1. WEAK SOLUTIONS AND REFLECTION

Refer to Section 1.2.5 of Prof. Demanet's notes, and to the definition of weak solution of the wave equation in the sense of Lions (slide 40, part 1), modified as follows

- 1D wave equation, rather that 3D;
- time interval is $(-\infty, \infty)$ rather than $[0, T]$;
- $\Omega = \mathbf{R}$

No boundary conditions are imposed, and the solution is permitted to have unbounded support - but the integrations present in the definitions are still over bounded domains, as the test functions ϕ have compact support.

1.1. STRONG SOLUTIONS ARE WEAK SOLUTIONS:

suppose that ρ, c, f are smooth, say $\rho, c \in C^1(\mathbf{R}), f \in C^0(\mathbf{R}^2)$. Show that an "ordinary" solution of the wave equation, that is, $u \in C^2(\mathbf{R}^2)$ satisfying

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial u}{\partial x} = f,$$

is a weak solution.

1.2. WEAK SOLUTIONS OBEY THE USUAL LAWS OF REFLECTION:

Suppose that ρ and $\kappa = \rho c^2$ are as described in Section 1.2.5. Suppose that u is as given in equations (1.19) and (1.20), reproduced here:

$$u(x,t) = f(x - c_1 t) + Rf(-x - c_1 t), \; x < 0 \; (1.19)$$

$$u(x,t) = Tf\left(\frac{c_1}{c_2}(x - c_2 t)\right), \; x > 0 \; (1.20)$$

with the other symbols having the meaning explained in Section 1.2.5. Show that u is a weak solution of the wave equation in the sense of Lions. [Since such weak solutions are uniquely defined by their data, it is *the* weak solution - thus the notion of weak solution captures the physics of reflection and transmission.] Hint: write the integral in the defn of weak solution as a limit of integrals over the complement of the slab $\{(x,t) : |x| \leq \epsilon\}$ as $\epsilon \to 0$. u is smooth away from $x = 0$, so integration by parts in the complement of the slab is permissible; integrate by parts to move all derivatives onto u, and *then* take the limit, watching what happens to the boundary contributions.

2. DISTRIBUTION SOLUTIONS

Back to 3D, $\Omega = \mathbf{R}^3$, unbounded time interval.

Suppose that the coefficients and solution are smooth enough that the wave equation is solved in the ordinary sense (both sides continuous, and equal pointwise). Observe that u satisfies

$$\int\int dtdx\, u \left\{ \frac{1}{\rho c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla\phi \right\} = \int\int dtdx f\phi$$

for any $\phi \in C_0^\infty(\mathbf{R}^4)$. This observation leads to a definition of *distribution solution*: a distribution $u \in \mathcal{D}'(\mathbf{R}^4)$ solves the wave equation with right-hand side $f \in \mathcal{D}'(\mathbf{R}^4)$ iff

$$\left\langle u, \left\{ \frac{1}{\rho c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla\phi \right\} \right\rangle = \langle f, \phi \rangle \text{ for all } \phi \in C_0^\infty(\mathbf{R}^4),$$

in which the angle brackets represent the duality pairing of distributions with test functions. This pairing amounts to the integral of the product when u is a locally integrable function, so the definition of distribution solution yet another generalization of the ordinary notion. [Converse also holds: a distribution solution is a weak solution in the sense of Lions if it happens to have the regularity described in slide 40, and a solution in the ordinary sense if it is of class C^2.]

2.1. DISTRIBUTION SOLUTION OF THE RADIATION PROBLEM

This problem describes a slightly different point of view on section 1.2.3 of Prof. Demanet's notes.

Suppose that ρ, c are constant, and $w \in C_0^\infty(\mathbf{R})$. Show that the locally integrable function u, given by

$$u(x,t) = \frac{w(t - \|x-y\|/c)}{4\pi\|x-y\|}$$

solves the *radiation problem*

$$\left(\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u \right)(x,t) = w(t)\delta(x-y); \quad u = 0, t << 0$$

in the sense of distributions.

[It is possible to take $w \to \delta$ along a Dirac sequence - u will also converge, in the sense of distributions, to the Green's function G of (1.15) (modulo a factor of c^2).]

Note: The notion of distribution solution makes sense if c is smooth, rather than constant, and in fact is a central tool in the analysis of linear PDE with smooth coefficients. The concept in its present form dates back to the mid-20th century work of Laurent Schwartz and others. See any good graduate text on PDE, such as Michael E. Taylor, *Partial Differential Equations*, Springer, New York, 1996. Weak solutions in the sense of Lions, on the other hand, provide a sensible notion of solution for some classes of linear PDEs with nonsmooth coefficients, in particular a natural and physically correct representation of waves in heterogeneous materials. The standard reference is J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, Springer, New York, 1972.