Seismic Inverse Scattering

William W. Symes

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Seismic inverse scattering = reflection seismology:

- probe Earth with artificially generated sound ("seismic") waves

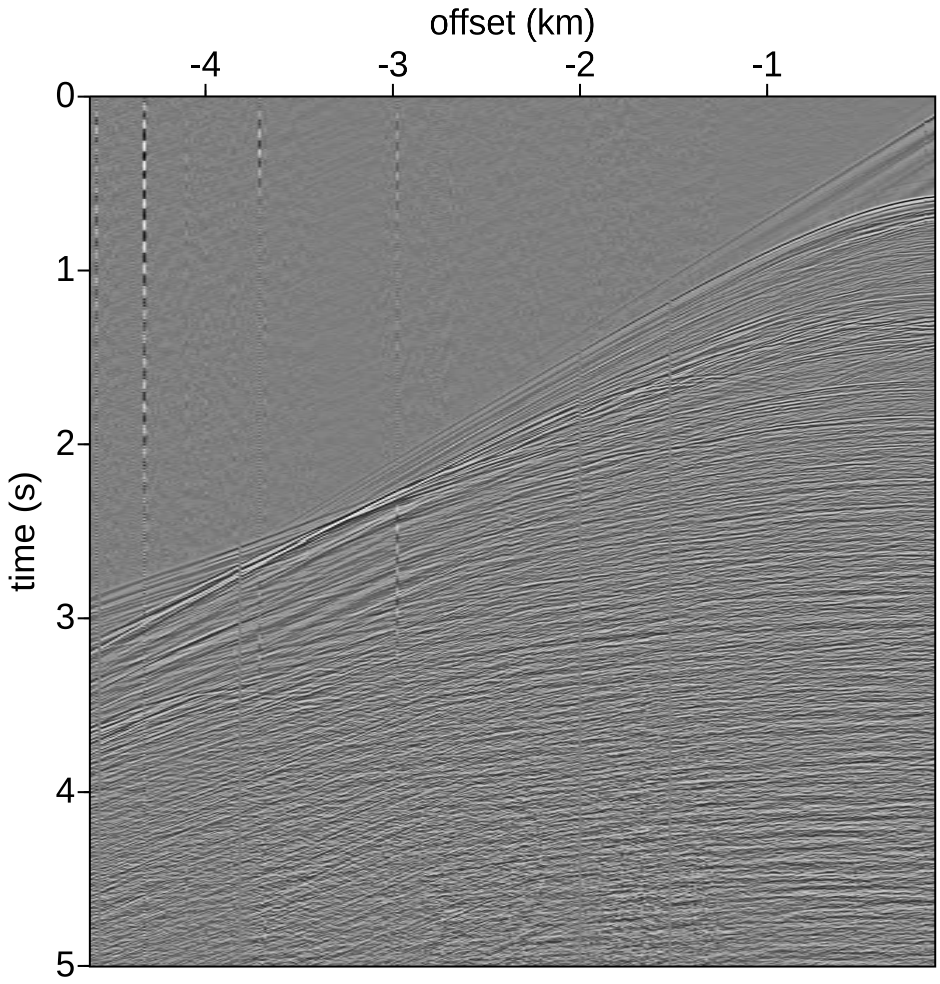
- Important tool in oil & gas exploration and production, civil engineering, environmental remediation, tectonic studies

- Similar technologies: radar, sonar, biomedical ultrasonics, ultrasonic materials testing

Fundamental problem:

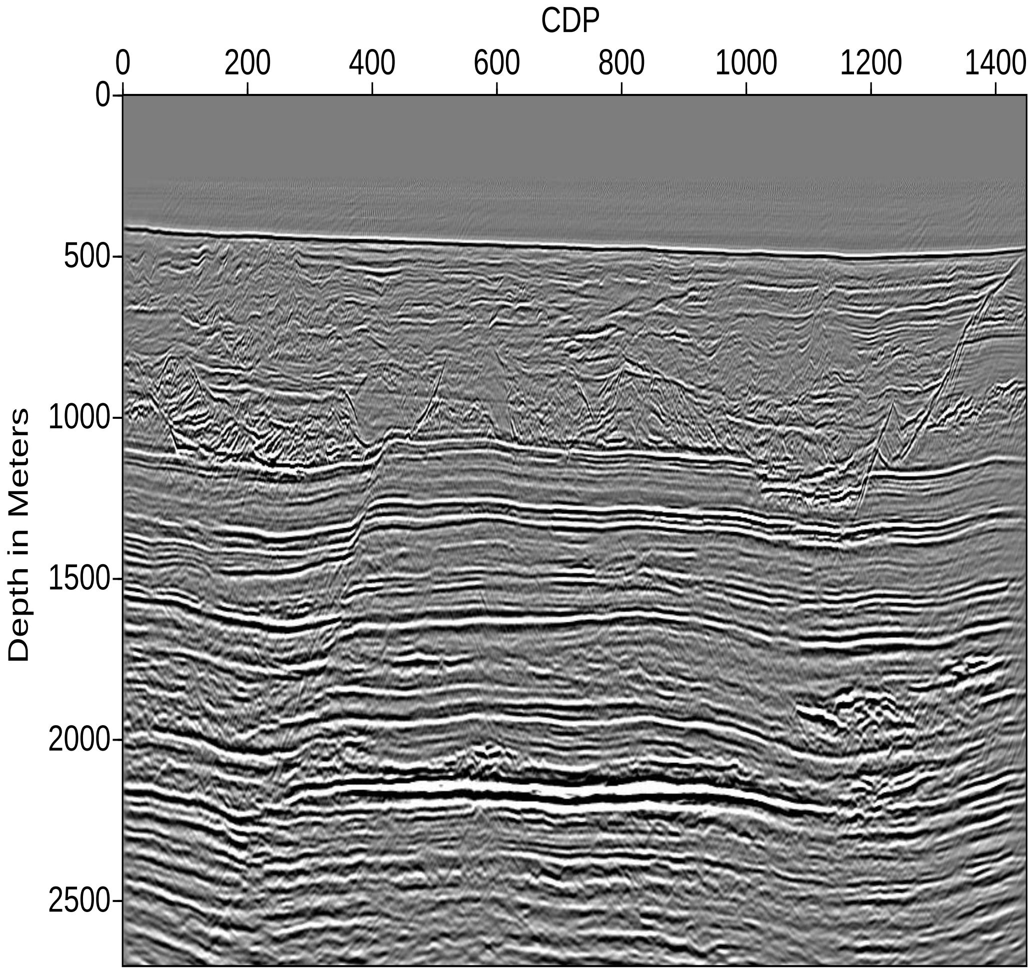
From measurements of ground motion near surface of Earth, determine structure of Earth's interior.

How do you turn lots of this...



(a field seismogram from the Gulf of Mexico - thanks: EPR Co.)

into this?



(a fair rendition of subsurface structure)

A mathematical view of reflection seismic processing:

- an inverse problem, based on a model of seismic wave propagation

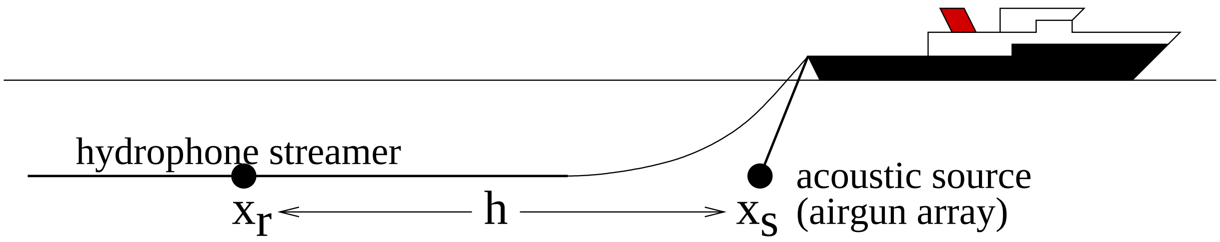
- contemporary practice uses *partial linearization* and high-frequency asymptotics

- recent progress in understanding capabilities, limitations of methods based on linearization/asymptotics in presence of *strong refraction*: applications of *microlocal analysis* with implications for practice

- limitations of linearization lead to many open problems

Marine reflection seismology apparatus:

- source (airgun array, explosives,...)
- receivers (hydrophone streamer, ocean bottom cable,...)
- recording and onboard processing



Land acquisition similar, but acquisition and processing are more complex. Vast bulk (90%+) of data acquired each year is marine.

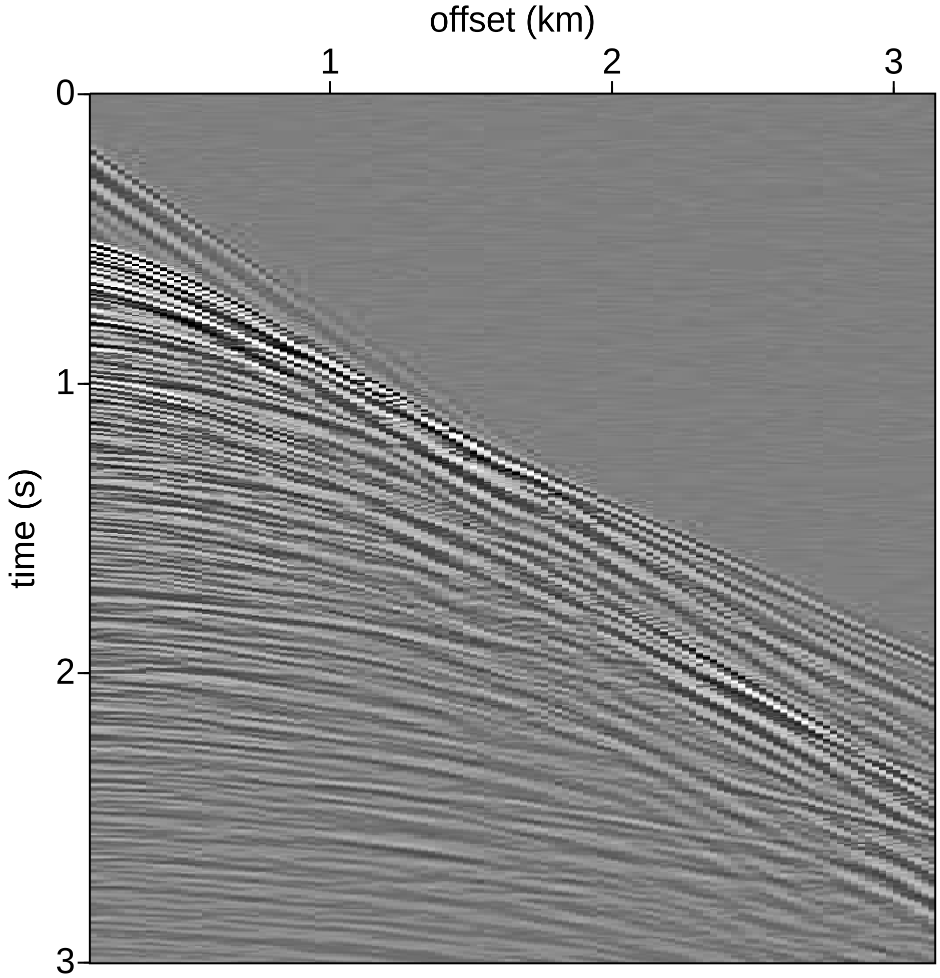
Data parameters: time t, source location $\mathbf{x}_s$, and receiver location $\mathbf{x}_r$ or *offset* $\mathbf{h} = \mathbf{x}_r - \mathbf{x}_s$, $h = |\mathbf{h}|$.

Idealized marine "streamer" geometry: $\mathbf{x}_s$ and $\mathbf{x}_r$ lie roughly on constant depth plane, source-receiver lines are parallel $\rightarrow$ 3 spatial degrees of freedom (eg. $\mathbf{x}_s, h$). [Other geometries are interesting, eg. ocean bottom cables, but streamer surveys still prevalent.]

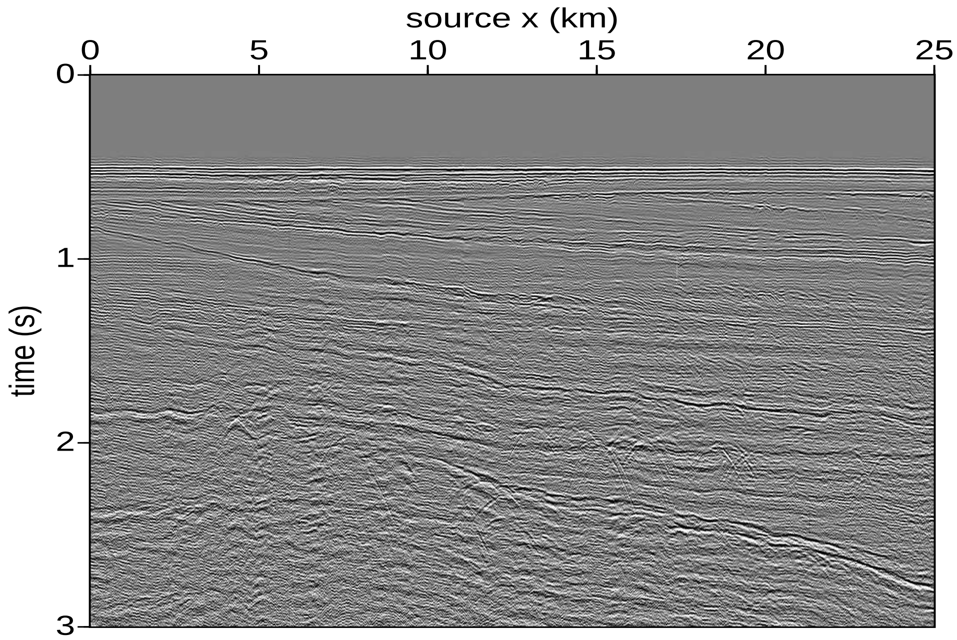
How much data? Contemporary surveys may feature

- Simultaneous recording by multiple streamers (up to 12!)
- Many (roughly) parallel ship tracks ("lines"), areal coverage
- single line ("2D") $\sim$ Gbyte; multiple lines ("3D") $\sim$ Tbyte

Main characteristic of data: *wave nature*, presence of **reflections** = amplitude coherence along trajectories in space-time.



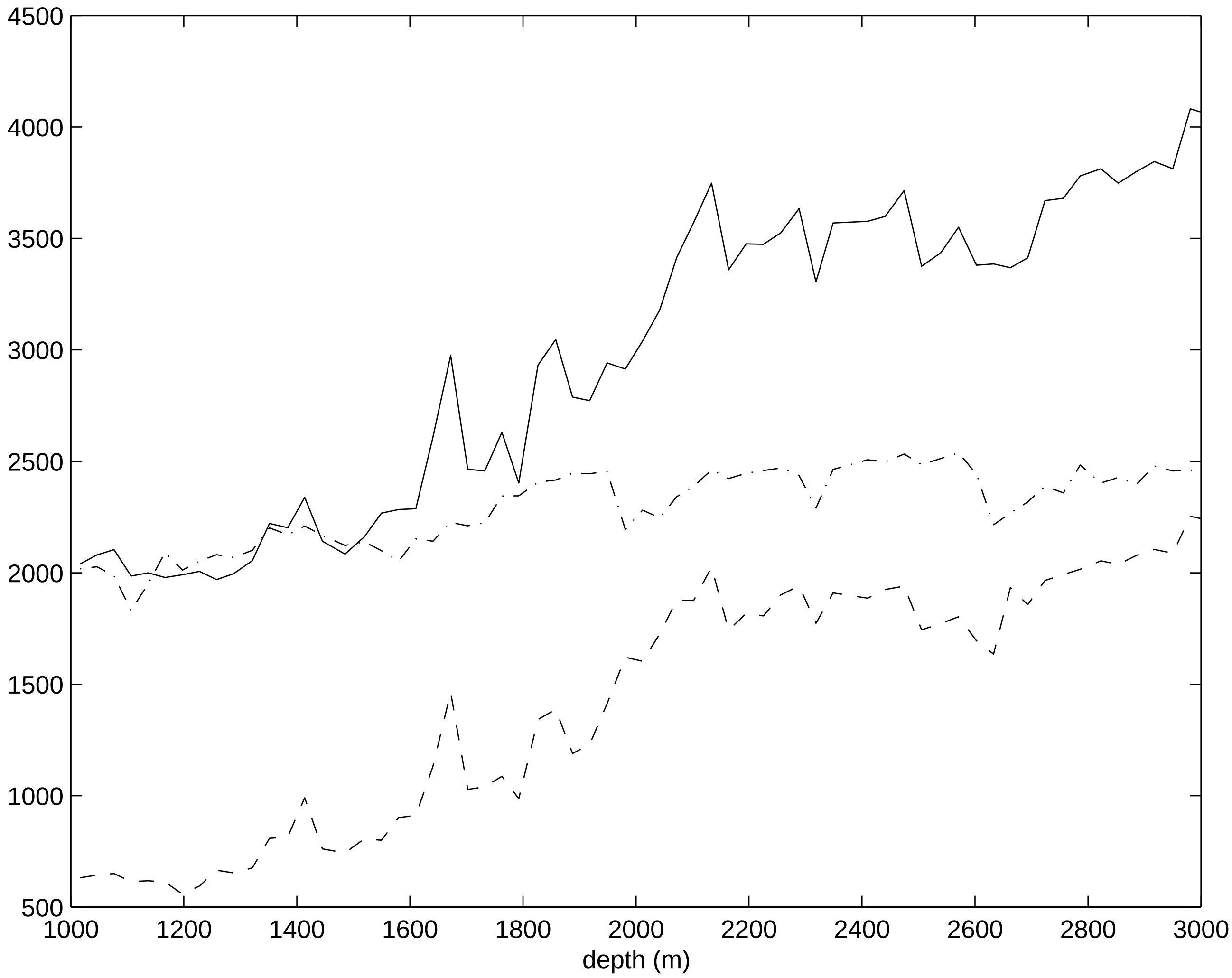
A field shot record from the North Sea, single streamer 2D survey: “Mobil AVO”, 25 m shot and recvr intervals, near offset 160 m, far offset 3135 m, source depth 4 m, recvr depth 6 m, 120 channels (thanks: Mobil R & D)



Near offset (= 160 m) gather from "Mobil AVO"

Characteristics of crustal rocks

- depths of interest in science and technology: 100 m = environmental engineering, 10 km = petroleum exploration, 100 km = crustal geophysics
- approximately linearly elastic - small forces, displacements proportional, but some frequency dependence
- direct, indirect measurement of some elastic parameters in wells - **logging**
- heterogeneity at all scales - km to mm to μm; wave velocities vary by factor of 2 - 5, density somewhat less
- sedimentary rocks contain many localized zones of rapid mechanical property changes along surfaces = **reflectors**



Blocked logs from well near “Mobil AVO” line. Solid: p-wave velocity (m/s), dashed: s-wave velocity (m/s), dash-dot: density (kg/m^3). “Blocked” means “averaged”, in this case over 30 m windows. Original sample rate of log tool < 1 m.

The Modeling Task: any model of the reflection seismogram must

- predict wave motion
- produce reflections from reflectors
- accomodate significant variation of wave velocity, material density,...

A *really good* model will also accomodate

- multiple wave modes, speeds
- material anisotropy
- attenuation, frequency dispersion of waves
- complex source, receiver characteristics

Acoustic Model (only compressional waves) - not *really good*, but good enough for today and basis of most contemporary processing. Relates $\rho(\mathbf{x})=$ material density, $\lambda(\mathbf{x}) =$ bulk modulus, $p(\mathbf{x},t)=$ pressure, $\mathbf{v}(\mathbf{x},t) =$ particle velocity, $\mathbf{f}(\mathbf{x},t)=$ force density (sound source):

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{f},$$

$$\frac{\partial p}{\partial t} = -\lambda \nabla \cdot \mathbf{v} \;(+\,\text{b.c.}'\text{s})$$

(compressional) wave speed $c = \sqrt{\frac{\lambda}{\rho}}$

acoustic field potential $u(\mathbf{x},t) = \int_{-\infty}^{t} ds\, p(\mathbf{x},s)$:

$$p = \frac{\partial u}{\partial t},\; \mathbf{v} = \frac{1}{\rho}\nabla u$$

Equivalent form: second order wave equation for potential

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla u = \int_{-\infty}^{t} dt\, \nabla \cdot \left(\frac{\mathbf{f}}{\rho}\right) \equiv \frac{f}{\rho}$$

Foundation: existence theory based on energy conservation, Lions, 1972. Strain energy, power densities:

$$\frac{1}{2\rho}\left\{\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2+\nabla u\cdot\nabla u\right\}\text{ and }\frac{1}{\rho}\frac{\partial u}{\partial t}\theta\cdot\nabla u$$

weak solutions of Dirichlet problem (similar treatment for other b. c.'s):

$$u\in C^1([0,T];L^2(\Omega))\cap C^0([0,T];H^1_0(\Omega))$$

satisfying for any $\phi\in C_0^\infty((0,T)\times\Omega)$,

$$\int_0^T\int_\Omega dt\,dx\,\left\{\frac{1}{\rho c^2}\frac{\partial u}{\partial t}\frac{\partial \phi}{\partial t}-\frac{1}{\rho}\nabla u\cdot\nabla\phi-\frac{1}{\rho}f\phi\right\}=0$$

Theorem Suppose that $\log\rho,\log c\in L^\infty(\Omega), f\in L^2(\Omega\times\mathbf{R})$. Then weak solutions of Dirichlet problem exist; initial data

$$u(\cdot,0)\in H^1_0(\Omega),\ \frac{\partial u}{\partial t}(\cdot,0)\in L^2(\Omega)$$

uniquely determine them.

Constant density acoustic model of seismic survey:

assume source force density is point radiator at source point $\mathbf{x}_s$ with known time dependence

$$f(t,\mathbf{x};\mathbf{x}_s) = w(t)\delta(\mathbf{x}-\mathbf{x}_s)$$

Acoustic potential, pressure depends on $\mathbf{x}_s$ also.

Forward map S = time history of pressure for each $\mathbf{x}_s$ at receiver locations $\mathbf{x}_r$ (predicted seismic data), depends on velocity field $c(\mathbf{x})$:

$$S[c] = \{p(\mathbf{x}_r, t; \mathbf{x}_s)\}$$

Reflection seismic inverse problem: given *observed seismic data* S^{obs}, find c so that

$$S[c] \simeq S^{\text{obs}}$$

This inverse problem is

- large scale - up to Tbytes, Pflops
- nonlinear

Almost all progress to date based on *partial linearization*: write $c = v(1+r)$ and treat r as relative first order perturbation about v, resulting in perturbation of presure field: $\delta p = \frac{\partial \delta u}{\partial t} = 0, t \le 0$, where

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta u = 2r\nabla^2 u$$

Define *linearized forward map* F by

$$F[v]r = \{\delta p(\mathbf{x}_r, t; \mathbf{x}_s)\}$$

Critical question: how good is this approximation? If there is any justice $F[v]r = DS[v][vr]$ is derivative of S - but in what sense?

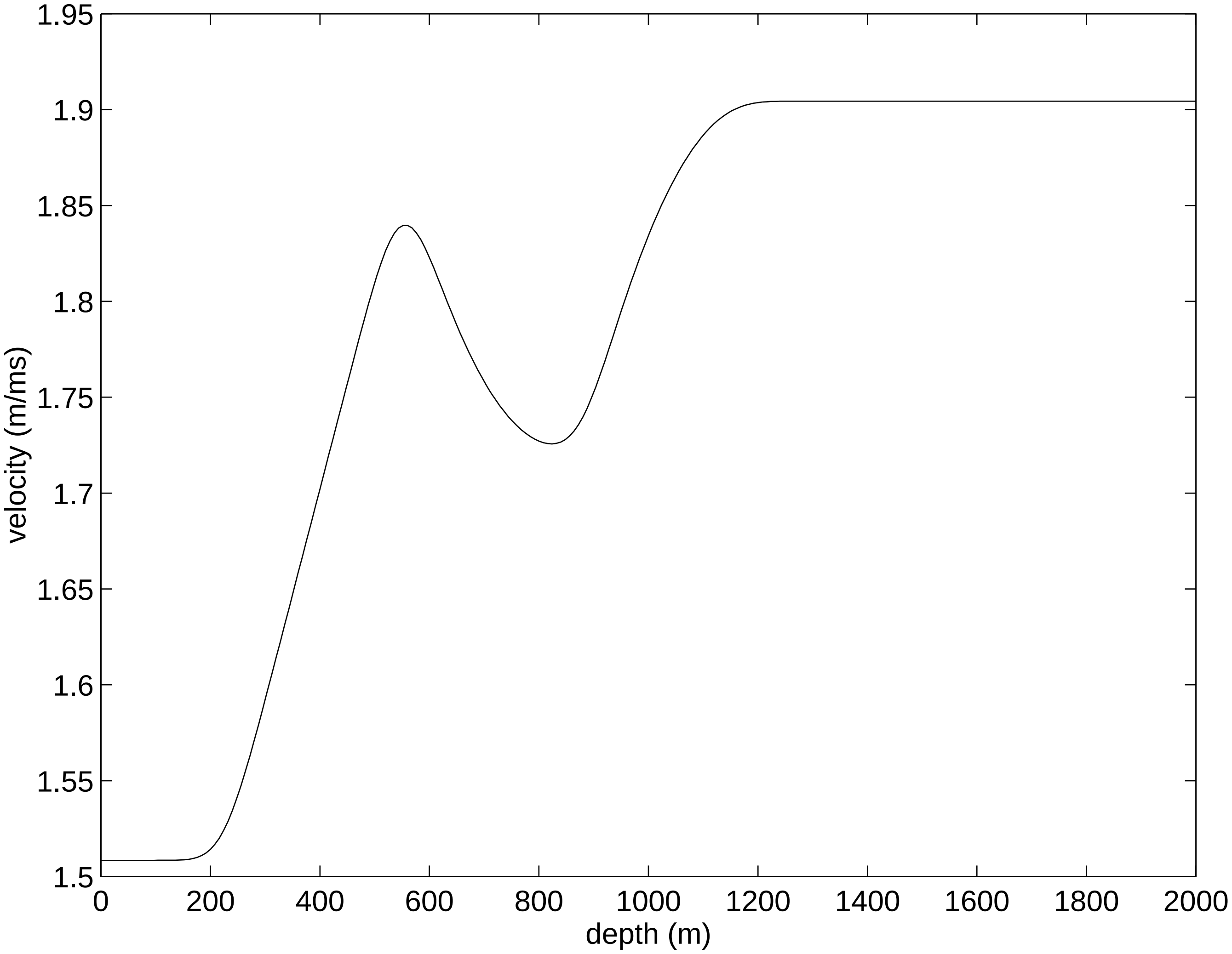
Physical intuition, numerical simulation, and not nearly enough mathematics: linearization error

$$S[v(1+r)] - (S[v] + F[v]r)$$

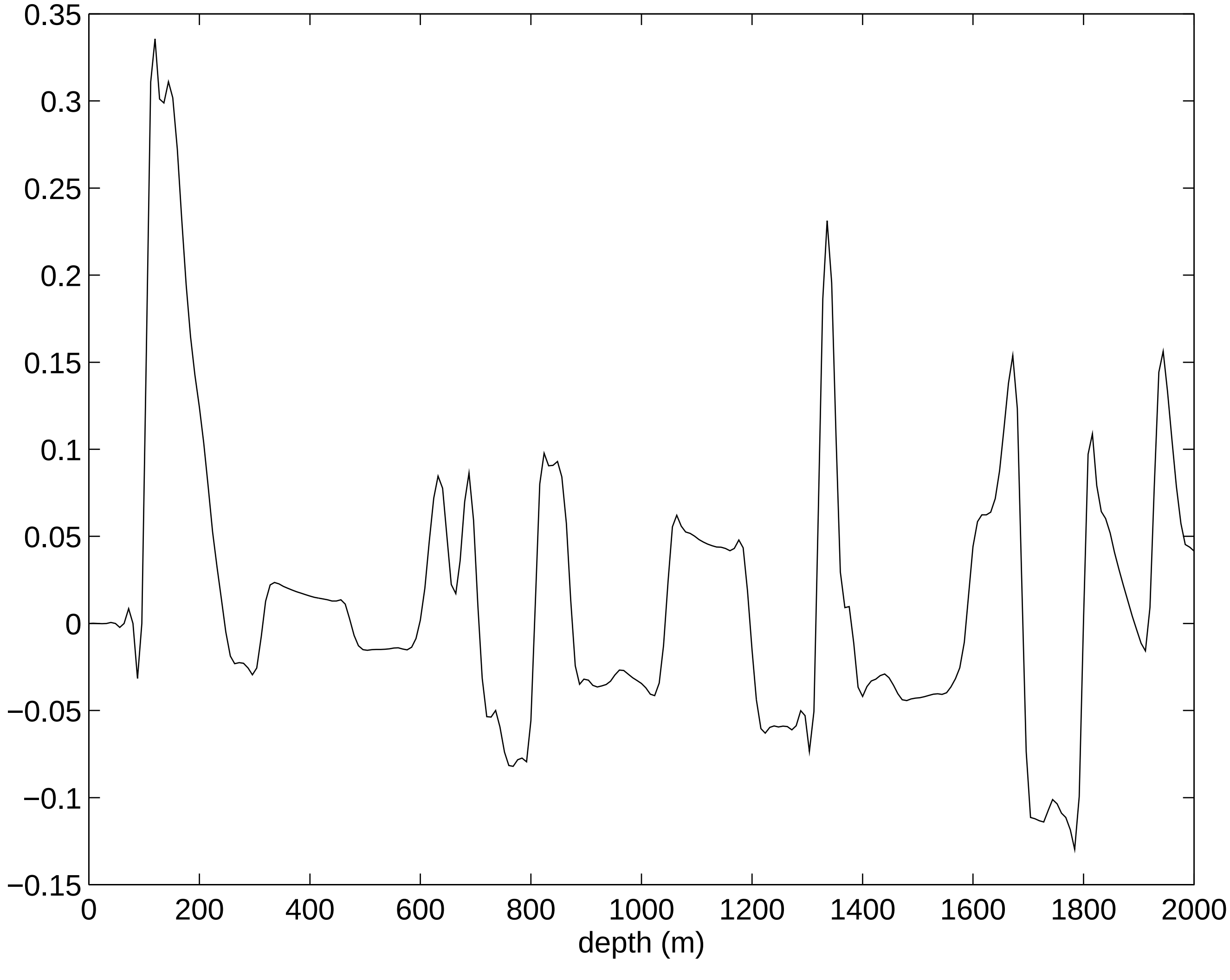
- *small* when v smooth, r rough or oscillatory on wavelength scale - well-separated scales

- *large* when v not smooth and/or r not oscillatory - poorly separated scales

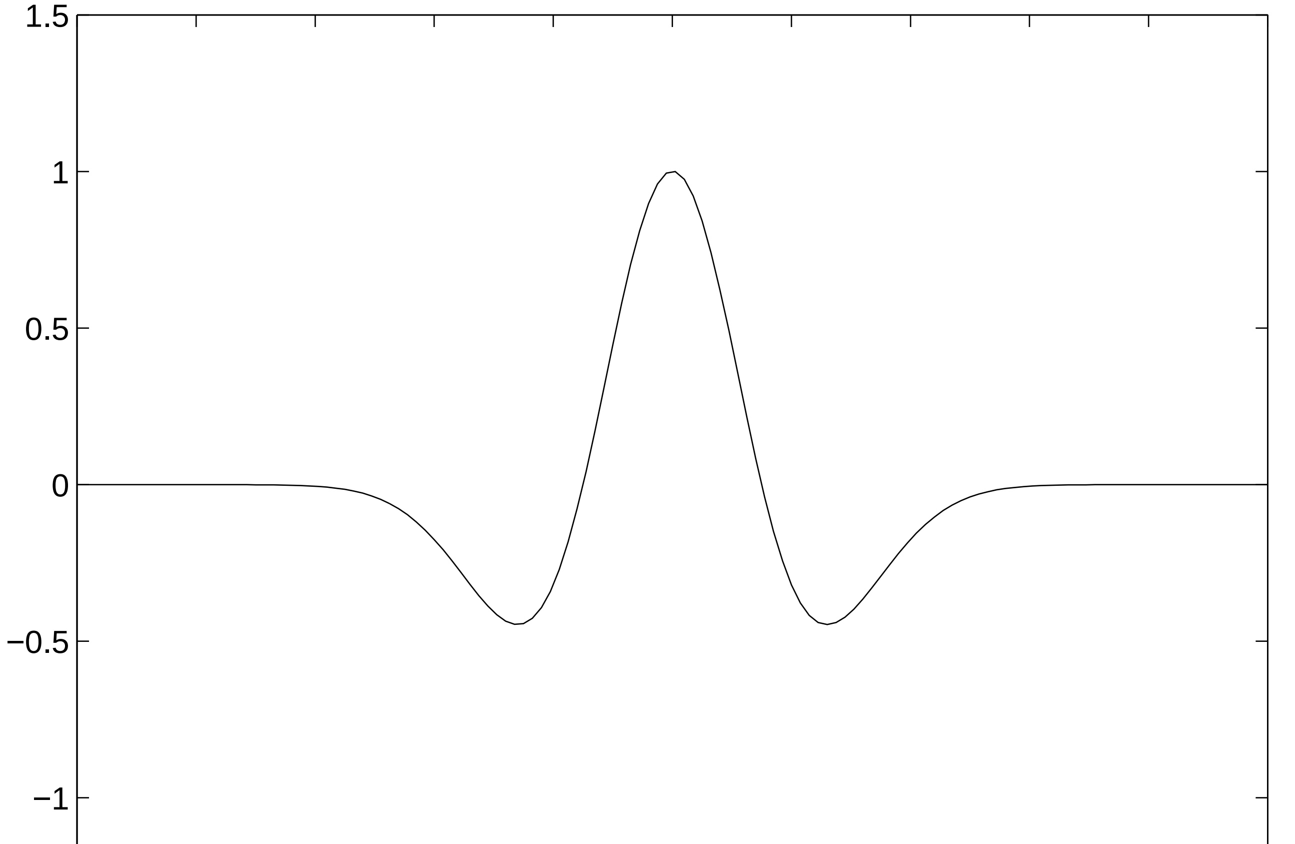
2D finite difference simulation: shot gathers with typical marine seismic geometry. Smooth $v(z)$, oscillatory $r(z)$ ("layered medium") extracted from "Marmousi" synthetic data set (Grau & Versteeg, 1994)



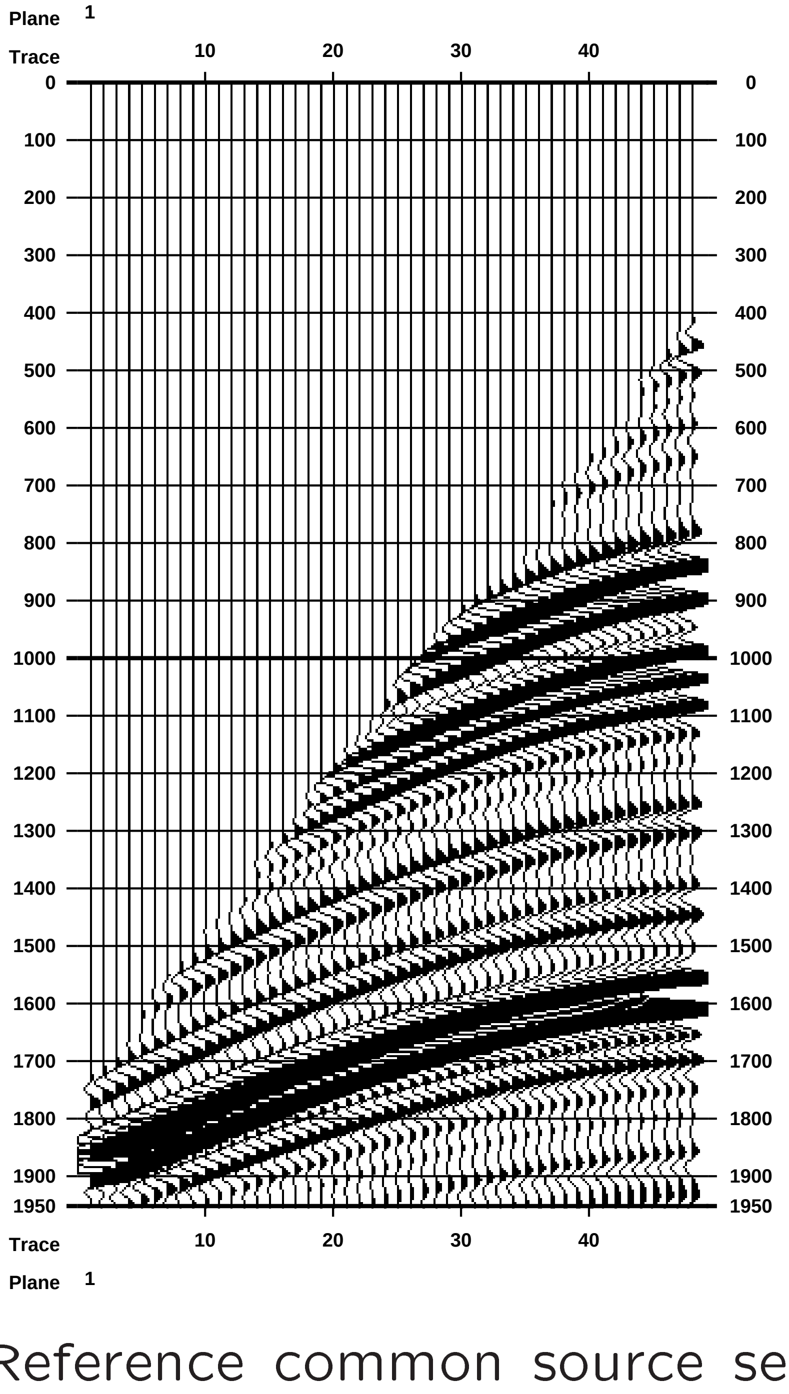
Velocity (function of depth only) used in numerical linearization study



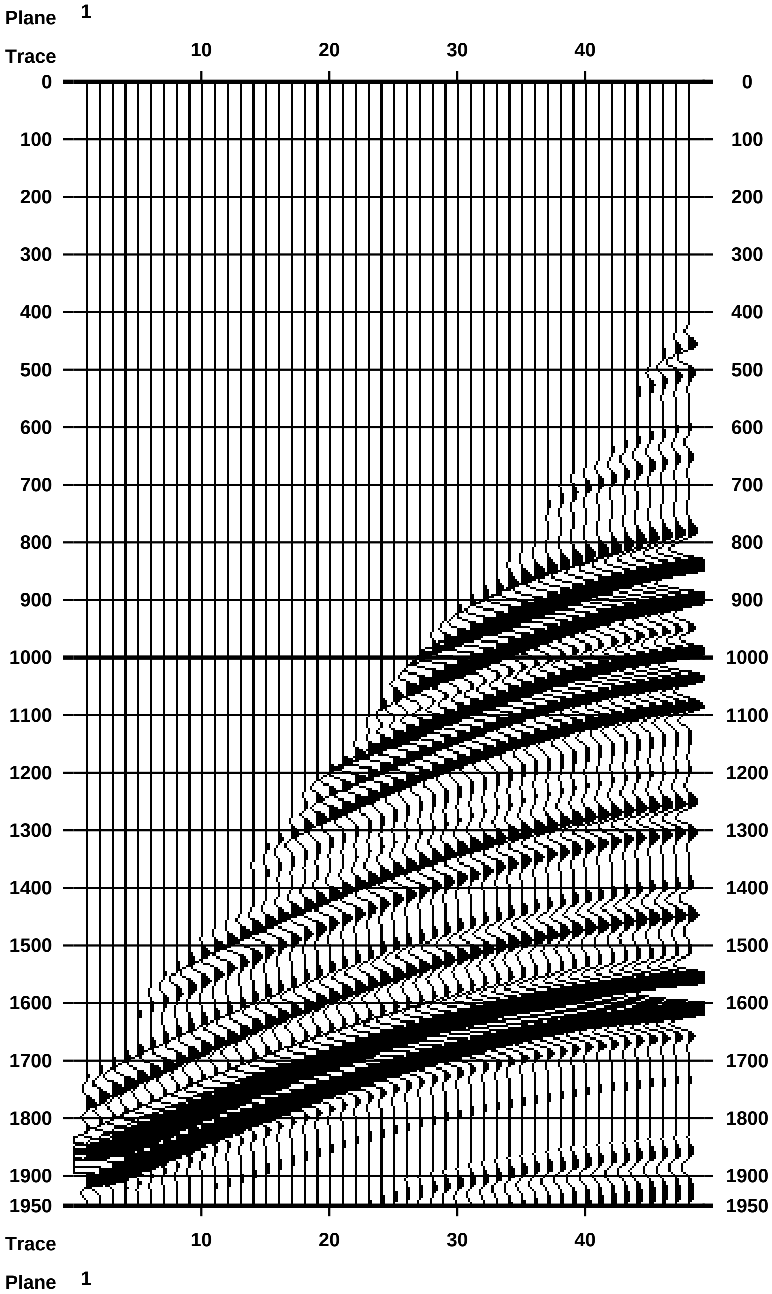
Relative velocity fluctuations ($\delta c/c$) used in numerical linearization study



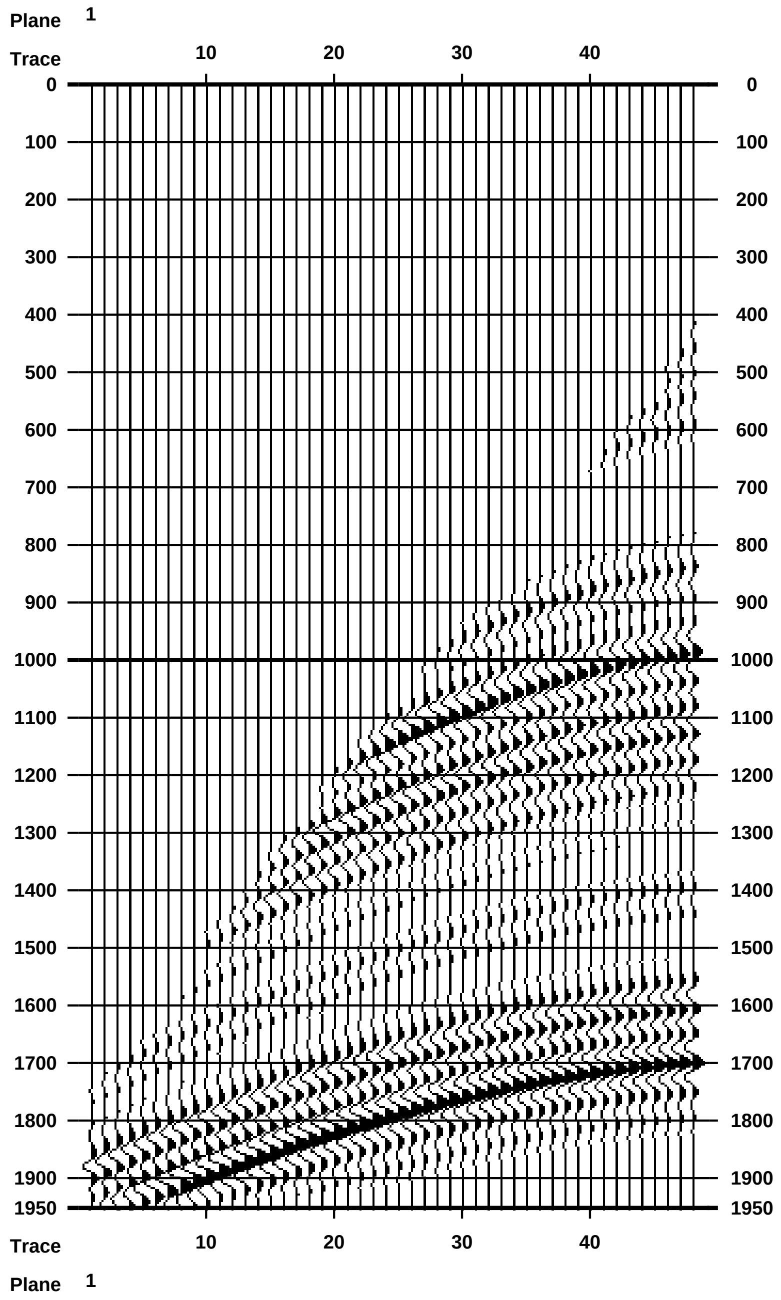
Source pulse $w(t)$ used in numerical linearization study



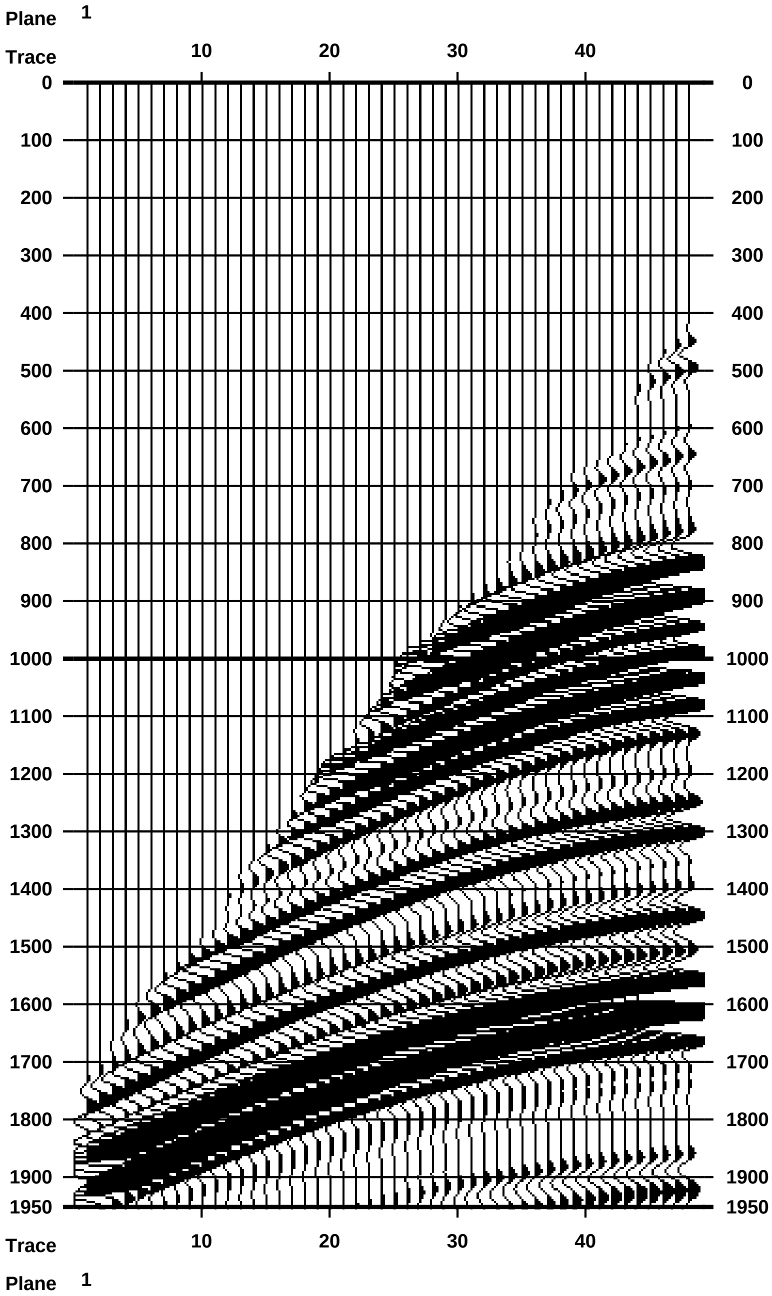
Reference common source seismogram $S[c + \delta c]$, after mute - offsets 150-1800 m 25 m.



Linearized seismogram $S[c] + DS[c]\delta c]$ muted in the same way.



Error seismogram $S[c+\delta c]-S[c]-S[c]\delta c]$ - note multiple reflections.



Effect of smooth perturbation: $S[1.02c+\delta c]-S[c+\delta c]-0.02DS[c+\delta c]c$ plotted on same scale.

Implications:

- Some geologies have well-separated scales - cf. sonic logs - and methods based on linearization will work well there. Other geologies do not - expect trouble!

- v smooth, r oscillatory $\Rightarrow F[v]r$ approximates *primary reflection* = result of wave interacting with material heterogeneity only once (single scattering); error consists of *multiple reflections*, which are "not too large" if r is "not too big".

- v nonsmooth, r smooth $\Rightarrow$ error consists of *time shifts* in waves which are very large perturbations as waves are oscillatory.

No mathematical results are known which justify/explain these observations in any rigorous way.

Partially linearized inverse problem = *velocity analysis problem*: given S^{obs} find v, r so that

$$S[v] + F[v]r \simeq S^{\text{obs}}$$

Linear subproblem = *imaging problem*: given S^{obs} and v, find r so that

$$F[v]r \simeq S^{\text{obs}} - S[v]$$

Last 20 years: much progress on imaging problem, much less on velocity analysis problem.

Importance of *high frequency asymptotics*: when linearization is accurate, properties of $F[v]$ dominated by those of $F_\delta[v]$ ($= F[v]$ with $w = \delta$) - Cohen & Bleistein, SIAM JAM 1977.

Critical fact about imaging problem: $F[v]$ is (essentially) a *Fourier Integral Operator* ("FIO": Rakesh, *Comm. PDE* 1988) - oscillatory integral operators with well defined singularity-mapping properties = systematic account of high frequency asymptotics of waves (and much else!), introduced by Hörmander ca. 1970. . From this fact follows

- geometry of $F[v]$ relating reflectors to reflections
- construction of approximate left inverses when these exist
- explanation of sense in which *migration* methods of seismic data processing provide approximate solutions to the imaging problem
- insight into problems posed by *strong refraction*, and eventually solutions of these

Reflectors (rapid changes in r) emulate *singularities*.

Harmonic analysis of singularities, *aprés* Hörmander: *Wave Front Set* $WF(u) \subset T^*(\mathbf{R}^n)$ of $u \in \mathcal{D}'(\mathbf{R}^n)$ is set of locations, directions in which u is not smooth -

> $(\mathbf{x}_0, \xi_0) \in WF(r) \;\Leftrightarrow\; \mathcal{F}(\phi r)(\tau\xi)$ is not rapidly decreasing as $\tau \to \infty$ for for any $\phi \in C^\infty(\mathbf{R}^n), \phi(\mathbf{x}_0) \neq 0$ and some $\xi/|\xi|$ in any nbhd of $\xi_0/|\xi_0|$

X, Y open in $\mathbf{R}^n, \mathbf{R}^m$, $F : \mathcal{E}'(X) \to \mathcal{D}'(Y)$ an FIO. To F belongs a *canonical relation* $C_F \subset T^*(X) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$, which describes the singularity mapping properties of F:

$$(\mathbf{y}, \eta) \in WF(Fu) \Rightarrow \text{for some}\,(\mathbf{x}, \xi) \in WF(u),$$

$$(\mathbf{x}, \xi, \mathbf{y}, \eta) \in C_F$$

The canonical relation of $F[v]$ (Rakesh 1988, Nolan 1997):

$Y = \{\mathbf{x}_s, t, \mathbf{x}_r\}$ (time $\times$ set of source-receiver pairs) submfd of $\mathbf{R}^7$ of dim. ≤ 5, $\Pi : T^*(\mathbf{R}^7) \rightarrow T^*Y$ the natural projection

$\mathrm{supp}\, r \subset X \subset \mathbf{R}^3$

$$((\mathbf{x}, \xi), (\mathbf{x}_s, t, \mathbf{x}_r, \xi_\mathbf{s}, \tau, \xi_\mathbf{r})) \in C_{F[v]}$$

$$\subset T^*(X) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$$

$$\Leftrightarrow$$

there are *rays of geometric optics* $(\mathbf{X}_s, \Xi_s)$, $(\mathbf{X}_r, \Xi_r)$ and times t_s, t_r so that

$$\Pi(\mathbf{X}_s(0), 0, \mathbf{X}_r(0), \Xi_s(0), \tau, \Xi_r(0)) =$$

$$(\mathbf{x}_s, 0, \mathbf{x}_r, \xi_s, \tau, \xi_r),$$

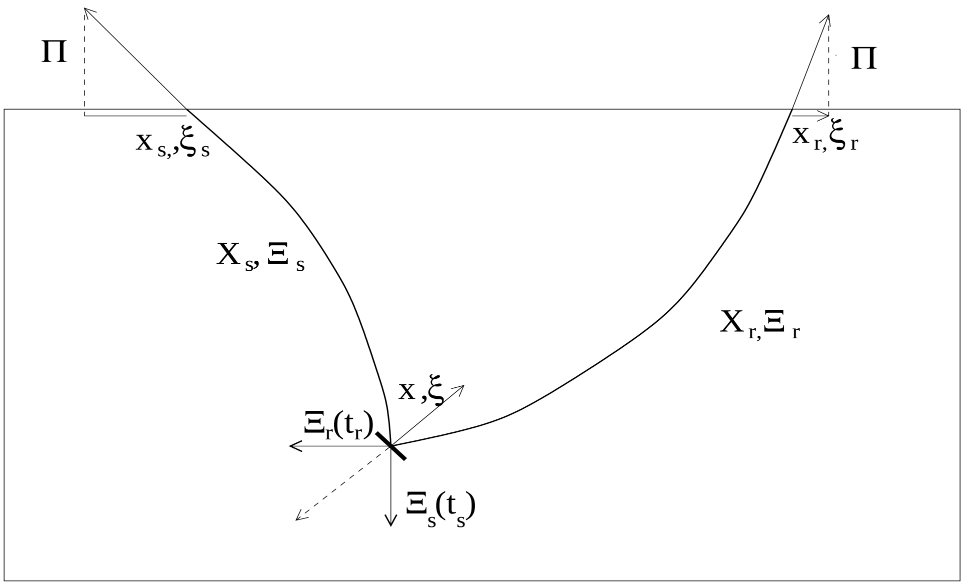
$\mathbf{X}_s(t_s) = \mathbf{X}_r(t_r) = \mathbf{x}$, $t_s + t_r = t$, $\Xi_s(t_s) + \Xi_r(t_r) || \xi$

Rays of geometric optics: solutions of Hamiltonian system

$$\frac{d\mathbf{X}}{dt} = \nabla_{\Xi} H(\mathbf{X}, \Xi), \ \frac{d\Xi}{dt} = -\nabla_{\mathbf{X}} H(\mathbf{X}, \Xi)$$

with $H(\mathbf{X}, \Xi) = 1 - v^2(\mathbf{X})|\Xi|^2 = 0$ (*null bicharacteristics*).

Since $\Xi_s(t_s)$, $\Xi_r(t_r)$ have same length, sum = bisector $\Rightarrow$ *incident ray from source* $\mathbf{X}_s, \Xi_s$ *and reflected ray to receiver* $\mathbf{X}_r, \Xi_r$ *make equal angles with reflector at* $\mathbf{x}$ *with normal* ξ.



Upshot: canonical relation of $F[v]$ simply enforces the equal-angles law of reflection.

These ideas lead to a usable definition of *image*: a reflectivity model r^* is an image of r if $WF(r^*) \subset WF(r)$ (the closer to equality, the better the image).

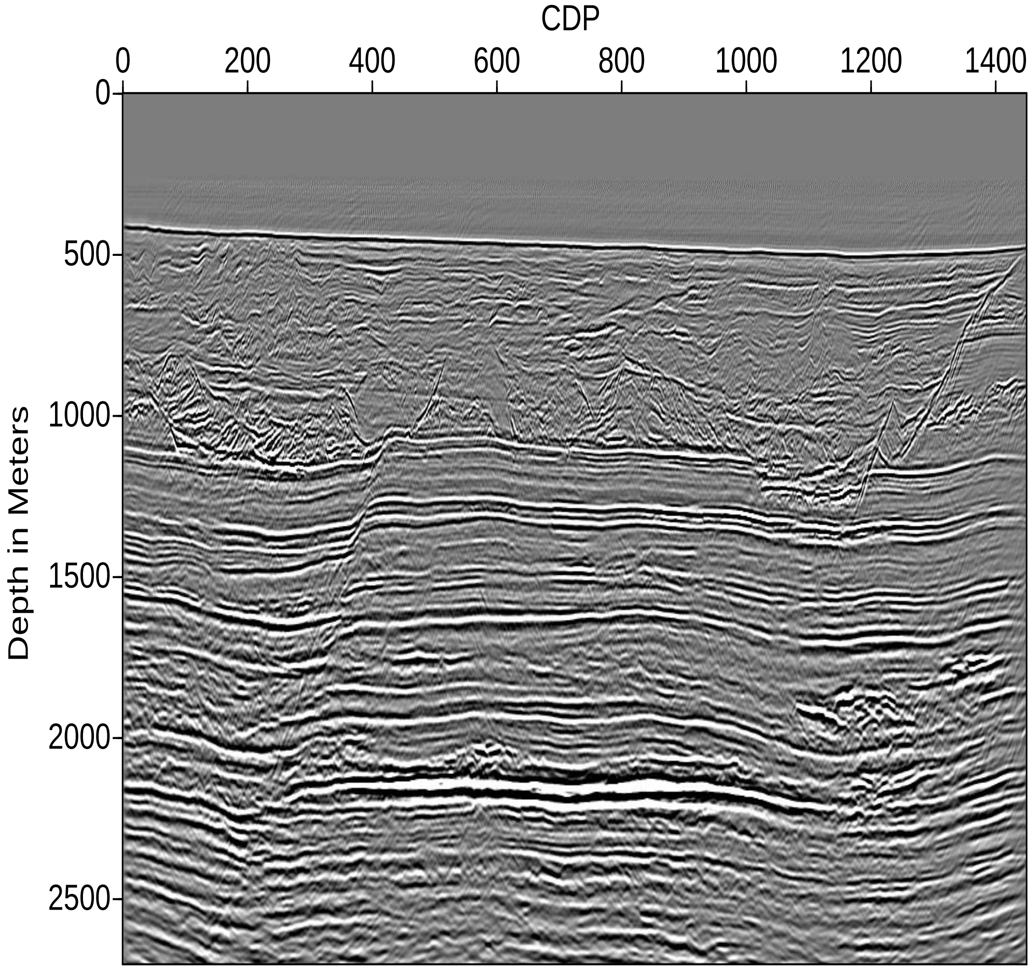
Beylkin, JMP 1985: suppose that each $\mathbf{x} \in X$ is connected to each source or receiver point $(\mathbf{x}_s, \mathbf{x}_r)$ by a unique ray. Then

- $WF(F[v]^* F[v] r) \subset WF(r)$
- if $S^{\mathrm{obs}} = S[v] + F[v]r$ (data consistent with linearized model), then $F[v]^*(S^{\mathrm{obs}} - S[v])$ is an image of r
- an operator $F[v]^\dagger$ exists for which $F[v]^\dagger(S^{\mathrm{obs}} - S[v]) - r$ is *smoother* than r, under some constraints on r - an *inverse modulo smoothing operators* or *parametrix*.

- both $F[v]^*$ and $F[v]^\dagger$ can be implemented computationally via efficient *Generalized Radon Transforms*.
- $F[v]^*$ and $F[v]^\dagger$ are variants of *Kirchhoff migration operators*, introduced into data processing in the 1970's.

Essential point: under "single ray" hypothesis, $F[v]^*F[v]$ is a *pseudodifferential operator ("ΨDO")* = FIO with C = identity. Calculus of ΨDOs contains parametrices for *elliptic* operators, i.e. ΨDOs P for which $WF(Pu) = WF(u)$. $F[v]^*F[v]$ is elliptic on part of T^*X and parametrix construction can be "microlocalized" there.

Many further developments by Beylkin, De-Hoop, Bleistein, Burridge, and others ⇒ production processing tools.

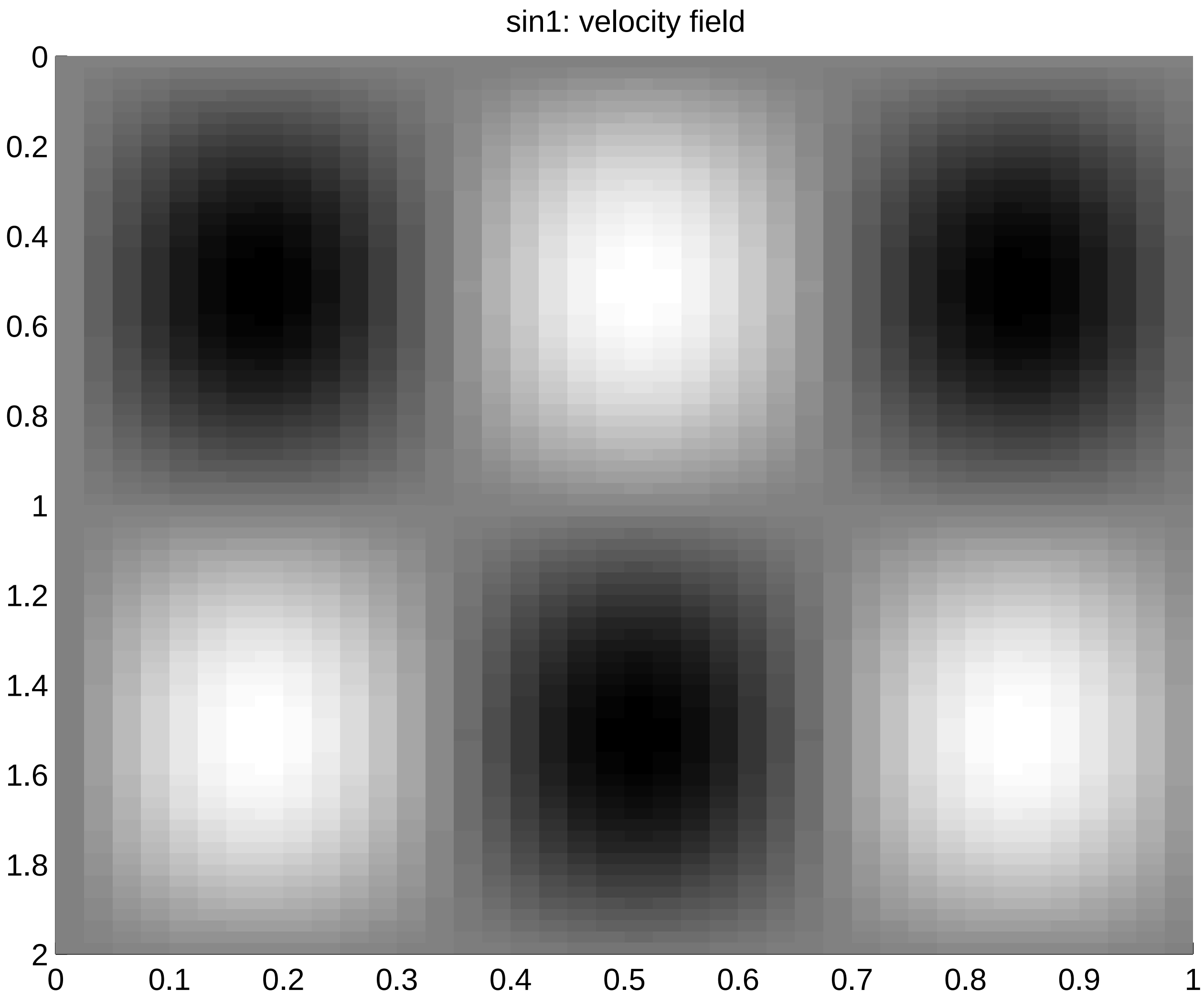


Example of GRT Inversion (application of $F[v]^{\dagger}$): K. Araya (1995), "2.5D" inversion of marine streamer data from Gulf of Mexico: 500 source positions, 120 receiver channels, 750 Mb.

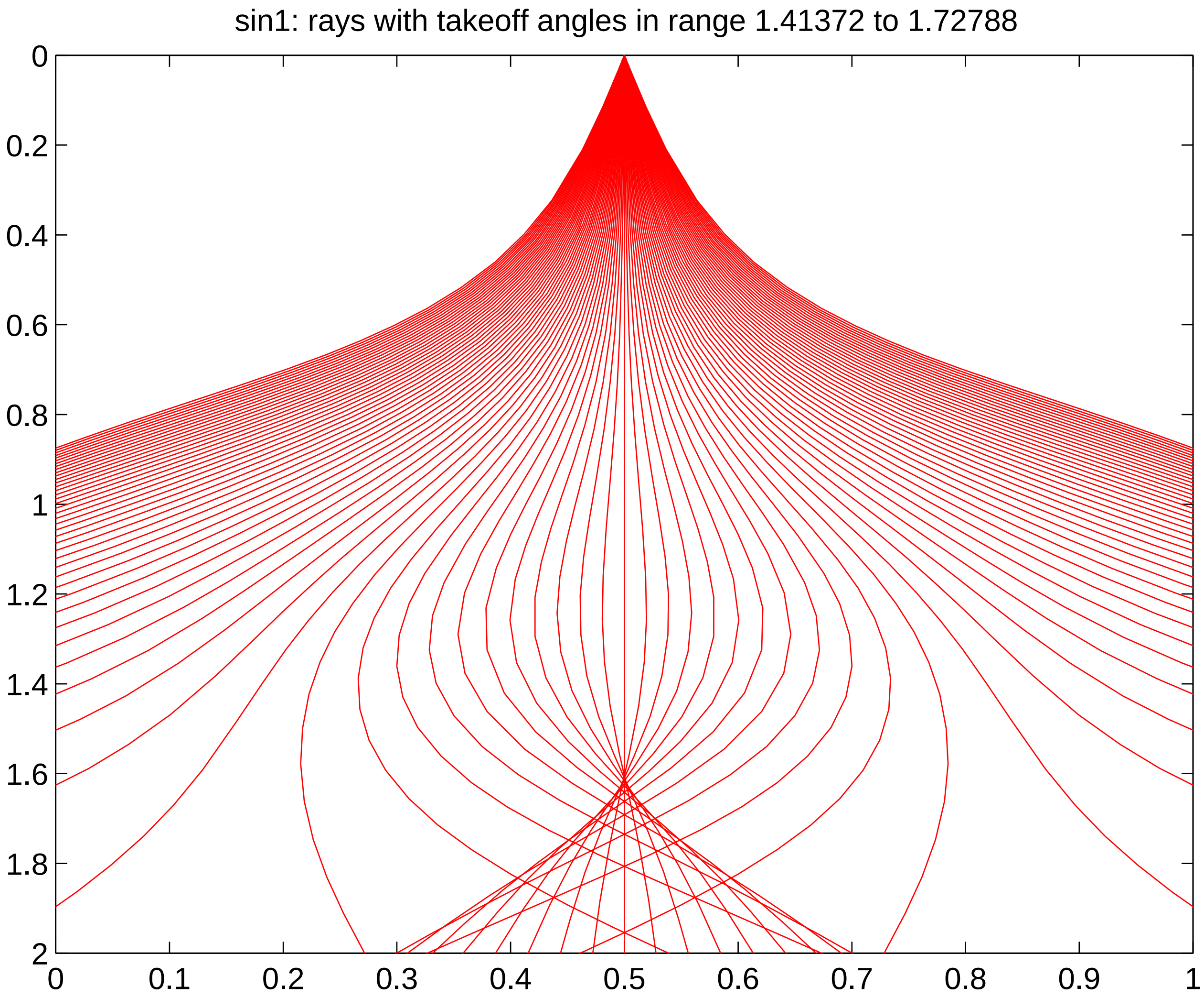
Not the end of the story:

- The "single ray" hypotheses generally fails in the presence of strong refraction.
- B. White, "The Stochastic Caustic" (1982): For "random but smooth" $v(\mathbf{x})$ with variance σ, points at distance $O(\sigma^{-2/3})$ from source have more than one ray connecting to source, with high probability 1 - *multipathing* associated with formation of *caustics* = ray envelopes.
- Formation of caustics invalidates asymptotic analysis on which Beylkin result is based.
- Strong refraction leading to multipathing and caustic formation typical of salt (4-5 km/s) intrusion into sedimentary rock (2-3 km/s) (eg. Gulf of Mexico), also chalk tectonics in North Sea and elsewhere - some of the most promising petroleum provinces!

NB: Rakesh's result accomodates strong refraction - $F[v]$ still FIO!



2D Example of strong refraction: Sinusoidal velocity field $v(x,z) = 1 + 0.2\sin\frac{\pi z}{2}\sin 3\pi x$



Rays in sinusoidal velocity field, source point = origin. Note formation of caustic, multiple rays to source point in lower center.

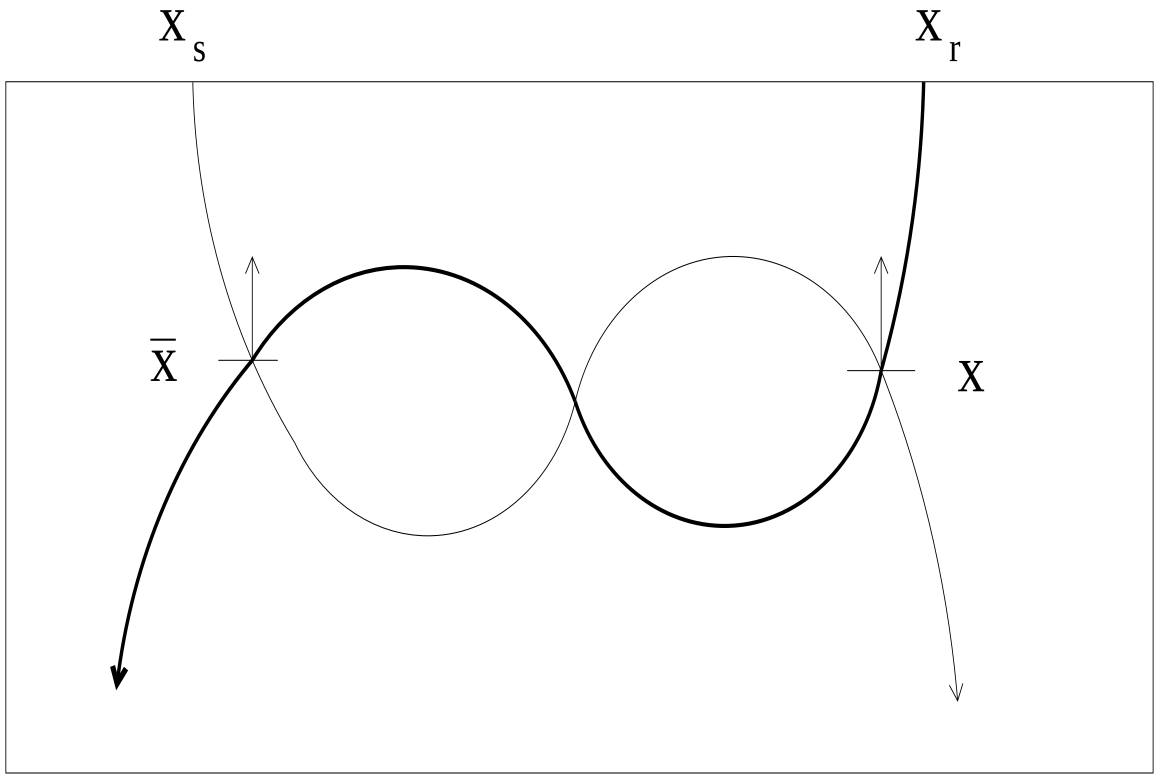
Smit, tenKroode and Verdel (1998): provided that

- source, receiver positions $(\mathbf{x}_s, \mathbf{x}_r)$ form an *open* 4D manifold ("complete coverage" - all source, receiver positions at least locally), and
- the *Traveltime Injectivity Condition* ("TIC") holds: $C_{F[v]}^{-1} \subset T^*Y - \{0\} \times T^*X - \{0\}$ is a *function* - that is, initial data for source and receiver rays and total travel time together determine reflector uniquely.

then $F[v]^*F[v]$ is ΨDO $\Rightarrow$ application of $F[v]^*$ produces image, and $F[v]^*F[v]$ has microlocal parametrix ("asymptotic inversion").

NB: application of $F[v]^*$ involves accounting for *all* rays connecting source and receiver with reflectors. Standard practice still attempts imaging with single choice of ray pair (shortest time, max energy,...). Operto et al (2000) give nice illustration that all rays must be included.

TIC is nontrivial constraint:



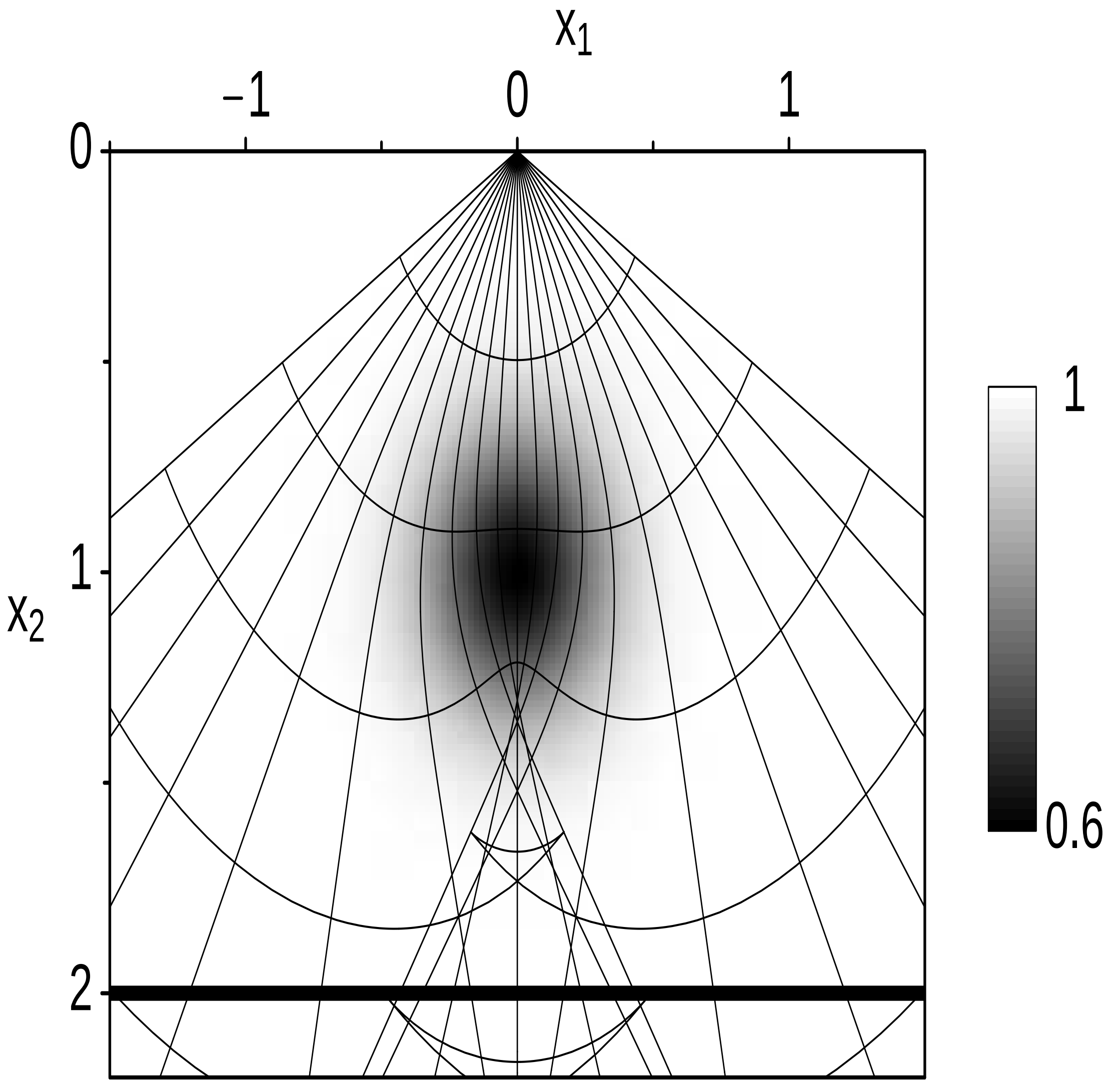
Symmetric waveguide: time $(\mathbf{x}_s \rightarrow \bar{\mathbf{x}} \rightarrow \mathbf{x}_r)$ same as time $(\mathbf{x}_s \rightarrow \mathbf{x} \rightarrow \mathbf{x}_r)$, so TIC fails.

Stolk (2000): under "complete coverage" hypothesis, v for which $F[v]^*F[v]$ is $=$ [ΨDO $+$ rel. smoothing op] form open, dense set.

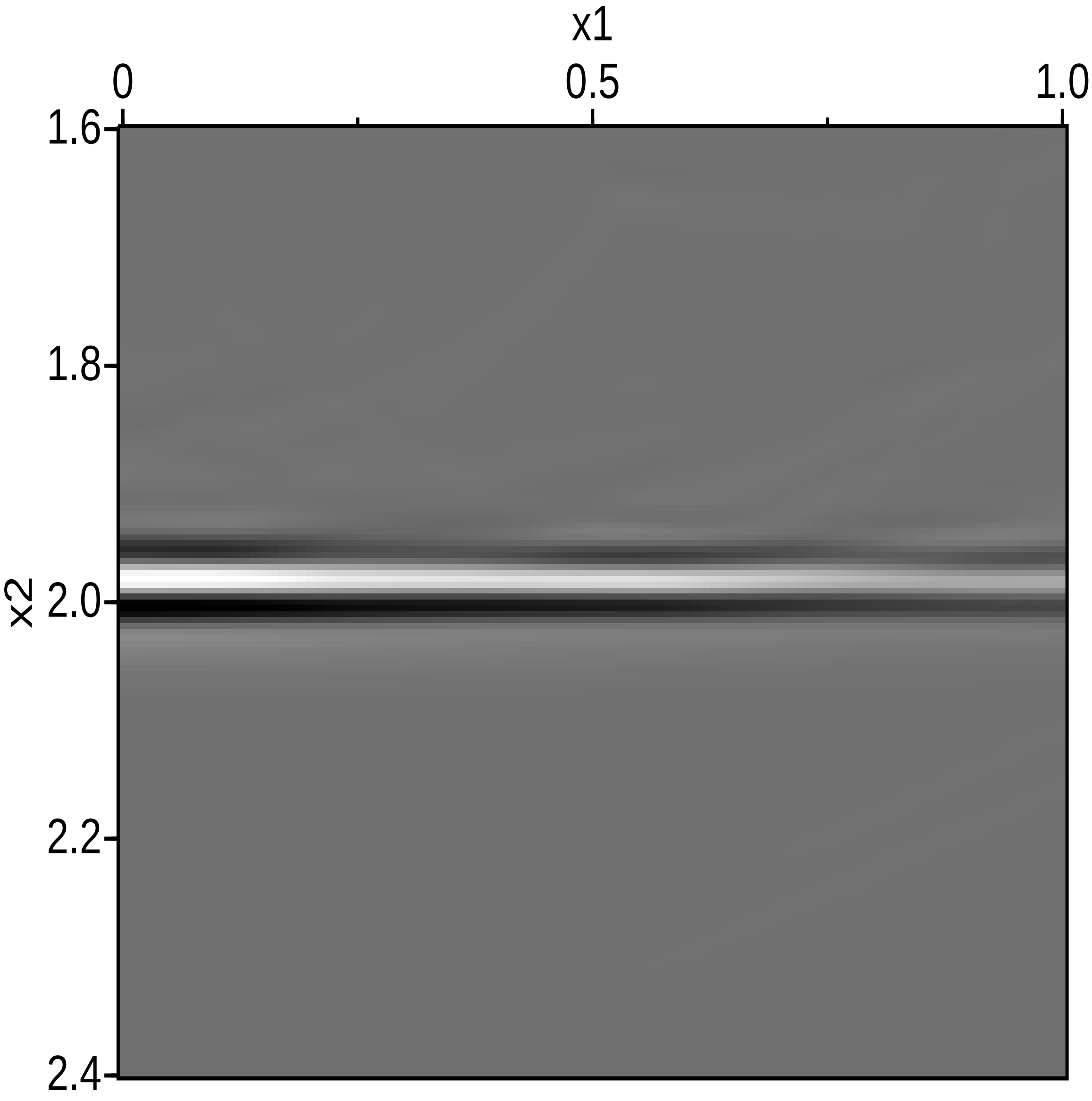
Limitation of Smit-tenKroode-Verdel: most idealized data acquisition geometries violate "complete coverage": for example, idealized marine streamer geometry (src-recvr submfd is 3D)

Nolan (1997): result remains true without "complete coverage" condition: requires only TIC plus addl condition so that projection $C_{F[v]} \to T^*Y$ is embedding - but examples violating TIC are much easier to construct when source-receiver submfd has positive codim.

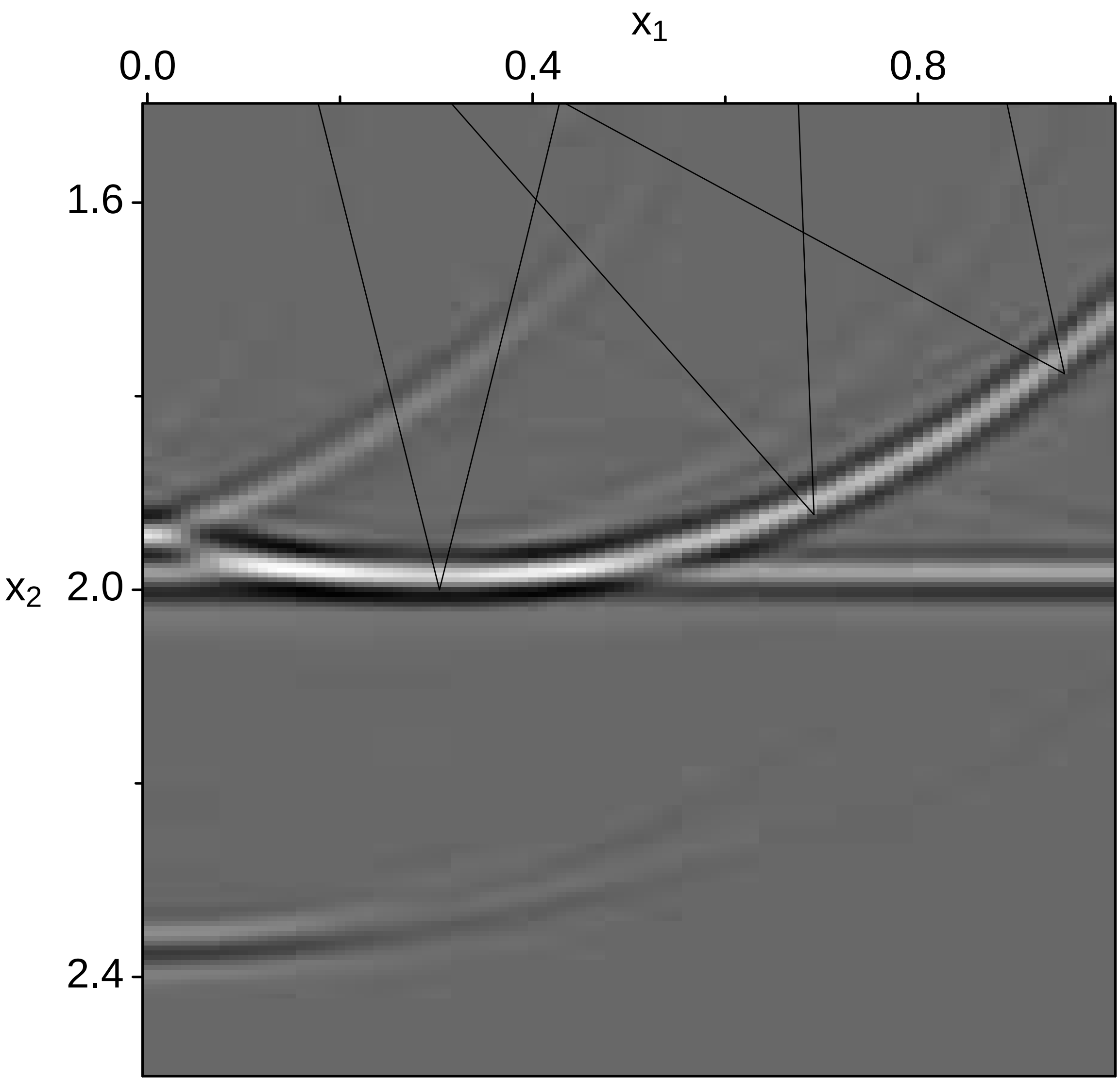
Synthetic 2D Example (Stolk and WWS, 2001): Strongly refracting acoustic lens (v) over horizontal reflector (r), $S^{\text{obs}} = F[v]r$. (i) for open source-receiver set, $F[v]^*S^{\text{obs}} =$ good image of reflector - within limits of finite frequency implied by numerical method, $F[v]^*F[v]$ acts like ΨDO; (ii) for *common offset* submfd (codim 1), TIC is violated and $WF(F[v]^*S^{\text{obs}})$ is larger than $WF(r)$.



Synthetic 2D Example (Stolk and WWS, 2001): grey scale plots of v (grey blob), r (horizontal bar), and rays from a source point overlain



Result of applying $F[v]^*$ to full data set - source-receiver pairs sample open set, TIC is satisfied, good image obtained.



Result of applying $F[v]^*$ to offset gather = all data with same offset $h = |\mathbf{x}_r - \mathbf{x}_s|$. Image overlain with ray pairs sharing same first factor in $C_{F[v]}^{-1}$ (TIC is violated).

Actual marine streamers: source-receiver positions sample *thin nbhd* of positive-codimension submfd ("narrow azimuth"), behaviour much closer to positive-codim theory (Nolan).

Positive-codim theory important in understanding construction of *covers* of $F[v]$, ie. manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \to \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Y)$ so that

$$\begin{array}{ccccc} & & \bar{F}[v] & & \\ & \mathcal{E}'(\bar{X}) & \rightarrow & \mathcal{D}'(Y) & \\ \chi & \uparrow & & \uparrow & \text{id} \\ & \mathcal{E}'(X) & \rightarrow & \mathcal{D}'(Y) & \\ & & F[v] & & \end{array}$$

commutes.

Covers express the redundancy in seismic surveys and play an important role in velocity analysis.

Recent development: discovery by Stolk and deHoop (2001) of cover construction for which $\bar{F}[v]\bar{F}[v]^*$ is ΨDO (note: $F[v]F[v]^*$ is *almost never* a ΨDO!).

Open problems and their practical meaning:

- understand structure of partially linearized ("velocity analysis") problem - principal bottleneck in data processing, better techniques needed for image formation in presence of strong refraction - global, nonlinear inverse problem.

- understand linearization error - better insight into limitations of linearization-based data processing

- devise approaches into full, nonlinear inverse problem when linearization is not accurate (mixed scales)

Key references:

Cohen, J. K. and Bleistein, N.: An inverse method for determining small variations in propagation speed, *SIAM J. Appl. Math. 32*, 1977, pp. 784-799.

Rakesh: A linearized inverse problem for the wave equation, *Comm. PDE 13*, 1988, pp. 573-601.

Beylkin, G.: Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform, *J. Math. Phys 26*, 1985, pp. 99-108.

Ten Kroode, A. P. E., Smit, D. J., and Verdel, A. R.: A microlocal analysis of migration *Wave Motion 28*, 1998, pp. 149-172.

Nolan, C. J. and Symes, W. W.: Global solution of a linearized inverse problem for the wave equation, *Comm. PDE 22*, 1997, pp. 919-952.

Stolk, C.: Microlocal analysis of a seismic linearized inverse problem, *Wave Motion 32*, 2000, pp. 267-290.