Mathematics of Seismic Imaging
Part III

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PIMS, July 2005

A step beyond linearization: velocity analysis

Velocity Analysis

Partially linearized seismic inverse problem ("velocity analysis"): given observed seismic data d, find smooth *velocity* $v \in \mathcal{E}(X), X \subset \mathbf{R}^3$ oscillatory *reflectivity* $r \in \mathcal{E}'(X)$ so that

$$F[v]r \simeq d$$

Acoustic partially linearized model: acoustic potential field u and its perturbation δu solve

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u = \delta(t)\delta(\mathbf{x} - \mathbf{x}_s), \ \left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta u = 2r\nabla^2 u$$

plus suitable bdry and initial conditions.

$$F[v]r = \left.\frac{\partial \delta u}{\partial t}\right|_Y$$

data acquisition manifold $Y = \{(\mathbf{x}_r, t; \mathbf{x}_s)\} \subset \mathbf{R}^7$, dimn $Y \leq 5$ (many idealizations here!).

$F[v] : \mathcal{E}'(X) \to \mathcal{D}'(Y)$ is a linear map (FIO of order 1), but dependence on v is quite nonlinear, so this inverse problem is nonlinear.

Agenda:

- reformulation of inverse problem via *extensions*
- "standard processing" extension and standard VA
- the surface oriented extension and standard MVA
- the ΨDO property and why it's important
- global failure of the ΨDO property for the SOE
- Claerbout's depth oriented extension has the ΨDO property
- differential semblance

Extensions

Extension of $F[v]$: manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \to \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Y)$ so that

$$\begin{array}{ccccc} & & \bar{F}[v] & & \\ & \mathcal{E}'(\bar{X}) & \to & \mathcal{D}'(Y) & \\ \chi & \uparrow & & \uparrow & \mathrm{id} \\ & \mathcal{E}'(X) & \to & \mathcal{D}'(Y) & \\ & & F[v] & & \end{array}$$

commutes.

Invertible extension: $\bar{F}[v]$ has a *right parametrix* $\bar{G}[v]$, i.e. $I - \bar{F}[v]\bar{G}[v]$ is smoothing. [The trivial extension - $\bar{X} = X, \bar{F} = F$ - is virtually never invertible.] Also χ has a *left inverse* η.

Reformulation of inverse problem: given d, find v so that $\bar{G}[v]d \in \mathcal{R}(\chi)$ (implicitly determines r also!).

Reformulation of inverse problem

Given d, find v so that $\bar{G}[v]d \in$ the range of χ.

Claim: if v is so chosen, then $[v, r]$ solves partially linearized inverse problem with $r = \eta\bar{G}[v]d$.

Proof: Hypothesis means

$$\bar{G}[v]d = \chi r$$

for some r (whence necessarily $r = \eta\bar{G}[v]d$), so

$$d \simeq \bar{F}[v]\bar{G}[v]d = \bar{F}[v]\chi r = F[v]r$$

Q. E. D.

Example 1: Standard VA extension

Treat each CMP *as if it were the result of an experiment performed over a layered medium*, but permit the layers to vary with midpoint.

Thus $v = v(z), r = r(z)$ for purposes of analysis, but at the end $v = v(\mathbf{x}_m, z), r = r(\mathbf{x}_m, z)$.

$$F[v]R(\mathbf{x}_m, h, t) \simeq A(\mathbf{x}_m, h, z(\mathbf{x}_m, h, t))R(\mathbf{x}_m, z(\mathbf{x}_m, h, t))$$

Here $z(\mathbf{x}_m, h, t)$ is the inverse of the 2-way traveltime

$$t(\mathbf{x}_m, h, z) = 2\tau(\mathbf{x}_m + (h, 0, z), \mathbf{x}_m)_{v=v(\mathbf{x}_m, z)}$$

computed with the layered velocity $v(\mathbf{x}_m, z)$, i.e.
$z(\mathbf{x}_m, h, t(\mathbf{x}_m, h, z')) = z'$.

That is, $F[v]$ *is a change of variable followed by multiplication by a smooth function.* NB: industry standard practice is to use vertical traveltime t_0 instead of z for depth variable.

Can write this as $F[v] = \bar{F}[v]S^*$, where $\bar{F}[v] = N[v]^{-1}M[v]^{-1}$ has right parametrix $\bar{G}[v] = M[v]N[v]$:

$N[v]$ = **NMO operator** $N[v]d(\mathbf{x}_m, h, z) = d(\mathbf{x}_m, h, t(\mathbf{x}_m, h, z))$

$M[v]$ = multiplication by A

S = **stacking operator**

$$Sf(\mathbf{x}_m, z) = \int dh\, f(\mathbf{x}_m, h, z), \;\; S^*r(\mathbf{x}_m, h, z) = r(\mathbf{x}, z)$$

Identify as extension: $\bar{F}[v], \bar{G}[v]$ as above, $X = \{\mathbf{x}_m, z\}, H = \{h\}, \bar{X} = X \times H, \chi = S^*, \eta = S$ - the invertible extension properties are clear.

Standard names for the Standard VA extension objects: $\bar{F}[v]$ = “inverse NMO”, $\bar{G}[v]$ = “NMO” [often the multiplication op $M[v]$ is neglected]; η = “stack”, χ = “spread”

How this is used for velocity analysis: Look for v that makes $\bar{G}[v]d \in \mathcal{R}(\chi)$

So what is $\mathcal{R}(\chi)$? $\chi[r](\mathbf{x}_m, z, h) = r(\mathbf{x}_m, z)$ Anything in range of χ is *independent of* h. Practical issues $\Rightarrow$ replace “independent of” with “smooth in”.

Flatten them gathers!

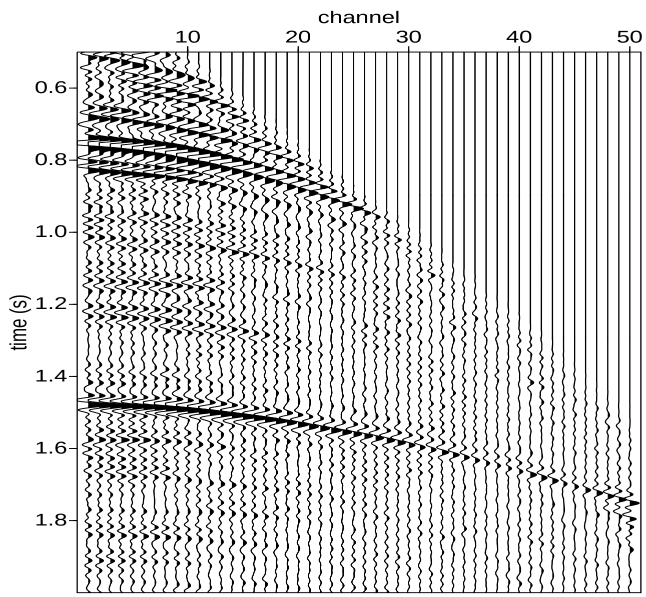
Inverse problem reduced to: adjust v to make $\bar{G}[v]d^{\text{obs}}$ smooth in h, i.e. *flat* in z, h display for each $\mathbf{x}_m$ (*NMO-corrected CMP*).

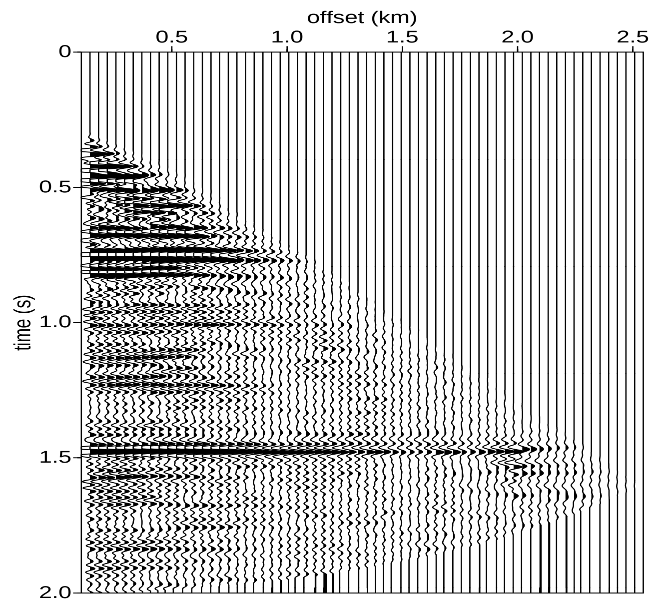
Replace z with t_0, v with v_{RMS} em localizes computation: reflection through $\mathbf{x}_m, t_0, 0$ *flattened* by adjusting $v_{\text{RMS}}(\mathbf{x}_m, t_0) \Rightarrow$ 1D search, do by visual inspection.

Various aids - NMO corrected CMP gathers, velocity spectra, etc.

See: Claerbout: *Imaging the Earth's Interior*

WWS: MGSS 2000 notes





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Left: part of survey (d) from North Sea (thanks: Shell Research), lightly preprocessed.
Right: restriction of $\bar{G}[v]d$ to $\mathbf{x}_m = \text{const}$ (function of depth, offset): shows rel. sm'ness in h (offset) for properly chosen v.

Example 2: Surface oriented or standard MVA extension

. Standard VA only works where Earth is "nearly layered". Where this fails, replace NMO by prestack migration.

Version based on common offset modeling/migration: Σ_h = set of half-offsets in data, $\bar{X} = X \times \Sigma_h$, $\chi[r](\mathbf{x},\mathbf{h}) = r(\mathbf{x})$.

$$\bar{F}[v]\bar{r}(\mathbf{x}_s,t,\mathbf{h}) = \frac{\partial^2}{\partial t^2}\int dx\,\bar{r}(\mathbf{x},\mathbf{h})\int ds\,G(\mathbf{x}_s+2\mathbf{h},t-s;\mathbf{x})G(\mathbf{x}_s,s;\mathbf{x})$$

Note that this operator is "block diagonal" in $\mathbf{h}$.

Properties of SOE

Beylkin (1985), Rakesh (1988): if $\|v\|_{C^2(X)}$ "not too big", then

- $\bar{F}$ has **the ΨDO property**: $\bar{F}\bar{F}^*$ is ΨDO
- singularities of $\bar{F}\bar{F}^* d \subset$ singularities of d
- straightforward construction of right parametrix $\bar{G} = \bar{F}^* Q$, Q = ΨDO, also as generalized Radon Transform - explicitly computable.

Range of χ (offset version): $\bar{r}(\mathbf{x}, \mathbf{h})$ independent of $\mathbf{h}$ $\Rightarrow$ "semblance principle": find v so that $\bar{G}[v]d^{\rm obs}$ is independent of $\mathbf{h}$. Practical limitations $\Rightarrow$ replace "independent of $\mathbf{h}$" by "smooth in $\mathbf{h}$".

Industrial MVA

Application of these ideas = industrial practice of migration velocity analysis.

Idea: twiddle v until $\bar{G}[v]d^{\rm obs}$ is smooth in $\mathbf{h}$.

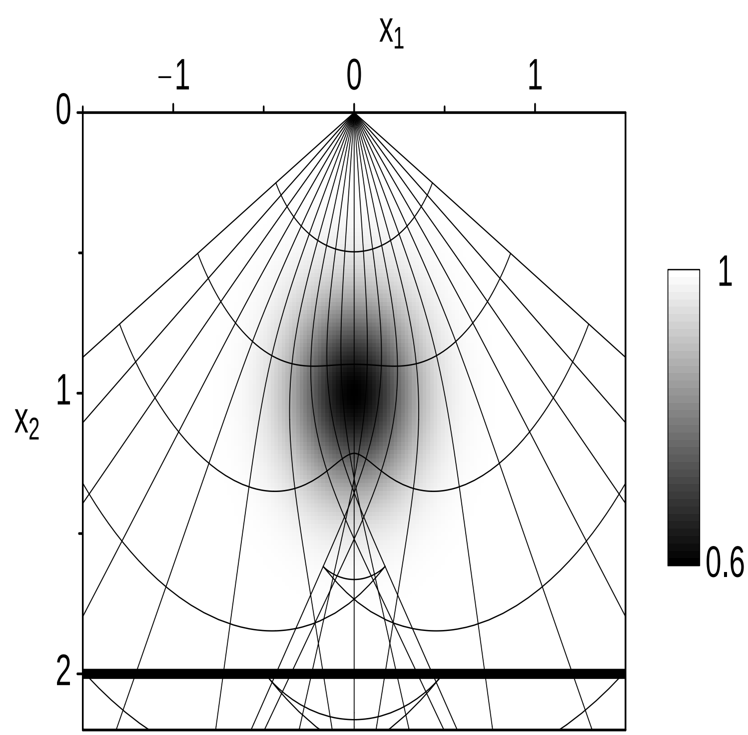
Since it is hard to inspect $\bar{G}[v]d^{\rm obs}(x, y, z, h)$, pull out subset for constant x, y = **common image gather** ("CIG"): display function of z, h for fixed x, y. These play same role as NMO corrected CMP gathers in layered case.

Try to adjust v so that selected CIGs are *flat* - just as in Standard VA. This is much harder, as there is no RMS velocity trick to localize the computation - each CIG depends globally on v.

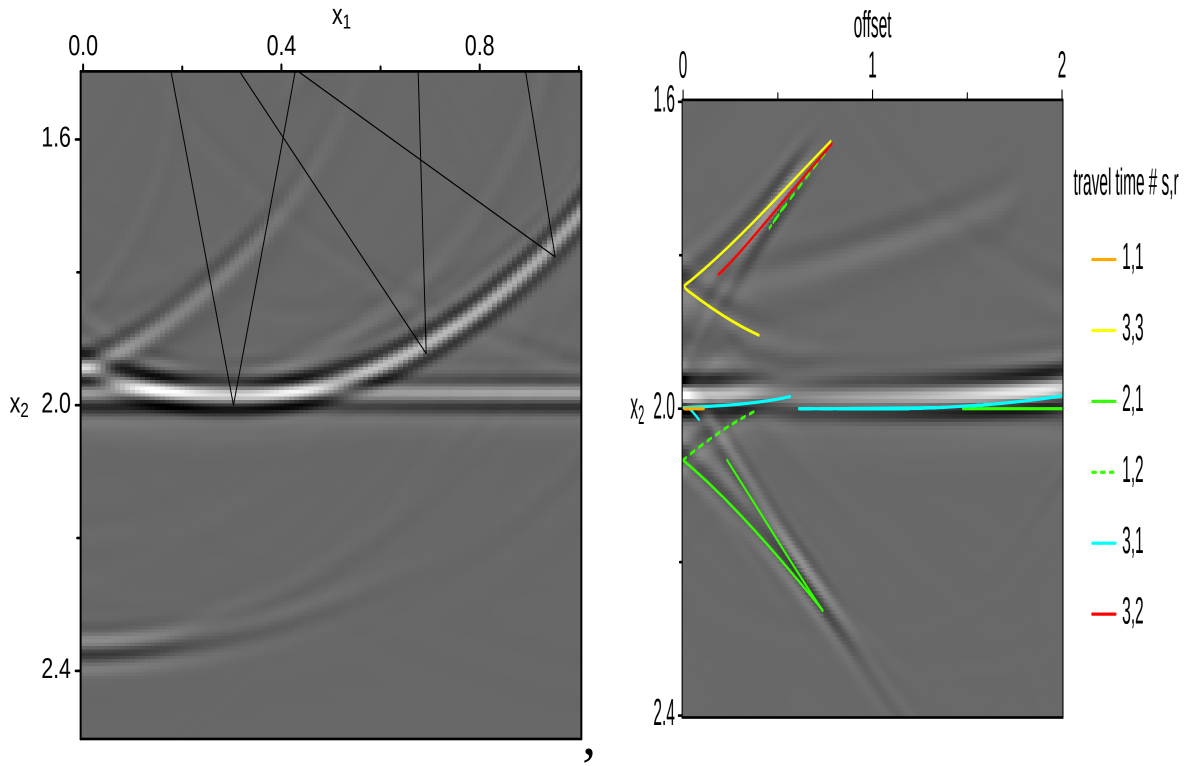
Description, some examples: Yilmaz, *Seismic Data Processing*.

Bad news

Nolan (1997), Stolk & WWS (2004): big trouble! In general, standard extension does **not** have the ΨDO property. Geometric optics analysis: for $\|v\|_{C^2(X)}$ "large", multiple rays connect source, receiver to reflecting points in X; block diagonal structure of $\bar{F}[v] \Rightarrow$ info necessary to distinguish multiple rays is *projected out*.



Example (Stolk & WWS, 2001): Gaussian lens over flat reflector at depth z ($r(\mathbf{x}) = \delta(x_1 - z)$, $x_1 =$ depth).



Left: Const. h slice of $\bar{G}d$: several refl. points corresponding to same singularity in d^{obs}.
Right: CIG (const. x, y slice) of $\bar{G}d$: not smooth in h!

Example 3: Claerbout's depth oriented extension

Standard MVA extension only works when Earth has simple ray geometry. Claerbout (1971) proposed alternative extension:

Σ_d = somewhat arbitrary set of vectors near 0 ("offsets"), $\bar{X} = X \times \Sigma_d$, $\chi[r](\mathbf{x},\mathbf{h}) = r(\mathbf{x})\delta(\mathbf{h})$, $\eta[\bar{r}](\mathbf{x}) = \bar{r}(\mathbf{x},0)$

$$\bar{F}[v]\bar{r}(\mathbf{x}_s,t,\mathbf{x}_r) = \frac{\partial^2}{\partial t^2}\int dx \int_{\Sigma_d} dh\,\bar{r}(\mathbf{x},\mathbf{h}) \int ds\,G(\mathbf{x}_s,t-s;\mathbf{x}+2\mathbf{h})G(\mathbf{x}_r,s;\mathbf{x})$$

$$= \frac{\partial^2}{\partial t^2}\int dx \int_{\mathbf{x}+2\Sigma_d} dy\,\bar{r}(\mathbf{x},\mathbf{y}-\mathbf{x}) \int ds\,G(\mathbf{x}_s,t-s;\mathbf{y})G(\mathbf{x}_r,s;\mathbf{x})$$

NB: in this formulation, there appears to be too many model parameters.

Shot record modeling

for each $\mathbf{x}_s$ solve

$$\bar{F}[v]\bar{r}(\mathbf{x}_r, t; \mathbf{x}_s) = u(\mathbf{x}, t; \mathbf{x}_s)|_{\mathbf{x}=\mathbf{x}_r}$$

where

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) u(\mathbf{x}, t; \mathbf{x}_s) = \int_{\mathbf{x}+2\Sigma_d} dy\, \bar{r}(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, t; \mathbf{x}_s)$$

$$\left(\frac{1}{v(\mathbf{y})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{y}}\right) G(\mathbf{y}, t; \mathbf{x}_s) = \delta(t)\delta(\mathbf{x}_s - \mathbf{y})$$

Finite difference scheme: form RHS for eqn 1, step u, G forward in t.

Computing $\bar{G}[v]$

Instead of parametrix, be satisfied with adjoint.

Reverse time adjoint computation - specify adjoint field as in standard reverse time prestack migration:

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) w(\mathbf{x},t;\mathbf{x}_s) = \int \, dx_r \, d(\mathbf{x}_r,t;\mathbf{x}_s)\delta(\mathbf{x}-\mathbf{x}_r)$$

with $w(\mathbf{x},t;\mathbf{x}_s) = 0, t >> 0$. Then

$$\bar{F}[v]^* d(\mathbf{x},\mathbf{h}) = \int \, dx_s \, \int \, dt \, G(\mathbf{x}+2\mathbf{h},t;\mathbf{x}_s) w(\mathbf{x},t;\mathbf{x}_s)$$

i.e. exactly the same computation as for reverse time prestack, except that crosscorrelation occurs at an offset $2\mathbf{h}$.

Nomenclature

NB: the "usual computation" of $\bar{G}[v]$ is either DSR or a variant of shot record computation of previous slide using depth extrapolation. $\mathbf{h}$ is usually restricted to be horizontal, i.e. $h_3 = 0$.

Common names: shot-geophone or survey-sinking migration (with DSR), or shot record migration.

"Downward continue sources and receivers, image at $t = 0, h = 0$"

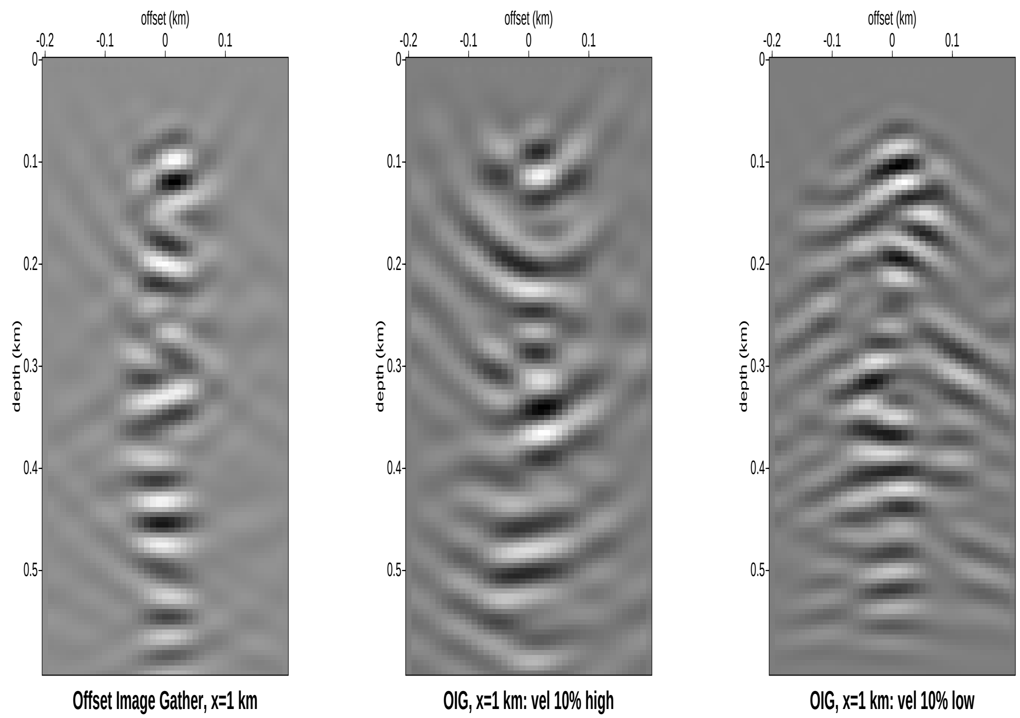
These are what is typically meant by "wave equation migration"!

What should be the character of the image when the velocity is correct?

Hint: for simulation of seismograms, the input reflectivity had the form $r(\mathbf{x})\delta(\mathbf{h})$.

Therefore guess that when velocity is correct, *image is concentrated near* $h = 0$.

Examples: 2D finite difference implementation of reverse time method. Correct velocity $\equiv 1$. Input reflectivity used to generate synthetic data: random! For output reflectivity (image of $\bar{F}[v]^*$), constrain offset to be horizontal: $\bar{r}(\mathbf{x}, \mathbf{h}) = \tilde{r}(\mathbf{x}, h_1)\delta(h_3)$. Display CIGs (i.e. x_1 =const. slices).



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Two way reverse time horizontal offset S-G image gathers of data from random reflectivity, constant velocity. From left to right: correct velocity, 10% high, 10% low.

Stolk and deHoop, 2001

Claerbout extension has the ΨDO property, at least when restricted to $\bar{r}$ of the form $\bar{r}(\mathbf{x},\mathbf{h}) = R(\mathbf{x},h_1,h_2)\delta(h_3)$, and under DSR assumption.

Sketch of proof (after Rakesh, 1988):

This will follow from *injectivity* of wavefront or *canonical relation* $C_{\bar{F}} \subset T^*(\bar{X}) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$ which describes singularity mapping properties of $\bar{F}$:

$$(\mathbf{x},\mathbf{h},\xi,\nu,\mathbf{y},\eta) \in C_{F_\delta[v]} \Leftrightarrow$$

$$\text{for some } u \in \mathcal{E}'(\bar{X}),\ (\mathbf{x},\mathbf{h},\xi,\nu) \in WF(u),\ \text{and } (\mathbf{y},\eta) \in WF(\bar{F}u)$$

Characterization of $C_{\bar{F}}$

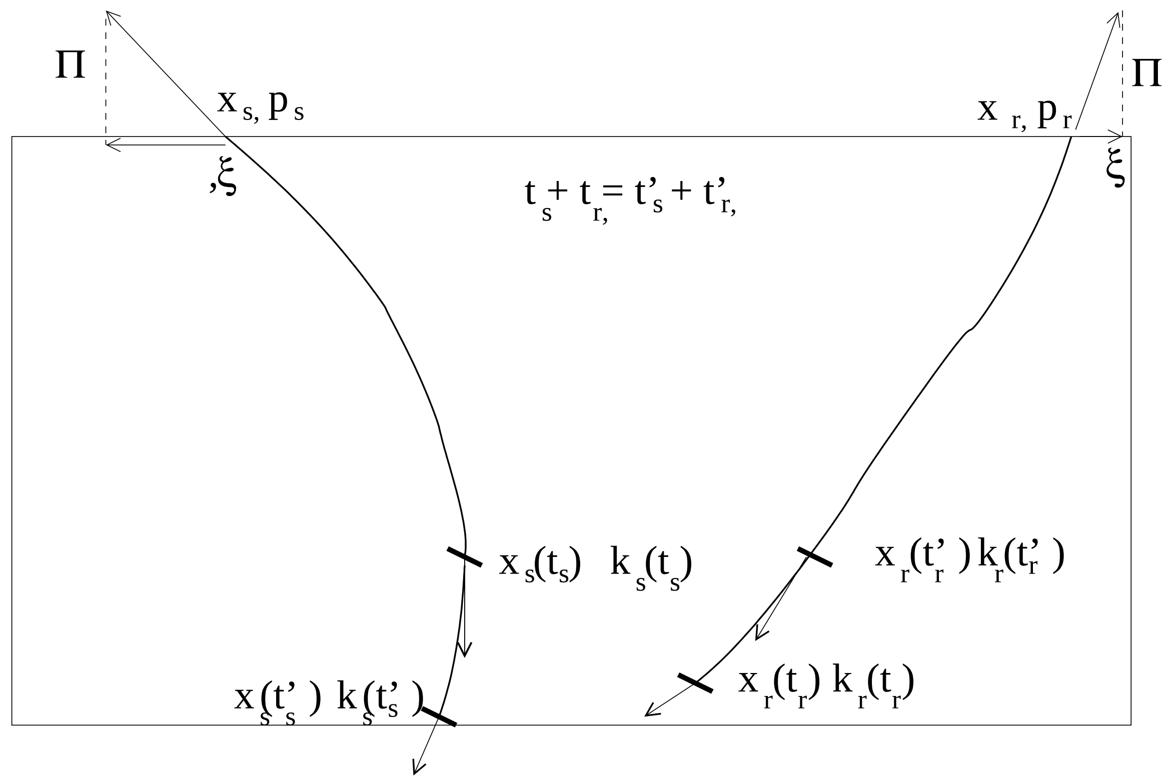
$$((\mathbf{x},\mathbf{h},\xi,\nu),(\mathbf{x}_s,t,\mathbf{x}_r,\xi_{\mathbf{s}},\tau,\xi_{\mathbf{r}})) \in C_{\bar{F}}[v] \subset T^*(\bar{X}) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$$

$\Leftrightarrow$ there are *rays of geometric optics* $(\mathbf{X}_s,\mathbf{\Xi}_s)$, $(\mathbf{X}_r,\mathbf{\Xi}_r)$ and times t_s, t_r so that

$$\Pi(\mathbf{X}_s(0),t,\mathbf{X}_r(0),\mathbf{\Xi}_s(0),\tau,\mathbf{\Xi}_r(0)) = (\mathbf{x}_s,t,\mathbf{x}_r,\xi_s,\tau,\xi_r),$$

$$\mathbf{X}_s(t_s) = \mathbf{x}, \mathbf{X}_r(t_r) = \mathbf{x}+2\mathbf{h},\ t_s+t_r = t,$$

$$\mathbf{\Xi}_s(t_s)+\mathbf{\Xi}_r(t_r)||\xi, \mathbf{\Xi}_s(t_s)-\mathbf{\Xi}_r(t_r)||\nu$$



Π
x s, p s
,ξ
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x r, p r
ξ
t s + t r, = t's + t'r,
x s(ts) k s(ts)
x r(t'r) k r(t'r)
x s(t's) k s(t's)
x r(tr) k r(tr)

Proof

Uses wave equations for u, G and

- Gabor calculus: computes wave front sets of products, pullbacks, integrals, etc. See Duistermaat, Ch. 1.
- Propagation of Singularities Theorem

and that's all! [No integral representations, phase functions,...]

Note intrinsic ambiguity: if you have a ray pair, move times t_s, t_r resp. t'_s, t'_r, for which $t_s + t_r = t'_s + t'_r = t$ then you can construct two points $(\mathbf{x}, \mathbf{h}, \boldsymbol{\xi}, \nu), (\mathbf{x}', \mathbf{h}', \boldsymbol{\xi}', \nu')$ which are candidates for membership in $WF(\bar{r})$ and which satisfy the above relations with the same point in the cotangent bundle of $T^*(Y)$.

No wonder - there are too many model parameters!

Stolk and deHoop fix this ambiguity by imposing two constraints:

- DSR assumption: all rays carrying significant reflected energy (source or receiver) are upcoming.
- Restrict $\bar{F}$ to the domain $\mathcal{Z} \subset \mathcal{E}'(\bar{X})$

$$\bar{r} \in \mathcal{Z} \;\Leftrightarrow\; \bar{r}(\mathbf{x}, \mathbf{h}) = R(\mathbf{x}, h_1, h_2)\delta(h_3)$$

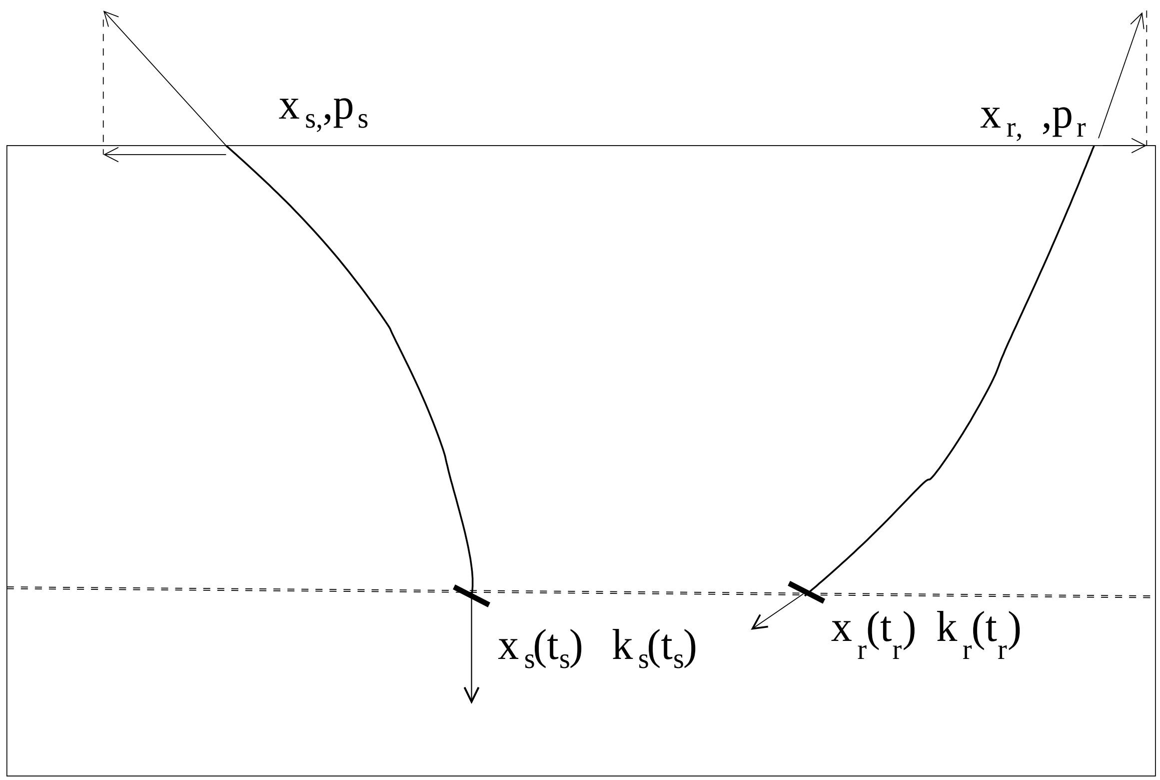
If $\bar{r} \in \mathcal{Z}$, then $(\mathbf{x}, \mathbf{h}, \boldsymbol{\xi}, \nu) \in WF(\bar{r}) \Rightarrow h_3 = 0$. So source and receiver rays in $C_{\bar{F}}$ must terminate at same depth, to hit such a point.

Because of DSR assumption, this fixes the traveltimes t_s, t_r.

Restricted to $\mathcal{Z}$, $C_{\bar{F}}$ is injective.

$\Rightarrow C_{\bar{F}^*\bar{F}} = I$

$\Rightarrow \bar{F}^*\bar{F}$ is ΨDO when restricted to $\mathcal{Z}$.


x s,,p s
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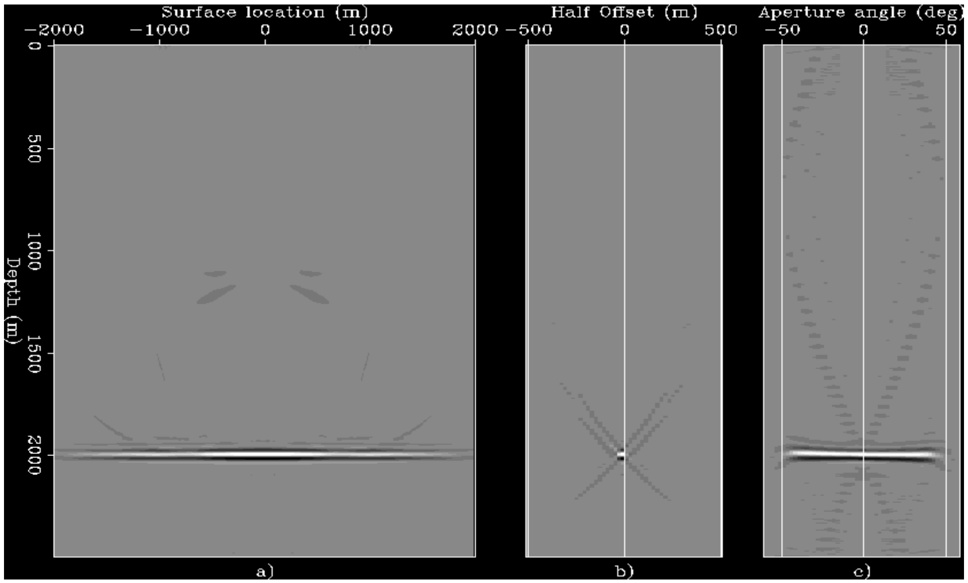

Lens data, shot-geophone migration [B. Biondi, 2002]

Left: Image via DSR. Middle: $\bar{G}[v]d$ - well-focused (at $\mathbf{h} = 0$), i.e. in range of χ to extent possible. Right: Angle CIG.

Quantitative VA

Suppose $W : \mathcal{E}'(\bar{X}) \rightarrow \mathcal{D}'(Z)$ annihilates range of χ:

$$\mathcal{E}'(X) \overset{\chi}{\rightarrow} \mathcal{E}'(\bar{X}) \overset{W}{\rightarrow} \mathcal{D}'(Z) \rightarrow 0$$

and moreover W is bounded on $L^2(\bar{X})$. Then

$$J[v;d] = \frac{1}{2}\|W\bar{G}[v]d\|^2$$

minimized when $[v, \eta\bar{G}[v]d]$ solves partially linearized inverse problem.

Construction of *annihilator* of $\mathcal{R}(F[v])$ (Guillemin, 1985):

$$d \in \mathcal{R}(F[v]) \Leftrightarrow \bar{G}[v]d \in \mathcal{R}(\chi) \Leftrightarrow W\bar{G}[v]d = 0$$

Annihilators, annihilators everywhere...

For Standard Extended Model, several popular choices:

- $W = (I - \Delta)^{-\frac{1}{2}}\nabla_{\mathbf{h}}$ ("differential semblance" - WWS, 1986)
- $W = I - \frac{1}{|H|}\int\, dh$ ("stack power" - Toldi, 1985)
- $W = I - \chi F[v]^{\dagger}\bar{F}[v] \Rightarrow$ minimizing $J[v,d]$ equivalent to least squares.

For Claerbout extension, differential semblance $W = h$.

But not many are good for much...

Since *problem is huge*, only W giving rise to differentiable $v \mapsto J[v,d]$ are useful - must be able to use Newton!!! Once again, idealize $w(t) = \delta(t)$.

Theorem (Stolk & WWS, 2003): $v \mapsto J[v,d]$ smooth $\Leftrightarrow$ W pseudodifferential.

i.e. only *differential semblance* gives rise to smooth optimization problem, *uniformly in source bandwidth*.

Numerical examples using synthetic and field data: WWS et al., Chauris & Noble 2001, Mulder & tenKroode 2002. deHoop et al. 2004.

Example: NMO-based Differential Semblance

$$J[v,d] = \frac{1}{2}\left\|\frac{\partial}{\partial h}N[v]d\right\|^2$$

(recall that $N[v]$ is the NMO operator = composition with $t(z,h)$)

Theory: under some circumstances, can show that *all stationary points are global minimizers* (WWS, TRIP annual reports ’99, ’01).

Example uses data from North Sea survey (thanks: Shell) with light preprocessing: cutoff (“mute”) and multiple suppression (predictive decon) to enhance conformance with model, low pass filter.

Minimization of J via quasi-Newton method.

Beyond Born

Nonlinear effects not included in linearized model: *multiple reflections*. Conventional approach: treat as *coherent noise*, attempt to eliminate - active area of research going back 40+ years, with recent important developments.

Why not model this "noise"?

Proposal: *nonlinear extensions* with $F[v]r$ replaced by $\mathcal{F}[c]$. Create annihilators in same way (now also nonlinear), optimize differential semblance.

Nonlinear analog of Standard Extended Model appears to be *invertible* - in fact extended nonlinear inverse problem is *underdetermined*.

Open problems: no theory. Also must determine $w(t)$ (Lailly SEG 2003).

And so on...

- Elasticity: theory of asymptotic Born inversion at smooth background in good shape (Beylkin & Burridge 1988, deHoop & Bleistein 1997). Theory of extensions, annihilators, differential semblance partially complete (Brandsberg-Dahl et al 2003).
- Anisotropy - work of deHoop (Brandsberg-Dahl et al 2003).
- Anelasticity - in the sedimentary section, $Q = 100 - 1000$, lower in gassy sediments and near surface. No mathematical results, but some numerics - Minkoff & WWS 1997, Blanch et al 1998.
- Source determination - actually always an issue. Some success in casting as an inverse problem - Minkoff & WWS 1997, Routh et al SEG 2003.
- ...