
Mathematics of Seismic Imaging

Part IV

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Geometric optics, Rakesh's construction, and
imaging and inversion in the presence of
multipathing.

Why Beylkin isn't enough

The theory developed by Beylkin and others cannot be the end of the story:

- The “single ray” hypotheses generally fails in the presence of strong refraction.
- B. White, “The Stochastic Caustic” (1982): For “random but smooth” $v(\mathbf{x})$ with variance σ , points at distance $O(\sigma^{-2/3})$ from source have more than one ray connecting to source, with probability 1 - *multipathing* associated with formation of *caustics* = ray envelopes.
- Formation of caustics invalidates asymptotic analysis on which Beylkin result is based.

Why it matters

- Strong refraction leading to multipathing and caustic formation typical of salt (4-5 km/s) intrusion into sedimentary rock (2-3 km/s) (eg. Gulf of Mexico), also chalk tectonics in North Sea and elsewhere - some of the most promising petroleum provinces!

Escape from simplicity - the Canonical Relation

How do we get away from “simple geometric optics”, SSR, DSR,... - all violated in sufficiently complex (and realistic) models? Rakesh *Comm. PDE* 1988, Nolan *Comm. PDE* 1997: global description of $F_\delta[v]$ as mapping reflectors \mapsto reflections.

$Y = \{\mathbf{x}_s, t, \mathbf{x}_r\}$ (time \times set of source-receiver pairs) submfd of \mathbf{R}^7 of dim. ≤ 5 ,
 $\Pi : T^*(\mathbf{R}^7) \rightarrow T^*Y$ the natural projection

$\text{supp } r \subset X \subset \mathbf{R}^3$

Canonical relation $C_{F_\delta[v]} \subset T^*(X) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$ describes singularity mapping properties of F :

$$(\mathbf{x}, \xi, \mathbf{y}, \eta) \in C_{F_\delta[v]} \Leftrightarrow$$

for some $u \in \mathcal{E}'(X)$, $(\mathbf{x}, \xi) \in WF(u)$, and $(\mathbf{y}, \eta) \in WF(Fu)$

Rays Construction of the Relation

Rays of geometric optics: solutions of Hamiltonian system

$$\frac{d\mathbf{X}}{dt} = \nabla_{\Xi} H(\mathbf{X}, \Xi), \quad \frac{d\Xi}{dt} = -\nabla_{\mathbf{X}} H(\mathbf{X}, \Xi)$$

with $H(\mathbf{X}, \Xi) = 1 - v^2(\mathbf{X})|\Xi|^2 = 0$ (*null bicharacteristics*).

Characterization of C_F :

$$((\mathbf{x}, \xi), (\mathbf{x}_s, t, \mathbf{x}_r, \xi_s, \tau, \xi_r)) \in C_{F_\delta[v]} \subset T^*(X) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$$

\Leftrightarrow there are *rays of geometric optics* (\mathbf{X}_s, Ξ_s) , (\mathbf{X}_r, Ξ_r) and times t_s, t_r so that

$$\Pi(\mathbf{X}_s(0), t, \mathbf{X}_r(t), \Xi_s(0), \tau, \Xi_r(t)) = (\mathbf{x}_s, t, \mathbf{x}_r, \xi_s, \tau, \xi_r),$$

$$\mathbf{X}_s(t_s) = \mathbf{X}_r(t - t_r) = \mathbf{x}, \quad t_s + t_r = t, \quad \Xi_s(t_s) - \Xi_r(t - t_r) \parallel \xi$$

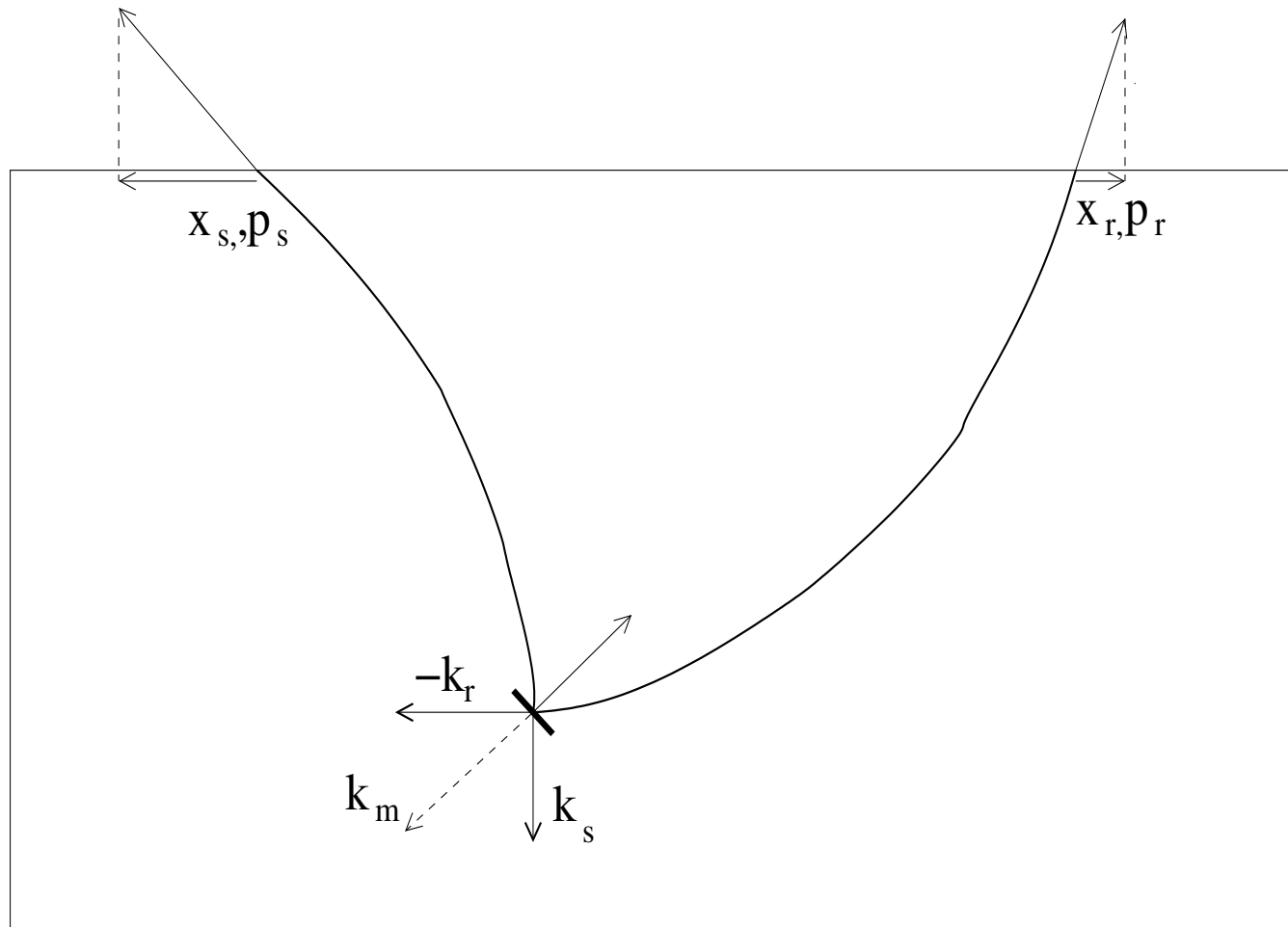
Since $\Xi_s(t_s)$, $-\Xi_r(t - t_r)$ have same length, sum = bisector \Rightarrow *velocity vectors of incident ray from source and reflected ray from receiver (traced backwards in time) make equal angles with reflector at \mathbf{x} with normal ξ .*

Upshot: canonical relation of $F_\delta[v]$ simply enforces the equal-angles law of reflection.

Further, *rays carry high-frequency energy*, in exactly the fashion that seismologists imagine.

Finally, *Rakesh's characterization of C_F is global*: no assumptions about ray geometry, other than no forward scattering and no grazing incidence on the acquisition surface Y , are needed.

The Picture



Proof: Plan of attack

Recall that

$$F[v]r(\mathbf{x}_r, t; \mathbf{x}_s) = \frac{\partial \delta u}{\partial t}(\mathbf{x}_r, t; \mathbf{x}_s)$$

where

$$\frac{1}{v^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla^2 \delta u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} r$$

$$\frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} - \nabla^2 u = \delta(t) \delta(\mathbf{x} - \mathbf{x}_s)$$

and $u, \delta u \equiv 0, t < 0$.

Need to understand (1) $WF(u)$, (2) relation $WF(r) \leftrightarrow WF(ru)$, (3) WF of soln of WE in terms of WF of RHS (this also gives (1)!).

Singularities of the Acoustic Potential Field

Main tool: **Propagation of Singularities** theorem of Hörmander (1970).

Given symbol $p(\mathbf{x}, \boldsymbol{\xi})$, order m , with asymptotic expansion, define *null bicharacteristics* (= rays) as solutions $(\mathbf{x}(t), \boldsymbol{\xi}(t))$ of Hamiltonian system

$$\frac{d\mathbf{x}}{dt} = \frac{\partial p}{\partial \boldsymbol{\xi}}(\mathbf{x}, \boldsymbol{\xi}), \quad \frac{d\boldsymbol{\xi}}{dt} = -\frac{\partial p}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\xi})$$

with $p(\mathbf{x}(t), \boldsymbol{\xi}(t)) \equiv 0$.

Theorem: Suppose $p(\mathbf{x}, D)u = f$, and suppose that for $t_0 \leq t \leq t_1$, $(\mathbf{x}(t), \boldsymbol{\xi}(t)) \notin WF(f)$. Then either $\{(\mathbf{x}(t), \boldsymbol{\xi}(t)) : t_0 \leq t \leq t_1\} \subset WF(u)$ or $\{(\mathbf{x}(t), \boldsymbol{\xi}(t)) : t_0 \leq t \leq t_1\} \subset T^*(\mathbf{R}^n) - WF(u)$.

Source to Field

RHS of wave equation for $u = \delta$ function in \mathbf{x}, t . WF set = $\{(\mathbf{x}, t, \boldsymbol{\xi}, \tau) : \mathbf{x} = \mathbf{x}_s, t = 0\}$ - i.e. no restriction on covector part.

$\Rightarrow (\mathbf{x}, t, \boldsymbol{\xi}, \tau) \in WF(u)$ iff a ray starting at $(\mathbf{x}_s, 0)$ passes over (\mathbf{x}, t) - i.e. (\mathbf{x}, t) lies on the “light cone” with vertex at $(\mathbf{x}_s, 0)$. Symbol for wave op is $p(\mathbf{x}, t, \boldsymbol{\xi}, \tau) = \frac{1}{2}(\tau^2 - v^2(\mathbf{x})|\boldsymbol{\xi}|^2)$, so Hamilton’s equations for null bicharacteristics are

$$\frac{d\mathbf{X}}{dt} = -v^2(\mathbf{X})\boldsymbol{\Xi}, \quad \frac{d\boldsymbol{\Xi}}{dt} = \nabla \log v(\mathbf{X})$$

Thus $\boldsymbol{\xi}$ is proportional to velocity vector of ray.

$[(\boldsymbol{\xi}, \tau)$ *normal* to light cone.]

Singularities of Products

To compute $WF(ru)$ from $WF(r)$ and $WF(u)$, use *Gabor calculus* (Duistermaat, Ch. 1)

Here r is really $(r \circ \pi)u$, where $\pi(\mathbf{x}, t) = \mathbf{x}$. Choose bump function ϕ localized near (\mathbf{x}, t)

$$\begin{aligned}\widehat{\phi(r \circ \pi)u}(\boldsymbol{\xi}, \tau) &= \int d\boldsymbol{\xi}' d\tau' \widehat{\phi r}(\boldsymbol{\xi}') \delta(\tau') \widehat{u}(\boldsymbol{\xi} - \boldsymbol{\xi}', \tau - \tau') \\ &= \int d\boldsymbol{\xi}' \widehat{\phi r}(\boldsymbol{\xi}') \widehat{u}(\boldsymbol{\xi} - \boldsymbol{\xi}', \tau)\end{aligned}$$

This will decay rapidly as $|(\boldsymbol{\xi}, \tau)| \rightarrow \infty$ unless (i) you can find $(\mathbf{x}', \boldsymbol{\xi}') \in WF(r)$ so that $\mathbf{x}, \mathbf{x}' \in \pi(\text{supp}\phi)$, $\boldsymbol{\xi} - \boldsymbol{\xi}' \in WF(u)$, i.e. $(\boldsymbol{\xi}, \tau) \in WF(r \circ \pi) + WF(u)$, or (ii) $\boldsymbol{\xi} \in WF(r)$ or $(\boldsymbol{\xi}, \tau) \in WF(u)$.

Possibility (ii) will not contribute, so effectively

$$WF((r \circ \pi)u) = \{(\mathbf{x}, t_s, \boldsymbol{\xi} + \boldsymbol{\Xi}_s(t_s), \cdot) : (\mathbf{x}, \boldsymbol{\xi}) \in WF(r), \mathbf{x} = \mathbf{X}_s(t_s)$$

for a ray $(\mathbf{X}_s, \boldsymbol{\Xi}_s)$ with $\mathbf{X}_s(0) = x_s$, some τ .

Wavefront set of Scattered Field

Once again use propagation of singularities: $(\mathbf{x}_r, t, \boldsymbol{\xi}_r, \tau_r) \in WF(\delta u) \Leftrightarrow$ on ray $(\mathbf{X}_r, \boldsymbol{\Xi}_r)$ passing through $WF(ru)$. Can argue that time of intersection is $t - t_r < t$.

That is,

$$\mathbf{X}_r(t) = \mathbf{x}_r, \mathbf{X}_r(t - t_r) = \mathbf{X}_s(t_s) = x,$$

$t = t_r + t_s$, and

$$\boldsymbol{\Xi}_r(t_s) = \boldsymbol{\xi} + \boldsymbol{\Xi}_s(t_s)$$

for some $\boldsymbol{\xi} \in WF(r)$. **Q. E. D.**

Rakesh's Thesis

Rakesh also showed that $F[v]$ is a *Fourier Integral Operator* = class of oscillatory integral operators, introduced by Hörmander and others in the '70s to describe the solutions of nonelliptic PDEs.

Phases and amplitudes of FIOs satisfy certain restrictive conditions. Canonical relations have geometric description similar to that of $F[v]$. Adjoint of FIO is FIO with inverse canonical relation.

Ψ DOs are special FIOs.

Composition of FIOs does *not* yield an FIO in general. Beylkin had shown that $F[v]^* F[v]$ is FIO (Ψ DO, actually) under simple ray geometry hypothesis - but this is only sufficient. Rakesh noted that this follows from general results of Hörmander: *simple ray geometry \Leftrightarrow canonical relation is graph of ext. deriv. of phase function.*

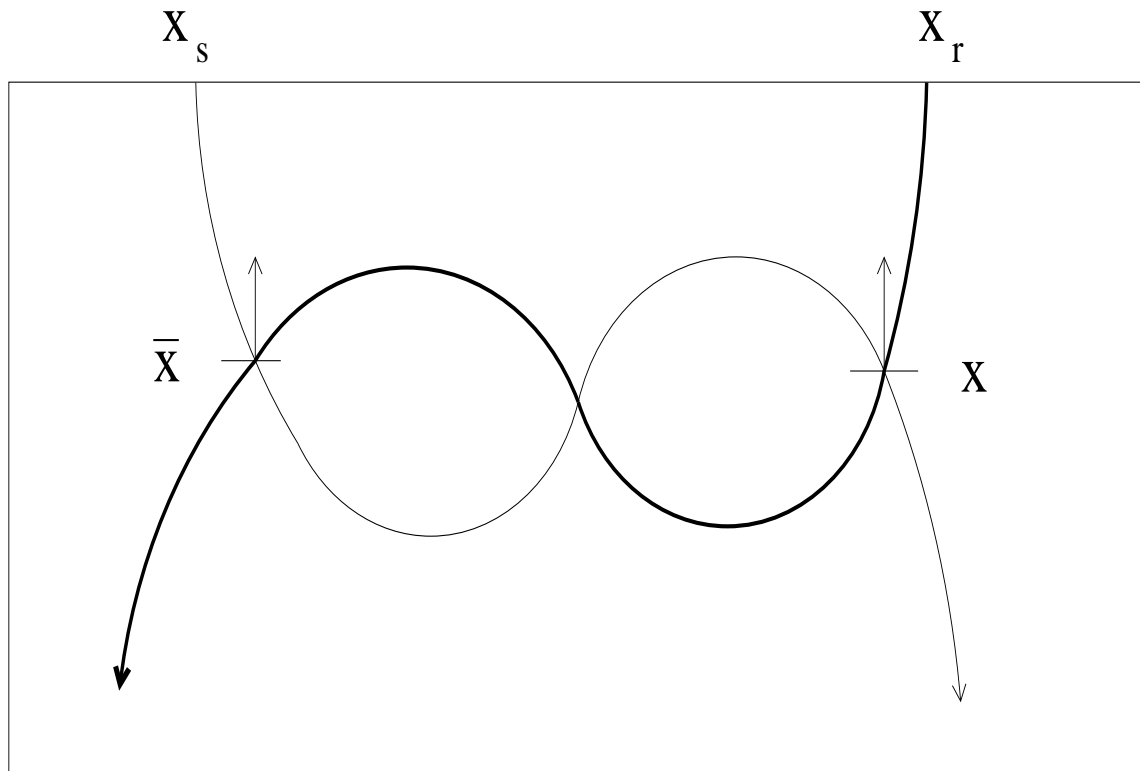
The Shell Guys and TIC

Smit, tenKroode and Verdel (1998): provided that

- source, receiver positions $(\mathbf{x}_s, \mathbf{x}_r)$ form an *open* 4D manifold (“complete coverage” - all source, receiver positions at least locally), and
- the *Traveltime Injectivity Condition* (“TIC”) holds: $C_{F[v]}^{-1} \subset T^*Y - \{0\} \times T^*X - \{0\}$ is a *function* - that is, initial data for source and receiver rays and total travel time together determine reflector uniquely.

then $F[v]^*F[v]$ is Ψ DO \Rightarrow application of $F[v]^*$ produces image, and $F[v]^*F[v]$ has microlocal parametrix (“asymptotic inversion”).

TIC is a nontrivial constraint!



Symmetric waveguide: time ($x_s \rightarrow \bar{x} \rightarrow x_r$) same as time ($x_s \rightarrow x \rightarrow x_r$), so TIC fails.

Stolk's Thesis

Stolk (2000): under “complete coverage” hypothesis, v for which $F[v]^*F[v]$ is $= [\Psi\text{DO} + \text{rel. smoothing op}]$ form open, dense set (without assuming TIC!).

NB: application of $F[v]^*$ involves accounting for *all* rays connecting source and receiver with reflectors. Standard practice still attempts imaging with single choice of ray pair (shortest time, max energy,...). Operto et al (2000) give nice illustration that all rays must be included.

Nolan's Thesis

Limitation of Smit-tenKroode-Verdel: most idealized data acquisition geometries violate “complete coverage”: for example, idealized marine streamer geometry (src-recvr submfd is 3D)

Nolan (1997): result remains true without “complete coverage” condition: requires only TIC plus addl condition so that projection $C_{F[v]} \rightarrow T^*Y$ is embedding - but examples violating TIC are much easier to construct when source-receiver submfd has positive codim.

Sinister Implication: When data is just a single gather - common shot, common offset - image may contain *artifacts*, i.e. spurious reflectors not present in model.

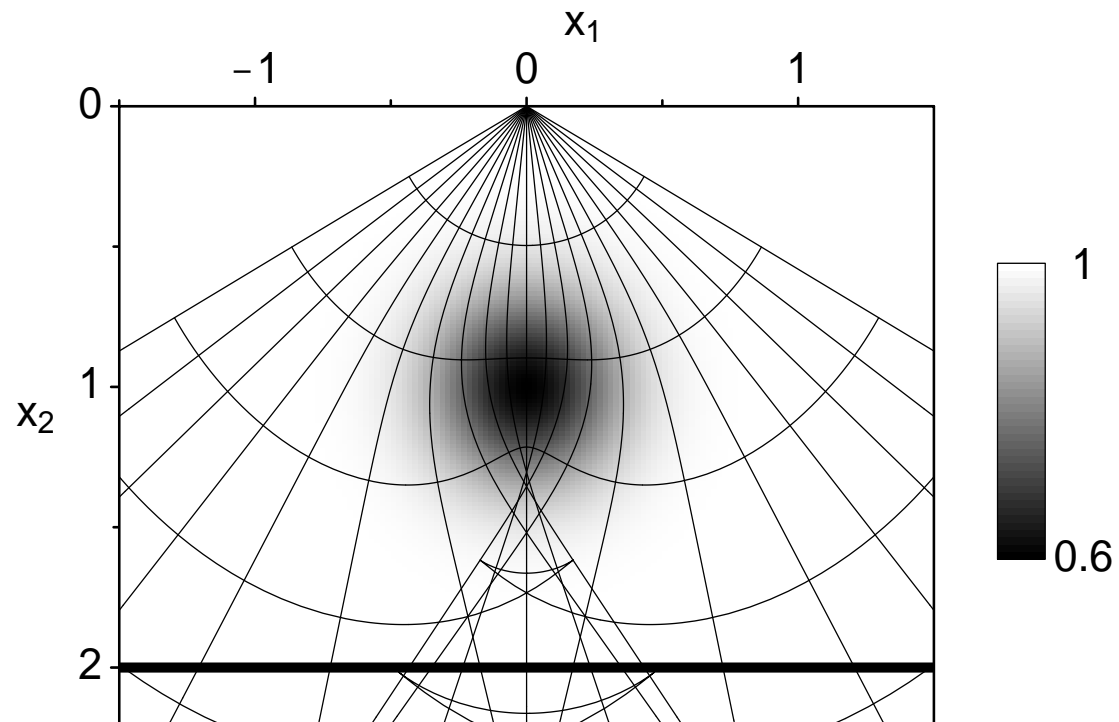
Horrible Example I

Synthetic 2D Example (Stolk and WWS, 2001 - *Geophysics* 2004)

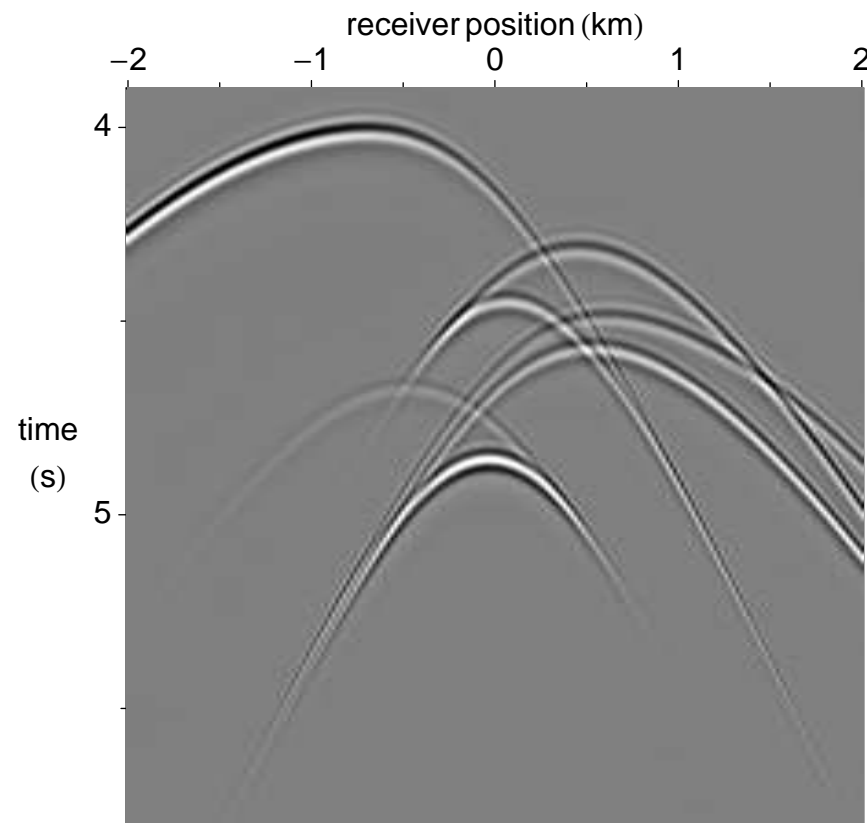
Strongly refracting acoustic lens (v) over horizontal reflector (r), $S^{\text{obs}} = F[v]r$.

(i) for open source-receiver set, $F[v]^* S^{\text{obs}} = \text{good image of reflector}$ - within limits of finite frequency implied by numerical method, $F[v]^* F[v]$ acts like ΨDO ;

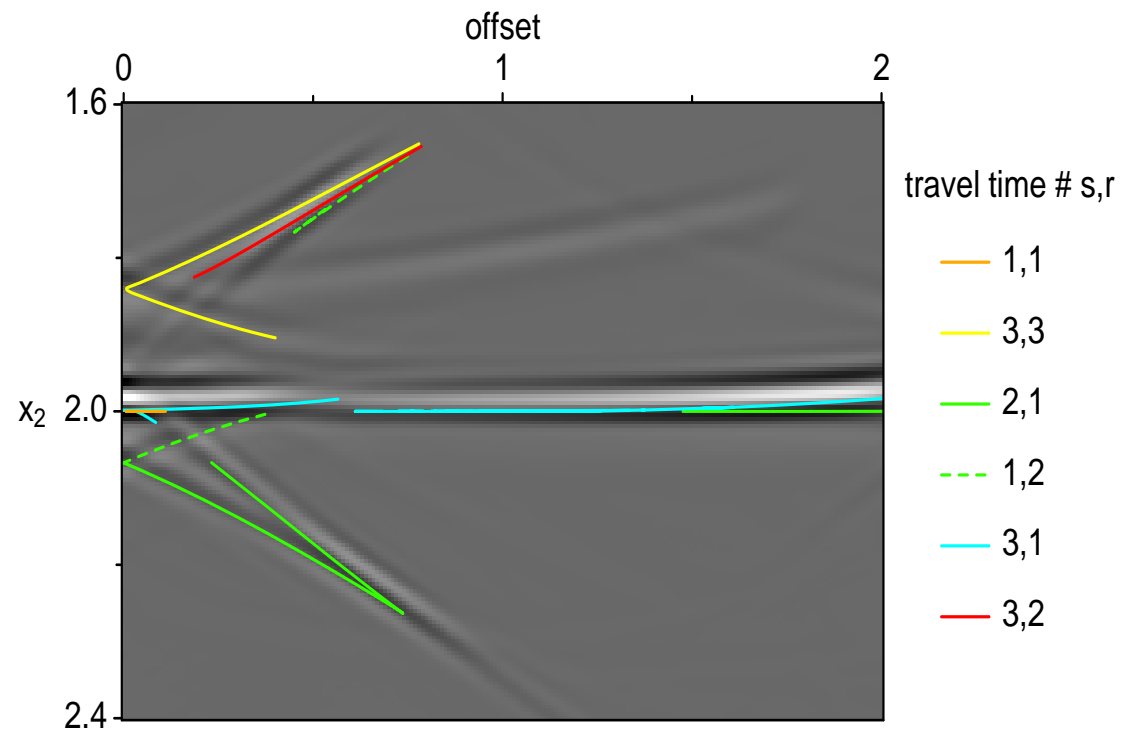
(ii) for *common offset* submfd (codim 1), TIC is violated and $WF(F[v]^* S^{\text{obs}})$ is larger than $WF(r)$.



Gaussian lens velocity model, flat reflector at depth 2 km, overlain with rays and wavefronts (Stolk & S. 2002 SEG).



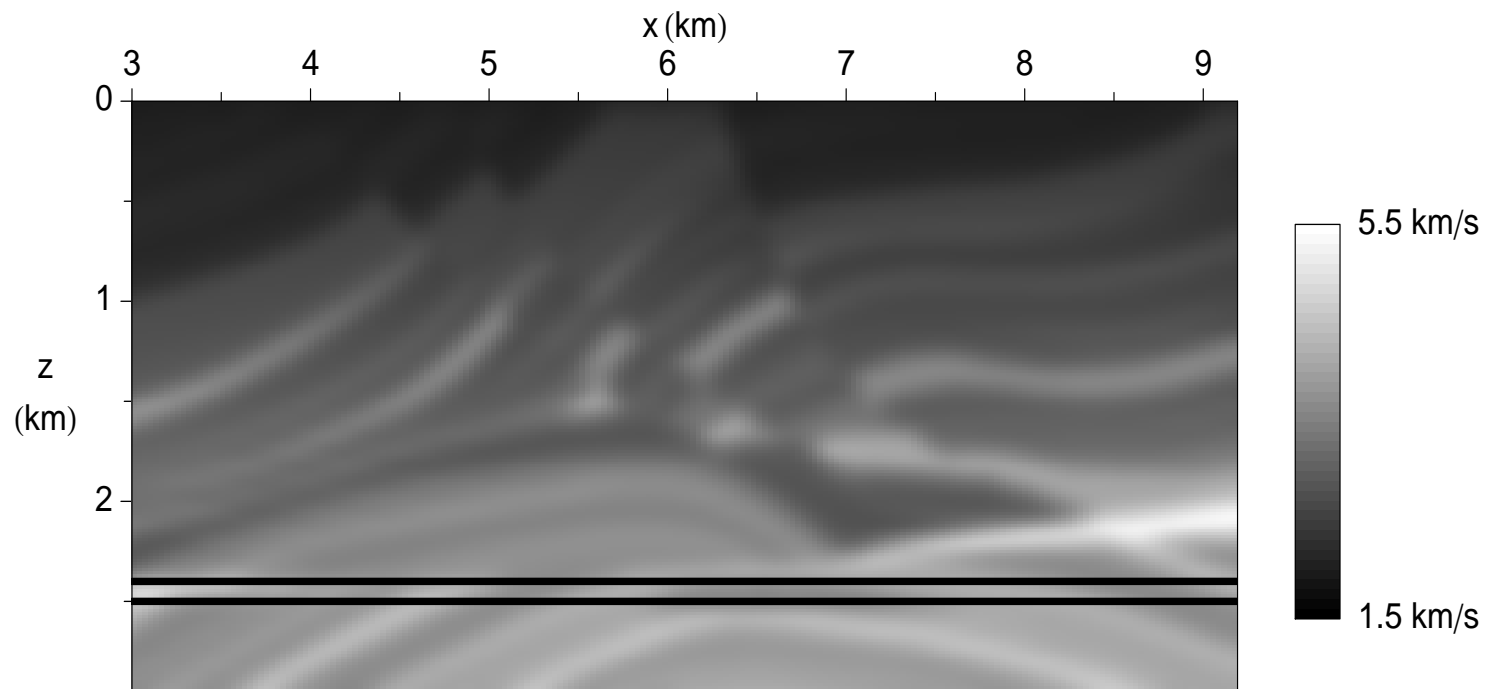
Typical shot gather - lots of arrivals

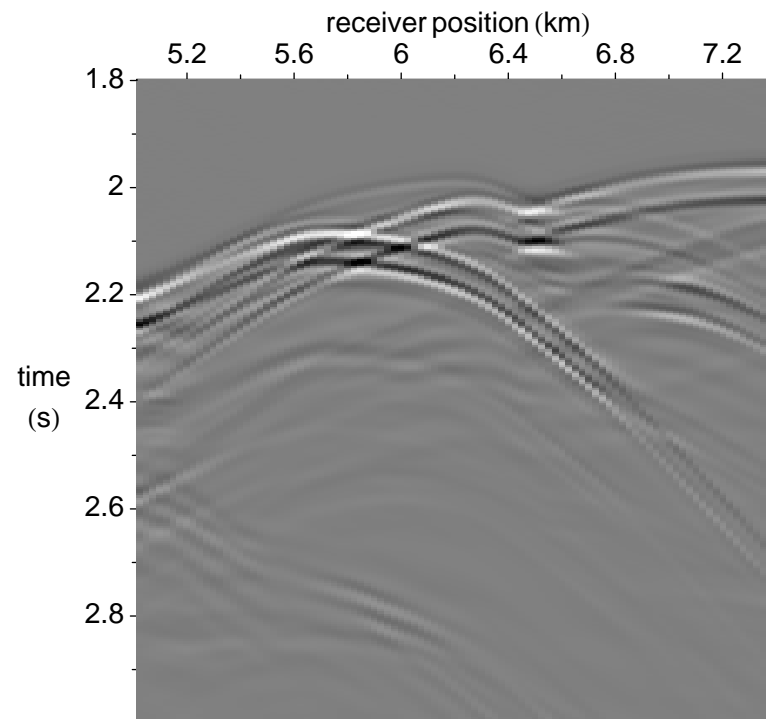


Offset common image gather (slice of $\tilde{F}[v]^*d$), with kinematically predicted reflector images overlain.

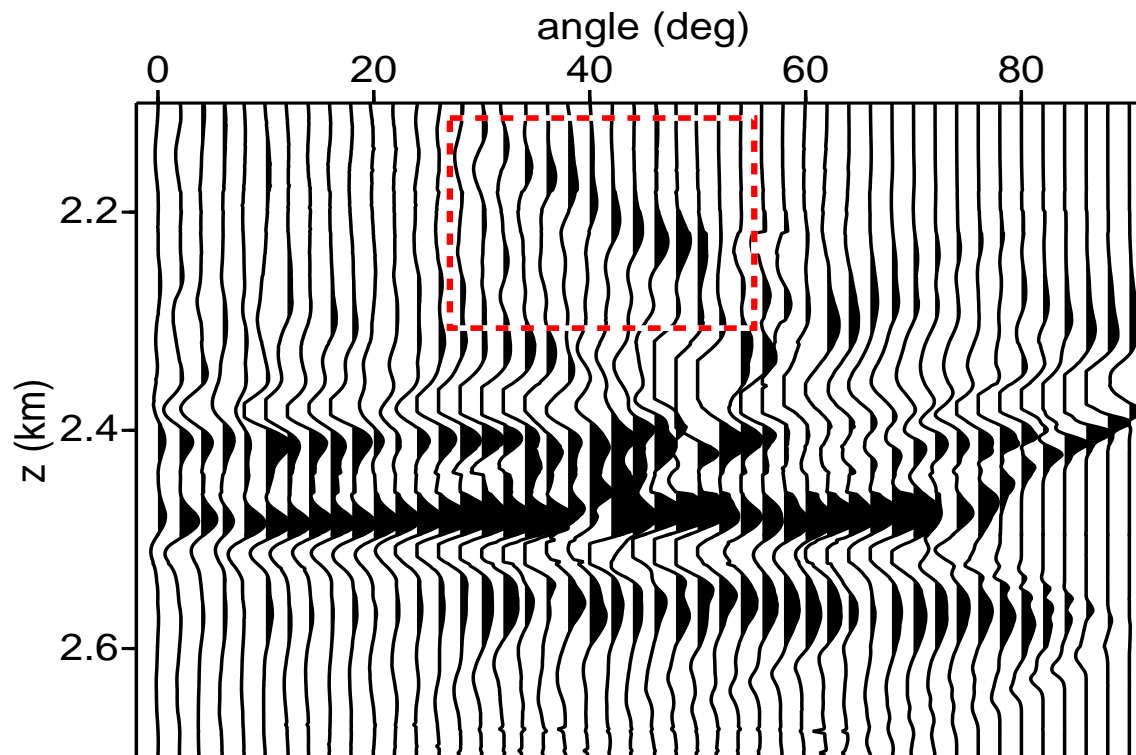
Horrible Example II

Stolk and Symes, *Geophysics* 2004: “Marmouflat” model = smoothed Marmousi (Versteeg & Grau 1991) with two flat reflectors.

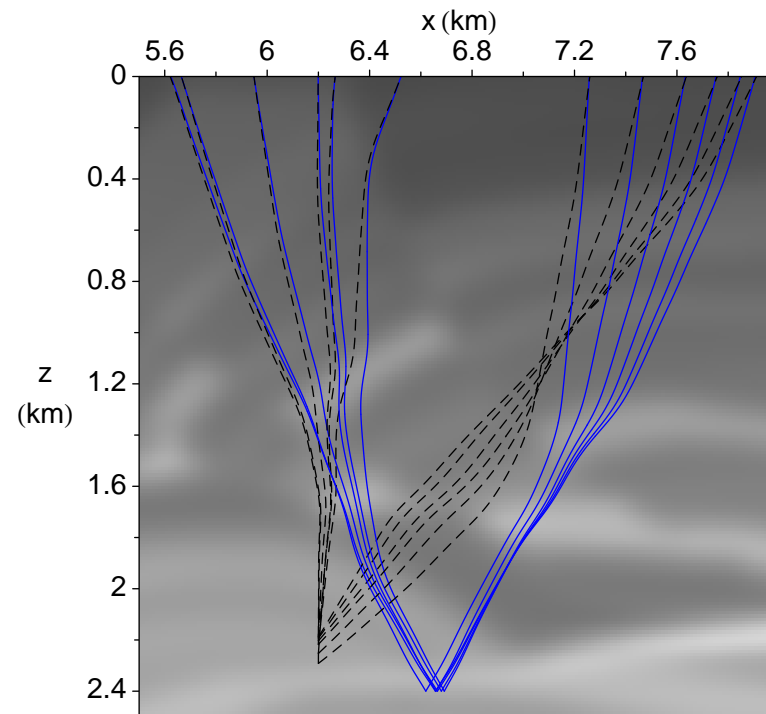




Typical shot gather: much evidence of multipathing, caustic formation.



Typical common scattering angle image gather: note nonflat event in box - results from data event migrating along *wrong ray pair*.



Blue rays = energy path producing data event. Black rays: energy path for migration, resulting in displaced, angle-dependent image artifact.

What it all means

Note that a gather scheme makes the scattering operator block-diagonal: for example with data sorted into common offset gathers $h = (x_r - x_s)/2$,

$$F[v] = [F_{h_1}[v], \dots, F_{h_N}[v]]^T, \quad d = [d_{h_1}, \dots, d_{h_N}]^T$$

Thus $F[v]^* d = \sum_i F_{h_i}[v]^* d_{h_i}$. Otherwise put: to form image, **migrate** i th gather (apply migration operator $F_{h_i}[v]^*$, then **stack** individual migrated images.

Horrible Examples show that individual migrated images may contain nonphysical apparent reflectors (artifacts).

Smit-tenKroode-Verdel, Nolan, Stolk: if TIC holds, then these artifacts are not stationary with respect to the gather parameter, hence *stack out* (interfere destructively) in final image.
