SIXTH PROBLEM SESSION EXERCISES

These exercises are dedicated to studying the solution of the eikonal equation for the travel time function τ. We will complete the proof outlined in lecture that the function constructed using a Hamiltonian system is in fact a solution of the eikonal equation.

Recall that the eikonal equation is

$$v(x)\ |\nabla \tau(x; x_s)| = 1.$$

We assume that v is a smooth strictly positive function. The so-called Hamiltonian is a function on phase space (i.e. $\mathbb{R}^n \times \mathbb{R}^n$ where n can be any integer ≥ 2 although the cases of interest for us are $n = 2$ and 3) defined by

$$H(X, \Xi) = \frac{1}{2}(v(X)^2 |\Xi|^2 - 1).$$

The Hamiltonian flow, which gives the rays when projected onto the X component, is defined by the solution of the Hamiltonian system

$$\dot{X}(t) = \nabla_\Xi H(X(t), \Xi(t)), \quad \dot{\Xi}(t) = -\nabla_X H(X(t), \Xi(t)). \tag{1}$$

To construct the solution of the eikonal equation we add initial conditions

$$X(0) = x_s, \quad \Xi(0) = \frac{\theta}{v(x_s)} \tag{2}$$

for $\theta \in \mathbb{S}^{n-1}$. By applying the inverse function theorem it is possible to show that the mapping

$$(\theta, t) \mapsto X(t)$$

is a diffeomorphism for $0 < t < A$, for some constant A, onto an neighborhood U of x_s minus x_s. Here (θ, t) should be thought of as giving polar coordinates with respect to the rays. Thus, for $x \in U \setminus x_s$ we can define

$$\tau(x; x_s) = t(x). \tag{3}$$

These exercises are aimed at showing that this function $\tau(\cdot; x_s)$ satisfies the eikonal equation on $U \setminus x_s$.

1: If X and Ξ are solutions of the Hamiltonian system (1) show that $H(X(t), \Xi(t))$ does not depend on t. If X and Ξ also have initial conditions in the form (2) show that

$$|\Xi(t)| = 1/v(X(t)), \quad \text{and} \quad \dot{X}(t) \cdot \Xi(t) = 1$$

for all t.

2: Suppose that θ in (2) is allowed to vary along a smooth curve $\mathbb{R} \ni \alpha \mapsto \theta_\alpha$ in $\mathbb{S}^{n-1}$. Then the solutions of (1) and (2) depend on α and so we have $X(t, \alpha)$ and $\Xi(t, \alpha)$. Show that

$$\frac{\partial}{\partial t}\left(\frac{\partial X}{\partial \alpha}(t, \alpha) \cdot \Xi(t, \alpha)\right) = 0.$$

Since

$$\frac{\partial X}{\partial \alpha}(0, \alpha) = 0$$

we can conclude that

$$\frac{\partial X}{\partial \alpha}(t, \alpha) \cdot \Xi(t, \alpha) = 0$$

for all t.

3: Suppose $x \in U \setminus x_s$ and $\theta(x)$ and $t(x)$ are the "polar coordinates" introduced above for x. Further, suppose the curve θ_α from the previous part satisfies $\theta_0 = \theta(x)$. Let τ be defined by (3). Show that

$$\nabla\tau(x; x_s) \cdot \frac{\partial X}{\partial \alpha}(t(x), 0) = 0$$

for all t, and that in fact any vector perpendicular to $\nabla\tau(x; x_s)$ can be realized as $\frac{\partial X}{\partial \alpha}(t(x), 0)$ for some curve θ_α. Combine this with the previous part to show that $\nabla\tau(x; x_s)$ and $\Xi(t(x))$ are parallel. This result is closely related to something called Gauss' lemma in differential geometry.

4: Finally, show that in fact τ defined by (3) satisfies the eikonal equation.