Mathematics of Seismic Imaging
Part III

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A step beyond linearization: velocity analysis
Velocity Analysis

Partially linearized seismic inverse problem (“velocity analysis”): given observed seismic data $d$, find smooth velocity $v \in \mathcal{E}(X), X \subset \mathbb{R}^3$ oscillatory reflectivity $r \in \mathcal{E}'(X)$ so that

$$F[v]r \simeq d$$

Acoustic partially linearized model: acoustic potential field $u$ and its perturbation $\delta u$ solve

$$\left( \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) u = \delta(t) \delta(x - x_s), \quad \left( \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \delta u = 2r \nabla^2 u$$

plus suitable bdry and initial conditions.

$$F[v]r = \frac{\partial \delta u}{\partial t} \bigg|_Y$$

data acquisition manifold $Y = \{(x_r, t; x_s)\} \subset \mathbb{R}^7$, dimn $Y \leq 5$ (many idealizations here!).
$F[v] : \mathcal{E}'(X) \rightarrow \mathcal{D}'(Y)$ is a linear map (FIO of order 1), but dependence on $v$ is quite nonlinear, so this inverse problem is nonlinear.

Agenda:

- reformulation of inverse problem via *extensions*
- “standard processing” extension and standard VA
- the surface oriented extension and standard MVA
- the ΨDO property and why it’s important
- global failure of the ΨDO property for the SOE
- Claerbout’s depth oriented extension has the ΨDO property
- differential semblance
Extensions

Extension of $F[v]$:

A manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \to \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Y)$ so that

\[
\begin{array}{ccc}
\mathcal{E}'(\bar{X}) & \to & \mathcal{D}'(Y) \\
\chi & \uparrow & \uparrow \text{id} \\
\mathcal{E}'(X) & \to & \mathcal{D}'(Y) \\
F[v] & \to &
\end{array}
\]

commutes.

Invertible extension: $\bar{F}[v]$ has a right parametrix $\bar{G}[v]$, i.e. $I - \bar{F}[v] \bar{G}[v]$ is smoothing. [The trivial extension - $\bar{X} = X$, $\bar{F} = F$ - is virtually never invertible.] Also $\chi$ has a left inverse $\eta$.

Reformulation of inverse problem: given $d$, find $v$ so that $\bar{G}[v]d \in \mathcal{R}(\chi)$ (implicitly determines $r$ also!).
Reformulation of inverse problem

Given $d$, find $v$ so that $\bar{G}[v]d \in$ the range of $\chi$.

Claim: if $v$ is so chosen, then $[v, r]$ solves partially linearized inverse problem with $r = \eta \bar{G}[v]d$.

Proof: Hypothesis means

$$\bar{G}[v]d = \chi r$$

for some $r$ (whence necessarily $r = \eta \bar{G}[v]d$), so

$$d \simeq \bar{F}[v] \bar{G}[v]d = \bar{F}[v] \chi r = F[v]r$$

Q. E. D.
Example 1: Standard VA extension

Treat each CMP as if it were the result of an experiment performed over a layered medium, but permit the layers to vary with midpoint.

Thus $v = v(z), r = r(z)$ for purposes of analysis, but at the end $v = v(x_m, z), r = r(x_m, z)$.

$$F[v]R(x_m, h, t) \simeq A(x_m, h, z(x_m, h, t))R(x_m, z(x_m, h, t))$$

Here $z(x_m, h, t)$ is the inverse of the 2-way traveltime

$$t(x_m, h, z) = 2\tau(x_m + (h, 0, z), x_m)_{v=v(x_m, z)}$$

computed with the layered velocity $v(x_m, z)$, i.e.

$z(x_m, h, t(x_m, h, z')) = z'$. 
That is, $F[v]$ is a change of variable followed by multiplication by a smooth function. NB: industry standard practice is to use vertical traveltime $t_0$ instead of $z$ for depth variable.

Can write this as $F[v] = \bar{F}[v]S^*$, where $\bar{F}[v] = N[v]^{-1}M[v]^{-1}$ has right parametrix $\bar{G}[v] = M[v]N[v]$: 

$N[v] =$ **NMO operator** $N[v]d(x_m, h, z) = d(x_m, h, t(x_m, h, z))$

$M[v] =$ multiplication by $A$

$S =$ **stacking operator**

$$Sf(x_m, z) = \int dh \ f(x_m, h, z), \ S^* r(x_m, h, z) = r(x, z)$$
Identify as extension: $\bar{F}[v], \bar{G}[v]$ as above, $X = \{x_m, z\}, H = \{h\}, \bar{X} = X \times H, \chi = S^*, \eta = S$ - the invertible extension properties are clear.

Standard names for the Standard VA extension objects: $\bar{F}[v]$ = “inverse NMO”, $\bar{G}[v]$ = “NMO” [often the multiplication op $M[v]$ is neglected]; $\eta = “stack”, \chi = “spread”$

**How this is used for velocity analysis:** Look for $v$ that makes $\bar{G}[v]d \in \mathcal{R}(\chi)$

So what is $\mathcal{R}(\chi)$? $\chi[r](x_m, z, h) = r(x_m, z)$ Anything in range of $\chi$ is independent of $h$. Practical issues $\Rightarrow$ replace “independent of” with “smooth in”.
Flatten them gathers!

Inverse problem reduced to: adjust $v$ to make $\tilde{G}[v]d^{obs}$ smooth in $h$, i.e. flat in $z, h$ display for each $x_m$ (NMO-corrected CMP).

Replace $z$ with $t_0$, $v$ with $v_{\text{RMS}}$ em localizes computation: reflection through $x_m, t_0, 0$ flattened by adjusting $v_{\text{RMS}}(x_m, t_0) \Rightarrow$ 1D search, do by visual inspection.

Various aids - NMO corrected CMP gathers, velocity spectra, etc.

See: Claerbout: *Imaging the Earth’s Interior*

WWS: MGSS 2000 notes
Left: part of survey \((d)\) from North Sea (thanks: Shell Research), lightly preprocessed.

Right: restriction of \(\bar{G}[v]d\) to \(x_m = \text{const}\) (function of depth, offset): shows rel. sm’ness in \(h\) (offset) for properly chosen \(v\).
Example 2: Surface oriented or standard MVA extension

Standard VA only works where Earth is “nearly layered”. Where this fails, replace NMO by prestack migration.

Version based on common offset modeling/migration: \( \Sigma_h = \text{set of half-offsets in data, } \bar{X} = X \times \Sigma_h, \chi[r](\mathbf{x}, \mathbf{h}) = r(\mathbf{x}). \)

\[
\bar{F}[v]\bar{r}(\mathbf{x}_s, t, \mathbf{h}) = \frac{\partial^2}{\partial t^2} \int d\mathbf{x} \bar{r}(\mathbf{x}, \mathbf{h}) \int ds \ G(\mathbf{x}_s + 2\mathbf{h}, t - s; \mathbf{x}) \ G(\mathbf{x}_s, s; \mathbf{x})
\]

Note that this operator is “block diagonal” in \( \mathbf{h}. \)
Properties of SOE

Beylkin (1985), Rakesh (1988): if $\|v\|_{C^2(X)}$ “not too big”, then

- $\tilde{F}$ has the $\Psi$DO property: $\tilde{F}\tilde{F}^*$ is $\Psi$DO
- singularities of $\tilde{F}\tilde{F}^*d \subset$ singularities of $d$
- straightforward construction of right parametrix $\tilde{G} = \tilde{F}^*Q$, $Q = \Psi$DO, also as generalized Radon Transform - explicitly computable.

Range of $\chi$ (offset version): $\tilde{r}(x, h)$ independent of $h \Rightarrow$ “semblance principle”: find $v$ so that $\tilde{G}[v]d^{\text{obs}}$ is independent of $h$. Practical limitations $\Rightarrow$ replace “independent of $h$” by “smooth in $h$”.
Industrial MVA

Application of these ideas = industrial practice of migration velocity analysis.

Idea: twiddle \( v \) until \( \bar{G}[v]d^{\text{obs}} \) is smooth in \( h \).

Since it is hard to inspect \( \bar{G}[v]d^{\text{obs}}(x, y, z, h) \), pull out subset for constant \( x, y = \text{common image gather} \) ("CIG"): display function of \( z, h \) for fixed \( x, y \). These play same role as NMO corrected CMP gathers in layered case.

Try to adjust \( v \) so that selected CIGs are flat - just as in Standard VA. This is much harder, as there is no RMS velocity trick to localize the computation - each CIG depends globally on \( v \).

Description, some examples: Yilmaz, Seismic Data Processing.
Bad news

Nolan (1997), Stolk & WWS (2004): big trouble! In general, standard extension does not have the \( \Psi \)DO property. Geometric optics analysis: for \( \|v\|_{C^2(X)} \) “large”, multiple rays connect source, receiver to reflecting points in \( X \); block diagonal structure of \( \bar{F}[v] \Rightarrow \) info necessary to distinguish multiple rays is \textit{projected out}. 
Example (Stolk & WWS, 2001): Gaussian lens over flat reflector at depth $z$ ($r(x) = \delta(x_1 - z)$, $x_1 = \text{depth}$).
**Left:** Const. $h$ slice of $\tilde{G}d$: several refl. points corresponding to same singularity in $d^{\text{obs}}$.

**Right:** CIG (const. $x, y$ slice) of $\tilde{G}d$: not smooth in $h$!
Example 3: Claerbout’s depth oriented extension

Standard MVA extension only works when Earth has simple ray geometry. Claerbout (1971) proposed alternative extension:

\[
\Sigma_d = \text{somewhat arbitrary set of vectors near 0 ("offsets"), } \bar{X} = X \times \Sigma_d, \chi[r](x, h) = r(x)\delta(h), \eta[\bar{r}](x) = \bar{r}(x, 0)
\]

\[
\bar{F}[v]\bar{r}(x_s, t, x_r) = \frac{\partial^2}{\partial t^2} \int dx \int_{\Sigma_d} dh \bar{r}(x, h) \int ds G(x_s, t - s; x + 2h)G(x_r, s; x)
\]

\[
= \frac{\partial^2}{\partial t^2} \int dx \int_{x+2\Sigma_d} dy \bar{r}(x, y - x) \int ds G(x_s, t - s; y)G(x_r, s; x)
\]

NB: in this formulation, there appears to be too many model parameters.
Shot record modeling

for each $x_s$ solve

$$\bar{F}[v] \bar{r}(x_r, t; x_s) = u(x, t; x_s)|_{x=x_r}$$

where

$$\left( \frac{1}{v(x)^2} \frac{\partial^2}{\partial t^2} - \nabla_x^2 \right) u(x, t; x_s) = \int_{x+2\Sigma_d} dy \bar{r}(x, y) G(y, t; x_s)$$

$$\left( \frac{1}{v(y)^2} \frac{\partial^2}{\partial t^2} - \nabla_y^2 \right) G(y, t; x_s) = \delta(t)\delta(x_s - y)$$

Finite difference scheme: form RHS for eqn 1, step $u, G$ forward in $t$. 
Computing $\bar{G}[v]$

Instead of parametrix, be satisfied with adjoint.

Reverse time adjoint computation - specify adjoint field as in standard reverse time prestack migration:

$$\left( \frac{1}{w(x)^2} \frac{\partial^2}{\partial t^2} - \nabla^2_x \right) w(x, t; x_s) = \int dx_r d(x_r, t; x_s) \delta(x - x_r)$$

with $w(x, t; x_s) = 0, t >> 0$. Then

$$\bar{F}[v]^* d(x, h) = \int dx_s \int dt G(x + 2h, t; x_s) w(x, t; x_s)$$

i.e. exactly the same computation as for reverse time prestack, except that crosscorrelation occurs at an offset $2h$. 
Nomenclature

NB: the “usual computation” of $\bar{G}[v]$ is either DSR or a variant of shot record computation of previous slide using depth extrapolation. $h$ is usually restricted to be horizontal, i.e. $h_3 = 0$.

Common names: shot-geophone or survey-sinking migration (with DSR), or shot record migration.

“Downward continue sources and receivers, image at $t = 0, h = 0$”

These are what is typically meant by “wave equation migration”!
What should be the character of the image when the velocity is correct?

Hint: for simulation of seismograms, the input reflectivity had the form $r(x)\delta(h)$.

Therefore guess that when velocity is correct, image is concentrated near $h = 0$.

Examples: 2D finite difference implementation of reverse time method. Correct velocity $\equiv 1$. Input reflectivity used to generate synthetic data: random! For output reflectivity (image of $\bar{F}[v]^*$), constrain offset to be horizontal: $\tilde{r}(x, h) = \tilde{r}(x, h_1)\delta(h_3)$. Display CIGs (i.e. $x_1 =$const. slices).
Two way reverse time horizontal offset S-G image gathers of data from random reflectivity, constant velocity. From left to right: correct velocity, 10% high, 10% low.
Claerbout extension has the $\Psi$DO property, at least when restricted to $\bar{r}$ of the form
\[ \bar{r}(x, h) = R(x, h_1, h_2)\delta(h_3), \]
and under DSR assumption.

Sketch of proof (after Rakesh, 1988):

This will follow from injectivity of wavefront or canonical relation $C_{\bar{F}} \subset T^*(\bar{X}) - \{0\} \times T^*(Y) - \{0\}$ which describes singularity mapping properties of $\bar{F}$:

\[ (x, h, \xi, \nu, y, \eta) \in C_{F_\delta[u]} \iff \]

for some $u \in \mathcal{E}'(\bar{X})$, $(x, h, \xi, \nu) \in WF(u)$, and $(y, \eta) \in WF(\bar{F}u)$
Characterization of $C_{\bar{F}}$

$$((x, h, \xi, \nu), (x_s, t, x_r, \xi_s, \tau, \xi_r)) \in C_{\bar{F}}[v] \subset T^*(\bar{X}) - \{0\} \times T^*(Y) - \{0\}$$

$\iff$ there are rays of geometric optics $(X_s, \Xi_s), (X_r, \Xi_r)$ and times $t_s, t_r$ so that

$$\Pi(X_s(0), t, X_r(0), \Xi_s(0), \tau, \Xi_r(0)) = (x_s, t, x_r, \xi_s, \tau, \xi_r),$$

$$X_s(t_s) = x, X_r(t_r) = x + 2h, t_s + t_r = t,$$

$$\Xi_s(t_s) + \Xi_r(t_r)||\xi, \Xi_s(t_s) - \Xi_r(t_r)||\nu$$
Proof

Uses wave equations for $u, G$ and

- Gabor calculus: computes wave front sets of products, pullbacks, integrals, etc.
  See Duistermaat, Ch. 1.
- Propagation of Singularities Theorem

and that’s all! [No integral representations, phase functions,...]
Note intrinsic ambiguity: if you have a ray pair, move times \( t_s, t_r \) resp. \( t'_s, t'_r \), for which \( t_s + t_r = t'_s + t'_r = t \) then you can construct two points \((x, h, \xi, \nu), (x', h', \xi', \nu')\) which are candidates for membership in \( WF(\bar{r}) \) and which satisfy the above relations with the same point in the cotangent bundle of \( T^*(Y) \).

No wonder - there are too many model parameters!

Stolk and deHoop fix this ambiguity by imposing two constraints:

- **DSR assumption**: all rays carrying significant reflected energy (source or receiver) are upcoming.
- **Restrict** \( \bar{F} \) **to the domain** \( \mathcal{Z} \subset \mathcal{E}'(\bar{X}) \)

\[
\bar{r} \in \mathcal{Z} \iff \bar{r}(x, h) = R(x, h_1, h_2)\delta(h_3)
\]
If $\bar{r} \in \mathcal{Z}$, then $(\mathbf{x}, h, \xi, \nu) \in WF(\bar{r}) \Rightarrow h_3 = 0$. So source and receiver rays in $C_{\bar{F}}$ must terminate at same depth, to hit such a point.

Because of DSR assumption, this fixes the traveltimes $t_s, t_r$.

**Restricted to $\mathcal{Z}$, $C_{\bar{F}}$ is injective.**

$\Rightarrow C_{\bar{F}^*\bar{F}} = I$

$\Rightarrow \bar{F}^*\bar{F}$ is $\Psi$DO when restricted to $\mathcal{Z}$. 
Lens data, shot-geophone migration [B. Biondi, 2002]
Left: Image via DSR. Middle: $\tilde{G}[\nu]d$ - well-focused (at $h = 0$), i.e. in range of $\chi$ to extent possible. Right: Angle CIG.
Quantitative VA

Suppose \( W : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Z) \) annihilates range of \( \chi \):

\[
\begin{array}{c}
\chi \\
\mathcal{E}'(X) \\
\to \\
\mathcal{E}'(\bar{X}) \\
\to \\
\mathcal{D}'(Z) \\
\to \\
0 \\
\end{array}
\]

and moreover \( W \) is bounded on \( L^2(\bar{X}) \). Then

\[
J[v; d] = \frac{1}{2} \| W \bar{G}[v] d \|^2
\]

minimized when \([v, \eta \bar{G}[v] d] \) solves partially linearized inverse problem.

Construction of annihilator of \( \mathcal{R}(F[v]) \) (Guillemin, 1985):

\[
d \in \mathcal{R}(F[v]) \iff \bar{G}[v] d \in \mathcal{R}(\chi) \iff W \bar{G}[v] d = 0
\]
Annihilators, annihilators everywhere...

For Standard Extended Model, several popular choices:

- \( W = (I - \Delta)^{-\frac{1}{2}} \nabla_h \) (“differential semblance” - WWS, 1986)
- \( W = I - \frac{1}{|H|} \int dh \) (“stack power” - Toldi, 1985)
- \( W = I - \chi F[v]^\dagger \bar{F}[v] \Rightarrow \text{minimizing } J[v, d] \text{ equivalent to least squares.} \)

For Claerbout extension, differential semblance \( W = h. \)
But not many are good for much...

Since *problem is huge*, only $W$ giving rise to differentiable $v \mapsto J[v, d]$ are useful - must be able to use Newton!!! Once again, idealize $w(t) = \delta(t)$.

**Theorem** (Stolk & WWS, 2003): $v \mapsto J[v, d]$ smooth $\iff W$ pseudodifferential.

i.e. only *differential semblance* gives rise to smooth optimization problem, *uniformly in source bandwidth*.

Example: NMO-based Differential Semblance

\[
J[v, d] = \frac{1}{2} \left\| \frac{\partial}{\partial h} N[v]d \right\|^2
\]

(recall that \(N[v]\) is the NMO operator = composition with \(t(z, h)\))

Theory: under some circumstances, can show that all stationary points are global minimizers (WWS, TRIP annual reports ’99, ’01).

Example uses data from North Sea survey (thanks: Shell) with light preprocessing: cutoff (“mute”) and multiple suppression (predictive decon) to enhance conformance with model, low pass filter.

Minimization of \(J\) via quasi-Newton method.
Beyond Born

Nonlinear effects not included in linearized model: *multiple reflections*. Conventional approach: treat as *coherent noise*, attempt to eliminate - active area of research going back 40+ years, with recent important developments.

Why not model this “noise”?

Proposal: *nonlinear extensions* with $F[v]r$ replaced by $F[c]$. Create annihilators in same way (now also nonlinear), optimize differential semblance.

Nonlinear analog of Standard Extended Model appears to be *invertible* - in fact extended nonlinear inverse problem is *underdetermined*.

Open problems: no theory. Also must determine $w(t)$ (Lailly SEG 2003).
And so on...


- Anisotropy - work of deHoop (Brandsberg-Dahl et al 2003).

- Anelasticity - in the sedimentary section, \( Q = 100 - 1000 \), lower in gassy sediments and near surface. No mathematical results, but some numerics - Minkoff & WWS 1997, Blanch et al 1998.


- ...