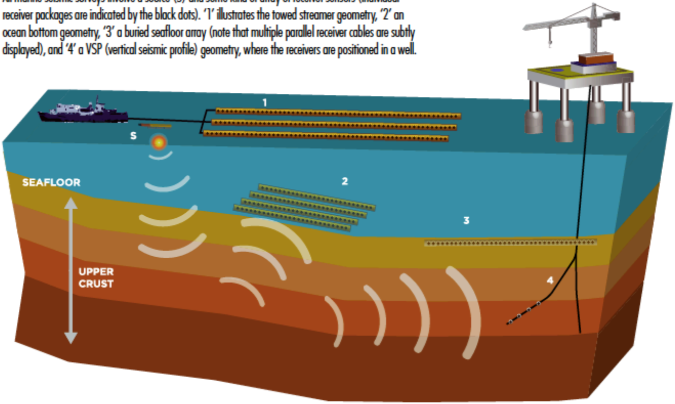
- time t - $0 \leq t \leq t_{\max}$, $t_{\max} = 5 - 30$s
- source location $\mathbf{x}_s$ - 100 - 100000 distinct values
- receiver location $\mathbf{x}_r$
- typically the same range of *offsets* $= \mathbf{x}_r - \mathbf{x}_s$ for each shot - *half offset* $\mathbf{h} = \frac{\mathbf{x}_r - \mathbf{x}_s}{2}$, $h = |\mathbf{h}|$: 100 - 500000 values (typical: 5000) - few m to 30 km (typical: 200 m - 8 km)
- data values: microphone output (volts), filtered version of local pressure (force/area)

RICE

Reflection seismology

Figure 1 *(credit: Jack Caldwell)*

All marine seismic surveys involve a source (S) and some kind of array or receiver sensors (individual receiver packages are indicated by the black dots). '1' illustrates the towed streamer geometry, '2' an ocean bottom geometry, '3' a buried seafloor array (note that multiple parallel receiver cables are subtly displayed), and '4' a VSP (vertical seismic profile) geometry, where the receivers are positioned in a well.



www.ogp.org.uk/pubs/448.pdf, 20.02.13

Reflection seismology

Acquisition "manifold":

Idealized marine "streamer" geometry: $\mathbf{x}_s$ and $\mathbf{x}_r$ lie roughly on constant depth plane, source-receiver lines are parallel $\rightarrow$ 3 spatial degrees of freedom (eg. $\mathbf{x}_s, h$): *codimension 1*.

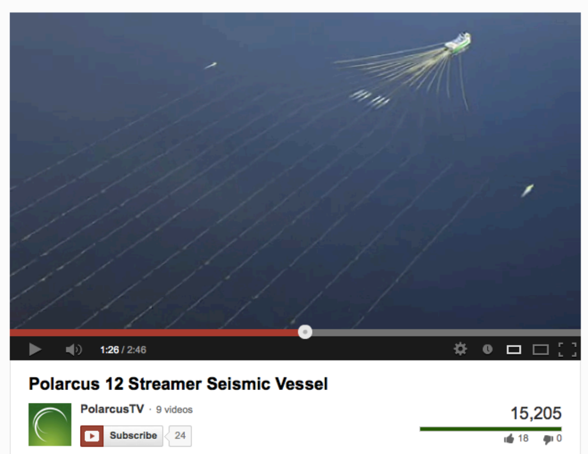
[Other geometries are interesting, eg. ocean bottom cables, but streamer surveys still prevalent.]

Reflection seismology

How much data? Contemporary surveys may feature

- Simultaneous recording by multiple streamers (up to 12!)
- Many (roughly) parallel ship tracks ("lines")
- Recent development: Wide Angle Towed Streamer (WATS) survey - uses multiple survey ships for areal sampling of source and receiver positions
- single line ("2D") $\sim$ Gbytes; multiple lines ("3D") $\sim$ Tbytes; WATS $\sim$ Pbytes

Reflection seismology



www.youtube.com/watch?v=vrOLWRVGosQ, 20.02.13

Distinguished data subsets

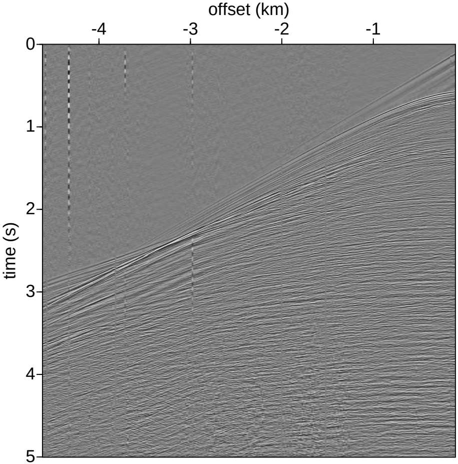
- *traces* = data for one source, one receiver: $t \mapsto d(\mathbf{x}_r, t; \mathbf{x}_s)$ - function of t, time series, single channel
- *gathers* or *bins* = subsets of traces, extracted from data after acquisition. Characterized by common value of an acquisition parameter

Distinguished data subsets

Examples:

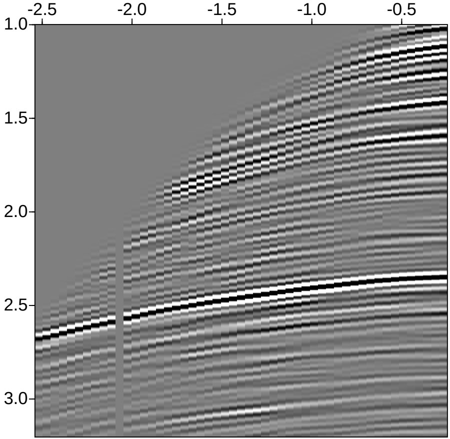
- shot (or common source) gather: traces w/ same shot location $\mathbf{x}_s$ (previous expls)
- offset (or common offset) gather: traces w/ same half offset h
- ...

Shot gather, Mississippi Canyon



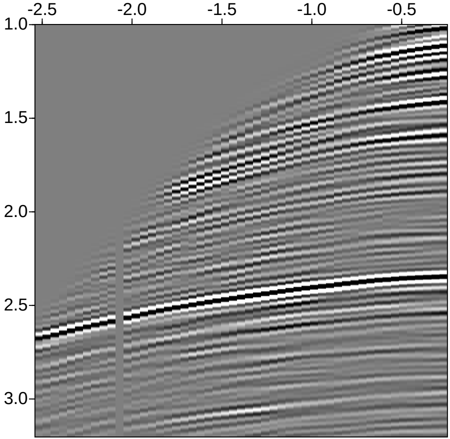
(thanks: Exxon)

Shot gather, Mississippi Canyon



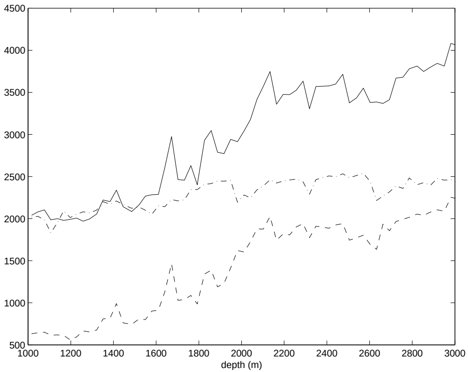
Lightly processed - bandpass filter 4-10-25-40 Hz, mute. Most striking visual characteristic: waves = coherent space-time structures ("reflections")

Shot gather, Mississippi Canyon



What features in the subsurface structure cause reflections? How to model?

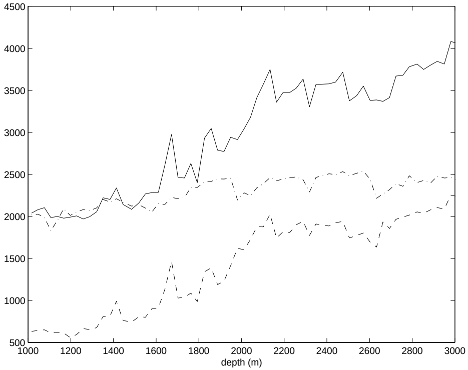
Well logs



Blocked logs from well in North Sea (thanks: Mobil R & D). Solid: p-wave velocity (m/s), dashed: s-wave velocity (m/s), dash-dot: density (kg/m^3).

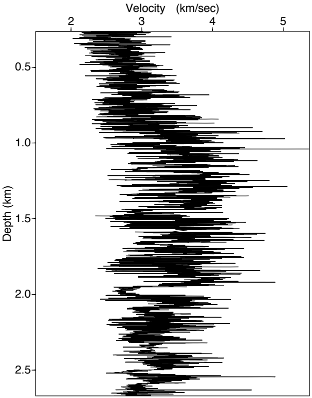
RICE

Well logs



"Blocked" means "averaged" (over 30 m windows). Original sample rate of log tool < 1 m. *Variance at all scales!*

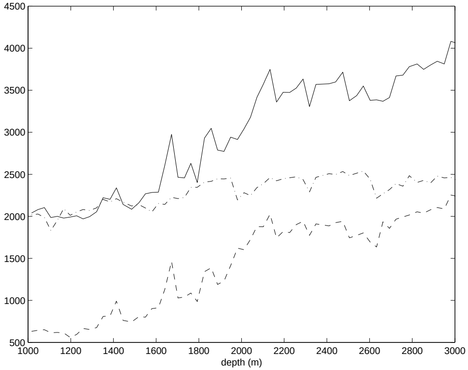
Well logs



P-wave velocity log from West Texas

Thanks: Total E&P USA

Well logs



- **Trends** = slow increase in velocities, density - scale of km
- **Reflectors** = jumps in velocities, density - scale of m or 10s of m

The Modeling Task

A useful model of the reflection seismology experiment must

- predict wave motion
- produce reflections from reflectors
- accomodate significant variation of wave velocity, material density,...

The Modeling Task

A *really good* model will also accomodate

- multiple wave modes, speeds
- material anisotropy
- attenuation, frequency dispersion of waves
- complex source, receiver characteristics

The Acoustic Model

Not *really good*, but good enough for this week and basis of most contemporary seismic imaging/inversion

- $\rho(\mathbf{x})$ = material density, $\kappa(\mathbf{x})$ = bulk modulus
- $p(\mathbf{x}, t)$= pressure, $\mathbf{v}(\mathbf{x}, t)$ = particle velocity,
- $\mathbf{f}(\mathbf{x}, t)$= force density, $g(\mathbf{x}, t)$ = constitutive law defect (external energy source model)

The Acoustic Model

Newton's law:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{f},$$

Constitutive (Hooke) law (stress-strain relation):

$$\frac{\partial p}{\partial t} = -\kappa \nabla \cdot \mathbf{v} + g$$

$+$ i. c.'s & b. c.'s.

wave speed $c = \sqrt{\frac{\kappa}{\rho}}$

The Acoustic Model

acoustic field potential $u(\mathbf{x},t) = \int_{-\infty}^{t} ds\, p(\mathbf{x},s)$:

$$p = \frac{\partial u}{\partial t},\ \mathbf{v} = \frac{1}{\rho}\nabla u$$

Equivalent form: second order wave equation for potential

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla u = \left\{ g + \int_{-\infty}^{t} dt\, \nabla \cdot \left(\frac{\mathbf{f}}{\rho}\right)\right\} \equiv \frac{f}{\rho}$$

plus initial, boundary conditions.

The Acoustic Model

Further idealizations:

- density ρ is constant,
- source force density is *isotropic point radiator with known time dependence* ("source pulse" $w(t)$, typically of compact support)

$$f(\mathbf{x}, t; \mathbf{x}_s) = w(t)\delta(\mathbf{x} - \mathbf{x}_s)$$

$\Rightarrow$ acoustic potential, pressure depends on source location $\mathbf{x}_s$ also.

Homogeneous acoustics

Suppose also that

- velocity c is constant

("homogeneous" acoustic medium - same stress-strain relation everywhere)

Explicit *causal* ($=$ vanishing for $t << 0$) solution for 3D [Proof: exercise!]:

$$u(\mathbf{x}, t) = \frac{w(t - r/c)}{4\pi r},\ r = |\mathbf{x} - \mathbf{x}_s|$$

Nomenclature: *outgoing spherical wave*

Homogeneous acoustics

Also explicit solution (up to quadrature) in 2D - a bit more complicated (Poisson's formula - exercise: find it! eg. in Courant and Hilbert)

Looks like expanding circular wavefront for typical $w(t)$

[MOVIE 1]

Observe: no reflections!!!

Homogeneous acoustics

Upshot: if acoustic model is at all appropriate, must use non-constant c to explain observations.

Natural mathematical question: how nonconstant can c be and still permit "reasonable" solutions of wave equation?

Heterogeneous acoustics

Weak solution of Dirichlet problem in $\Omega \subset \mathbf{R}^3$ (similar treatment for other b. c.'s):

$$u \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H^1_0(\Omega))$$

satisfying for any $\phi \in C_0^\infty((0,T) \times \Omega)$,

$$\int_0^T \int_\Omega dt\, dx \left\{ \frac{1}{\rho c^2} \frac{\partial u}{\partial t} \frac{\partial \phi}{\partial t} - \frac{1}{\rho} \nabla u \cdot \nabla \phi + \frac{1}{\rho} f \phi \right\} = 0$$

Heterogeneous acoustics

Theorem (Lions, 1972) $\log \rho, \log c \in L^{\infty}(\Omega)$, $f \in L^2(\Omega \times \mathbf{R}) \Rightarrow$ weak solutions of Dirichlet problem exist; initial data

$$u(\cdot,0) \in H^1_0(\Omega),\ \frac{\partial u}{\partial t}(\cdot,0) \in L^2(\Omega)$$

uniquely determine them.

Key Ideas in Proof

1. Conservation of energy: first assume that $f \equiv 0$, set

$$E[u](t) = \frac{1}{2}\int_{\Omega}\left(\frac{1}{\rho c^2}p(\cdot,t)^2 + \rho|\mathbf{v}(\cdot,t)|^2\right)$$

$=$ elastic strain energy (potential $+$ kinetic)

$$= \frac{1}{2}\int_{\Omega}\left(\frac{1}{\rho c^2}\left(\frac{\partial u}{\partial t}(\cdot,t)\right)^2 + \frac{1}{\rho}|\nabla u(\cdot,t)|^2\right)$$

Key Ideas in Proof

u smooth enough $\Rightarrow$ integrations by parts & differentiations under integral sign make sense $\Rightarrow$

$$\frac{dE[u]}{dt} = 0$$

Key Ideas in Proof

General case ($f \neq 0$): with help of Cauchy-Schwarz $\leq$,

$$\frac{dE[u]}{dt}(t) \leq \mathrm{const.} \left(E[u](t) + \int_0^t ds \int_\Omega dx\, f^2(\mathbf{x}, s) \right)$$

whence for $0 \leq t \leq T$,

$$E[u](t) \leq \mathrm{const.} \left(E[u](0) + \int_0^t ds \int_\Omega dx\, f^2(\mathbf{x}, s) \right)$$

(Gronwall's $\leq$)

const on RHS bounded by T, $\|\log \rho\|_{L^\infty(\Omega)}, \|\log c\|_{L^\infty(\Omega)}$

RICE

Key Ideas in Proof

Poincaré's $\leq \Rightarrow$ "*a priori* estimate"

$$\left\| \frac{\partial u}{\partial t}(\cdot, t) \right\|_{L^2(\Omega)^2} + \| u(\cdot, t) \|^2_{H^1(\Omega)}$$

$$\leq \text{const.} \left(\left\| \frac{\partial u}{\partial t}(\cdot, 0) \right\|_{L^2(\Omega)^2} + \| u(\cdot, 0) \|^2_{H^1(\Omega)} \right.$$

$$\left. + \int_{-\infty}^{t} ds \int_{\Omega} dx \, f^2(\mathbf{x}, s) \right)$$

Key Ideas in Proof

Derivation presumed more smoothness than weak solutions have, ex def. First serious result:

Weak solutions obey same *a priori* estimate

Proof via approximation argument.

Corollary: Weak solutions uniquely determined by $t = 0$ data

Key Ideas in Proof

2. Galerkin approximation: Pick increasing sequence of subspaces

$$W^0 \subset W^1 \subset W^2 \subset \ldots \subset H_0^1(\Omega)$$

so that

$$\cup_{n=0}^{\infty} W^n \,\mathrm{dense\,in}\, L^2(\Omega)$$

Typical example: piecewise linear Finite Element subspaces on sequence of meshes, each refinement of preceding.

Key Ideas in Proof

Galerkin principle: find $u^n \in C^2([0,T], W^n)$ so that for any $\phi^n \in C^1([0,T], W^n)$,

$$\int_0^T \int_\Omega dt\, dx \left\{ \frac{1}{\rho c^2} \frac{\partial u^n}{\partial t} \frac{\partial \phi^n}{\partial t} - \frac{1}{\rho} \nabla u^n \cdot \nabla \phi^n + \frac{1}{\rho} f \phi^n \right\}$$

$$= 0$$

Key Ideas in Proof

In terms of basis $\{\phi^n_m : m = 0, ..., N^n\}$ of W^n, write

$$u^n(t, \mathbf{x}) = \sum_{m=0}^{N^n} U^n_m(t)\phi^n_m(\mathbf{x})$$

Then integration by parts in $t \Rightarrow$ coefficient vector $U^n(t) = (U^n_0(t), ..., U^n_{N^n})^T$ satisfies ODE

$$M^n \frac{d^2 U^n}{dt^2} + K^n U^n = F^n$$

where

$$M^n_{i,j} = \int_\Omega \frac{1}{\rho c^2}\phi^n_i \phi^n_j,\ \ K^n_{i,j} = \int_\Omega \frac{1}{\rho}\nabla\phi^n_i \cdot \nabla\phi^n_j$$

and sim for F^n

Key Ideas in Proof

Assume temporarily that
$f \in C^0([0,T], L^2(\Omega)) \subset L^2([0,T] \times \Omega)$ - then
$F^n \in C^0([0,T], W^n)$, so...

basic theorem on ODEs $\Rightarrow$ existence of Galerkin approximation u^n.

Energy estimate for Galerkin approximation -

$$E[u^n](t) \le \mathrm{const.} \left(E[u^n](0) + \int_0^t \|f(\cdot,t)\|^2_{L^2(\Omega)} \right)$$

constant independent of n.

RICE

Key Ideas in Proof

Alaoglu Thm $\Rightarrow \{u^n\}$ weakly precompact in $L^2([0,T], H^1_0(\Omega))$, $\{\partial u^n/\partial t\}$ weakly precompact in $L^2([0,T], L^2(\Omega))$, so can select weakly convergent sequence, limit $u \in L^2([0,T], H^1_0(\Omega))$, $\{\partial u/\partial t\} \in L^2([0,T], L^2(\Omega))$.

Key Ideas in Proof

Final cleanup of Galerkin existence argument:

- u is weak solution (necessarily the weak solution!)
- remove regularity assumption on f via density of $C^0([0,T],L^2(\Omega))$ in $L^2([0,T]\times\Omega)$, energy estimate

More *time* regularity of $f \Rightarrow$ more *time* regularity of u. If you want more *space* regularity, then coefficients must be more regular! (examples later)

See Stolk 2000 for details, Blazek et al. 2008 for similar results re symmetric hyperbolic systems

RICE

Reflection seismic inverse problem

Forward map $\mathcal{F}$ = time history of pressure for each source location $\mathbf{x}_s$ at receiver locations $\mathbf{x}_r$, *as function of c*

Reality: $\mathbf{x}_s$ samples finitely many points near surface of Earth ($z = 0$), active receiver locations $\mathbf{x}_r$ may depend on source locations and are also discrete

but: sampling is *reasonably fine* (see plots!) so...

Idealization: $(\mathbf{x}_s, \mathbf{x}_r)$ range over 4-diml closed submfd with boundary Σ, source and receiver depths constant.

Reflection seismic inverse problem

(predicted seismic data), depends on velocity field $c(\mathbf{x})$:

$$\mathcal{F}[c] = p|_{\Sigma \times [0,T]}$$

Inverse problem: given *observed seismic data* $d \in L^2(\Sigma \times [0, T])$, find c so that

$$\mathcal{F}[c] \simeq d$$

(NB: generalizations to elasticity etc., vector data...)

Reflection seismic inverse problem

This inverse problem is

- large scale - Tbytes of data, Pflops to simulate forward map
- nonlinear
- yields to no known direct attack (no "solution formula")
- indirect approach: formulate as optimization problem (find "best fit" model)

Reflection seismic inverse problem

Optimization - typically least squares (Tarantola, Lailly,... 1980's $\rightarrow$ present):

Given d, find c to minimize

$$\|\mathcal{F}[c] - d\|^2 \text{ [+regularization]}$$

over suitable class of c

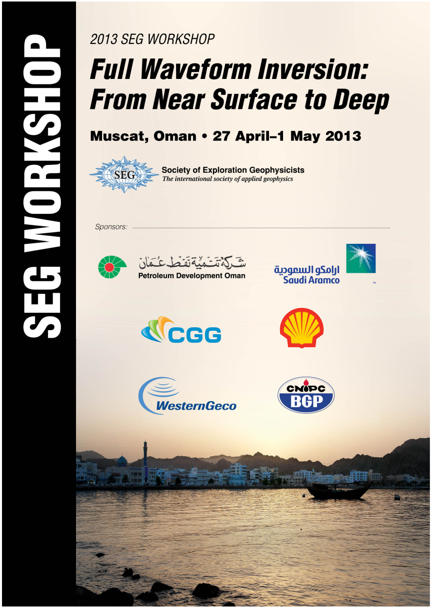
Contemporary alias: *full waveform inversion* ("FWI")

Reflection seismic inverse problem

Changing attitudes to FWI:

- 2002: called "academic approach" by prominent exploration geophysicist
- 2013: every major oil and service company has significant R & D effort, some deployment
- SEG 2002: 2 technical sessions (out of > 50) inversion and other topics
- SEG 2012: 9 technical sessions on seismic FWI
- 3 major workshops in 2012-13

Reflection seismic inverse problem



Reflection seismic inverse problem

Size, cost $\Rightarrow$ Newton relative $\Rightarrow$ compute *gradient* (perhaps *Hessian*) - *adjoint state method* (Ch. 3)

$\Rightarrow$ *linearization* must make sense, i.e. $\mathcal{F}$ must be differentiable in some sense