The Inverse Problem of Seismic Velocities

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Abstract Ruminations about Inverse Problems

The Seismic Reflection Experiment and the Acoustic Model

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Example
The usual set-up:

- $\mathcal{M}$ = a set of *models*
- $\mathcal{D}$ = a Hilbert space of (potential) data
- $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{D}$

[These things are collectively “the model”.]

Inverse problem: given $d \in \mathcal{D}$, find $m \in \mathcal{M}$ so that $\mathcal{F}[m] \simeq d$. 


Chief requirement of a “solution”: must be able to


2. Find - does the characterizing problem admit an effective numerical solution?

Common pattern for 2.: solution is extremum of variational principle, for instance

\[ m = \text{argmin} \| \mathcal{F}[m] - d \| \]
Example, topic of this talk: reflection seismology. Naturally formulated as inverse problem using various physical descriptions of seismic wave motion (acoustic, elastic, viscoelastic,...)

Typical problem size for adequately sampled 3D reflection seismic survey: \( \dim(M) \sim 10^9, \dim(D) \sim 10^{12} \)

\( \Rightarrow \) any computational “solution” must admit algorithms that scale well with problem size - if iterative, then iteration count should be ess. independent of dimension.

Optimization \( \Rightarrow \) Newton’s method \( \Rightarrow \) must be satisfied with any stationary point.
Takeaway messages of this talk:

Straightforward data fitting (eg. by least squares) does not work well for this class of problems

“Relaxed” variational formulation via model *extensions* leads to effective numerical algorithms

For simplest cases, can show that all stationary points are approximate global minimizers.

Numerical evidence for more than this, but *many* open questions
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Marine Seismic Reflection Experiment

Airguns = source of sound. Streamer consists of hydrophone receiver groups. Each group records a trace (time series of pressure) for each shot = excitation of source. Source-receiver distance = offset.
North Sea Survey (thanks: Shell).

Processing applied:

- bandpass filter 3-8-25-35 Hz (data was oscillatory to begin with!);
- cutoff or mute to remove non-reflection energy (direct, diving, head waves);
- predictive deconvolution to suppress multiple reflections.
Mechanical properties of sedimentary rocks

- $v_p$ varies significantly.
- Heterogeneity at all scales - km to mm to $\mu$m.

Well ($v_p$) log from Texas borehole
(thanks: P. Janak, Total E&P, USA)
Earth "=" \( \Omega \subset \mathbb{R}^3 \), wave velocity \( v : \Omega \rightarrow \mathbb{R}, v > 0 \).

Wave equation for acoustic potential response to isotropic point radiator at \( \mathbf{x}_s \), time dependence \( w(t) \):

\[
\left( \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} - \nabla^2 \right) u(t, \mathbf{x}; \mathbf{x}_s) = w(t)\delta(\mathbf{x} - \mathbf{x}_s)
\]

plus appropriate initial and boundary conditions. NB: to model oscillatory nature of data, \( w \) must be oscillatory -

\[
\hat{w}(\omega) = O(\omega^p), \quad p \geq 1.
\]

Lions, late ’60’s: proper notion of weak solution, well posed for \( v \in \mathcal{M} = \{ \log v \in L^\infty(\Omega) \} \), RHS in \( L^2([0, T] \times \Omega) \)
Point Source Acoustics - the minimal model

Forward map: \( \mathcal{F} : \mathcal{M} \rightarrow \mathcal{D} = L^2([0, T] \times \Sigma), \)
\( \Sigma \subset \{x_3 = 0\} \times \{x_3 = 0\} \) open, samples pressure in support of
\( \phi \in C_0^\infty(\Sigma): \) for \( (t, x_r, x_s) \in [0, T] \times \Sigma, \)

\[
\mathcal{F}[\nu](t, x_r; x_s) = \left( \phi \frac{\partial u}{\partial t} \right)(t, x_r; x_s)
\]

If \( \nu = \nu_0 \) known & constant in \( \{x_3 < z\} \) for some \( z > 0, \)
\( w \in L^2(\mathbb{R}), \) slight extension of Lions’ argument shows \( \mathcal{F} \)
well-defined.

Stolk 2000: continuous, differentiable “with loss of derivative”. 
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Least Squares Data Fitting

Natural formulation, in view of defn of $D$: choose $v$ by

$$v = \arg\min \left( \|F[v] - d\|_{L^2([0,T] \times \Sigma)}^2 + R[v] \right)$$

(“mean square error”) in which $R$ (regularization functional) supplies additional stability.

Promoted heavily by Tarantola and others in the 1980’s on grounds of Bayesian justification (maximum likelihood solution given Gaussian data error statistics)

Recently revived as major industry interest (“Full Waveform Inversion”, FWI) - all-day workshop with attendance $> 300$ at SEG 09.
A Sad Story:
Data is oscillatory - $O(100)$ wavelengths
Small changes in velocity ⇒
small changes in data isosurfaces
⇒ large changes in mean square error ⇒ saturation ⇒ many stationary points
Least Squares Data Fitting

Upshot:

- FWI via iterative optimization method recovers very detailed subsurface models, at least in numerical tests with model data, **when starting model is sufficiently accurate** (Tarantola and coworkers 80’s, 90’s; Bunks 95; much recent work)
- Fails when starting model is not sufficiently accurate (stalls at stationary point with poor data fit)
- Hard to tell what “sufficiently accurate” means - no *a priori* test
- Continuation from low to high frequency / depth permits convergence with less accurate starting model (Kolb et al 1986, Bunks 95, Pratt 2004, recent from Shin and coauthors) - however no guarantees
Solution via Model Extension

Extension of $\mathcal{F}$:

- $\chi : \mathcal{M} \to \tilde{\mathcal{M}}$, 
- $\overline{\mathcal{F}} : \tilde{\mathcal{M}} \to \tilde{\mathcal{D}}$, 
- $\phi : \tilde{\mathcal{D}} \to \mathcal{D}$

so that

\[
\begin{array}{ccc}
\tilde{\mathcal{F}} & \downarrow \phi & \\
\tilde{\mathcal{M}} & \to & \tilde{\mathcal{D}} \\
\chi & \uparrow & \\
\mathcal{M} & \to & \mathcal{D} \\
\mathcal{F} & \\
\end{array}
\]

commutes - that is,

\[
\phi[\tilde{\mathcal{F}}[\chi[v]]] = \mathcal{F}[v], \ v \in \mathcal{M}
\]
Solution via Model Extension

Example:

\( \tilde{M} \subset \text{self-adjoint positive definite bounded operators on } L^2(\Omega) \)

[Remark: action-at-a-distance], \( \tilde{D} \subset D'(R \times \Sigma) \). For \( \tilde{v} \in \tilde{M} \),

\[
\mathcal{F}[\tilde{v}](t, x_r; x_s) = \left( \phi \frac{\partial u}{\partial t} \right)(t, x_r; x_s)
\]

in which \( u \) is causal solution of

\[
\left( \tilde{v}^{-2} \frac{\partial^2 u}{\partial t^2} - \nabla^2 \right) u(t, x; x_s) = \delta(t)\delta(x - x_s)
\]

Minor modification of Lions’ construction \( \Rightarrow \) well-posed when \( \tilde{v} \) acts as multiple of identity on functions supported near \( x_s \).

\( \chi : M \subset L^\infty(\Omega) \rightarrow \tilde{M} \) multiplier: \( \chi[v]u = vu \). \( \phi : \tilde{D} \rightarrow D \) by \( \phi[d] = w *_t d \) - filters out low freqs

Range of \( \chi \) = “physical” models
Invertible extension: \( \bar{F} \) has approximate left inverse \( \bar{G} \) (on \( \mathcal{R}(\bar{F}) \))

**NB:** trivial extension - \( \bar{\mathcal{M}} = \mathcal{M}, \bar{F} = F, \chi = \phi = id \) - virtually never invertible.

Example: considerable numerical evidence (but little theory, except for space dimn = 1) strongly suggests that example extension is invertible.
Reformulation of inverse problem: seek $\bar{v} \in \bar{M}, \bar{d} \in \phi^{-1}[d]$ so that

$$\bar{F}[\bar{v}] \simeq \bar{d}, \bar{v} \in \mathcal{R}(\chi)$$

Then $\bar{v} = \chi[v]$ and $v$ is an (approx.) solution of original inverse problem

Only advantageous if $\bar{F}$ is invertible, with approximate inverse $\bar{G}$ - then problem becomes:

find $\bar{d} \in \phi^{-1}[d]$ so that $\bar{G}[\bar{d}] \in \mathcal{R}(\chi)$
Solution via Model Extension

Practical importance - back to Main Example:

Range of \( \chi \) consists of multiplication ops by \( L^\infty \) functions,

\[ \Rightarrow \text{distribution kernels } v(x, y) \text{ supported on (near) } \subset \text{ diagonal } x = y \]

\[ \Rightarrow \text{VISUALLY OBVIOUS in plot of } v(x, y)!!! \]

\[ \Rightarrow \text{industry standard algorithms: tweak (mostly by hand) parameters of } v(x, y) \text{ until support } \text{focuses} \text{ on diagonal} \]
Suppose $W : \tilde{M} \rightarrow \tilde{M}$ annihilates range of $\chi$:

$$\chi \quad W$$

$$\mathcal{M} \rightarrow \tilde{M} \rightarrow \tilde{M} \rightarrow 0$$

Define

$$A \equiv W \circ \bar{G} : \bar{D} \rightarrow \tilde{M}$$

Then for $\bar{d} \in \phi^{-1}(d)$,

$$A[\bar{d}] = 0 \Rightarrow \bar{G}[\bar{d}] = \chi v \Rightarrow d \simeq \phi[\bar{F}[\bar{G}[\bar{d}]]] = \bar{F}[\chi[v]] = \mathcal{F}[v]$$

Thus inverse problem equivalent to:

find $\bar{d} \in \phi^{-1}[d]$ so that $A[\bar{d}] = 0$
Back to the main example: range of \( \chi \) consists of multiplication ops by \( L^\infty \) functions, which commute with other multiplication ops - so can choose

\[
W[\bar{v}] = [\bar{v}, x]
\]

in which \( x \) represents multiplication by coordinate vector.

Write \( \bar{v} \) formally as integral operator with kernel \( \bar{v}(x, y) \). Then

\[
W[\bar{v}]u(x) = \int_\Omega d_\Omega \bar{v}(x, y)(x - y)u(y)
\]

multiplication of \( \bar{v} \) by offset \( x - y \)
Automation

Why should you care?

For very simple model problem using drastic approximations to \( \mathcal{F} \) and so on:

- least squares data fitting has stationary points unrelated to solution of inverse problem, even for noiseless data
- for proper choice of \( W \), hence \( A \), parametrization of \( \phi^{-1}[d] \), and Hilbert norm \( \| \cdot \| \) in \( \tilde{M} \), all stationary points of

\[
\tilde{d} \mapsto \| A[\tilde{d}] \|^2
\]

are approximate global minimizers (WWS).
- numerical experiments with synthetic, field data suggest that same is true in some generality
Conventional simplification: replace $\bar{F}$ with *linearization at smooth physical models*, in terms of $\bar{r} = \delta \bar{v}^{-2}$:

\[
\tilde{M}_1 = \left\{ (v, \bar{r}) : \log v \in C^\infty(\Omega), \bar{r} \in \mathcal{B}_{\text{symm}}(L^2(\Omega)) \right\}, \\
\tilde{D}_1 = \{ v : \log v \in C^\infty(\Omega) \} \times D
\]

define linearization $D\tilde{F} : \tilde{M}_1 \to \tilde{D}_1$ by

\[
D\tilde{F}[v, \bar{r}] = \left( v, \phi \frac{\partial \delta u}{\partial t} \right)
\]

in which

\[
\left( \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} - \nabla^2 \right) \delta u(t, x; x_s) = -\bar{r} \left[ \frac{\partial^2 u}{\partial t^2}(t, \cdot; x_s) \right]
\]
Notes:

- geometric optics analysis $\Rightarrow$ inclusion of low frequencies in extended data $\iff$ inclusion of smooth “background” velocity model in extended data
- $\phi[v, d] = d$, so search in $\phi^{-1}[d]$ is search over smooth background velocities
- Adjoint of $\bar{r} \mapsto D\bar{F}[v, \bar{r}]$ is shot-geophone migration operator
- with H-S norm, $\bar{r} \mapsto D\bar{F}[v, \bar{r}]$ is “nearly unitary”, so adjoint is closely related to inverse, often used instead
Recall annihilator of physical model perturbations: \( W[\tilde{r}] = [\tilde{r}, \textbf{x}] \) - in terms of kernel.

“prestack imaging operator”: approximate inverse \( \mathcal{I}[\nu] \) of \( \tilde{r} \mapsto D\mathcal{F}[\nu, \tilde{r}] \)

Idealized extended inversion algorithm boils down to: minimize (over \( \nu \)) operator norm of \( W[\mathcal{I}[\nu]d] \).

All implementations so far: take advantage of smoothness of numerical approximations to replace operator norm with H-S norm:

\[
J_{DS}[\nu, d] = \frac{1}{2} \int d\textbf{x} \int d\textbf{y} |\mathcal{I}[\nu]d(\textbf{x}, \textbf{y})(\textbf{x} - \textbf{y})|^2
\]

Estimate \( \nu \) by minimizing \( J_{DS} \): “differential semblance”, “annihilator-based waveform tomography”,... (Stolk-de Hoop 01, Shen et al 03, 05, Kabir et al. 06, Shen & WWS 08,...)
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Shen & WWS *Geophysics* 08 (similar expl: Kabir et al. 07 SEG)

Based on penalty version of differential semblance: compute $\mathcal{I}[v]$ by fitting extended data in least-squares sense, with $J_{DS}[v,d]$ as penalty.

Various approximations -

- use adjoint in place of inverse of $D\bar{F}$
- approximate adjoint by solving wave equation as evolution in depth (see Stolk-de Hoop *Wave Motion* 05, 06)
- compute gradient of $J_{DS}$ by adjoint state method (Shen’s thesis)
- iterative quasi-Newton optimization algorithm - limited memory BFGS with adjoint state gradients
Field Data Example

Gas chimney example - thanks Shell

Marine 2D line - preliminary imaging with regional velocity model shows gas-induced distortion ("sag").

Reflection tomography (traveltime inversion) partially removes sag effect, but interpreters not happy.

Differential Semblance to rescue - 20 iterations of Newton-like optimization algorithm produces more "geological" velocity \( (v) \), image (diagonal of \( I[v]d \)) - interpreters happier.
Field Data Example

Initial velocity model - regional trends with depth
Field Data Example

Image at initial model
Field Data Example

Model produced by Reflection Tomography
Field Data Example

Reflection Tomography image
Field Data Example

Model produced by diff’l semblance (20 LBFGS iterations)
Field Data Example

DiffI semblance image (diagonal of $I[v]d$)
Field Data Example

Angle domain common image gathers ("ADCIGs" - Sava & Fomel 03) - Radon transform of 2D slice ("depth-offset") of $\mathcal{I}[v]d$, should be flat at correct velocity - internal measure of consistency between $v$, $d$

Initial velocity - dramatic failure to flatten.

Reflection tomography - much better, but still not flat at larger depths.

Differential semblance - better yet
Field Data Example

ADCIGs, initial model
Field Data Example

ADCIGs, reflection tomography
Field Data Example

ADCIGs, differential semblance
Summary

- Because of oscillatory/wave character of data, seismic inverse problem for wave velocity poorly suited for data fitting formulation - data misfit norm has many stationary points far from solution
- Model extensions permit reformulation in larger model domain ("relaxation", infeasible point method)
- Example extension: velocity coefficient in wave equation as possibly nondiagonal SPD operator
- Simplest examples of extended variational principle ("differential semblance") yield quasi-convex optimization problem - all stationary points are global mins
- With sufficiently many layers of approximation, applicable at field scale
- Almost every mathematical question is open
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