Mathematics of Seismic Imaging
Part I

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PIMS, July 2005

A mathematical view

...of reflection seismic imaging, as practiced in the petroleum industry:

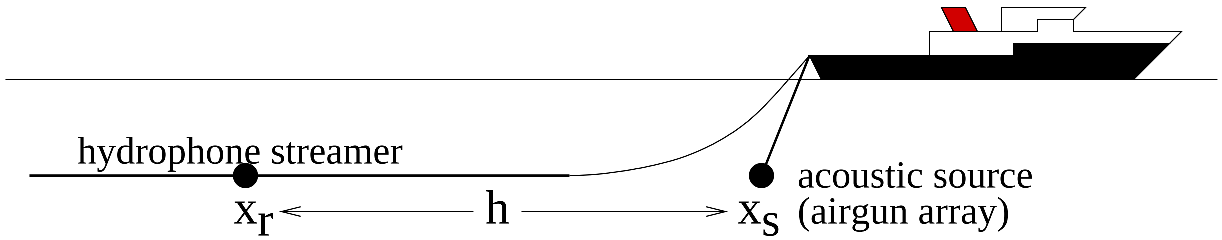
- an inverse problem, based on a model of seismic wave propagation
- contemporary practice relies on *partial linearization* and high-frequency asymptotics
- recent progress in understanding capabilities, limitations of methods based on linearization/asymptotics in presence of *strong refraction*: applications of *microlocal analysis* with implications for practice
- limitations of linearization lead to many open problems

Agenda

1. The reflection seismic experiment, nature of data and of Earth mechanical fields, the acoustic model, linearization and its limitations, definition of imaging based on high frequency asymptotics, geometric optics analysis of the model-data relationship and the GRT representation, zero-offset migration, standard processing = layered imaging
2. Analysis of GRT migration, asymptotic inversion, difficulties due to multipathing, global theory of imaging, ”wave equation” imaging;
3. The partially linearized inverse problem (“velocity analysis”), extended models, importance of invertibility, geometric optics of extensions, some invertible extensions, automating the solution of the partially linearized inverse problem via differential semblance.

Marine reflection seismology

- acoustic source (airgun array, explosives,...)
- acoustic receivers (hydrophone streamer, ocean bottom cable,...)
- recording and onboard processing



Land acquisition similar, but acquisition and processing are more complex. Vast bulk (90%+) of data acquired each year is marine.

Data parameters: time t, source location $\mathbf{x}_s$, and receiver location $\mathbf{x}_r$ or *half offset* $\mathbf{h} = \frac{\mathbf{x}_r - \mathbf{x}_s}{2}$, $h = |\mathbf{h}|$.

Idealized marine "streamer" geometry: $\mathbf{x}_s$ and $\mathbf{x}_r$ lie roughly on constant depth plane, source-receiver lines are parallel $\rightarrow$ 3 spatial degrees of freedom (eg. $\mathbf{x}_s, h$): *codimension 1*. [Other geometries are interesting, eg. ocean bottom cables, but streamer surveys still prevalent.]

How much data? Contemporary surveys may feature

- Simultaneous recording by multiple streamers (up to 12!)
- Many (roughly) parallel ship tracks ("lines"), areal coverage
- single line ("2D") $\sim$ Gbyte; multiple lines ("3D") $\sim$ Tbyte

NB: *In these lectures, will largely ignore sampling issues and treat data as continuously sampled.* First of many approximations...

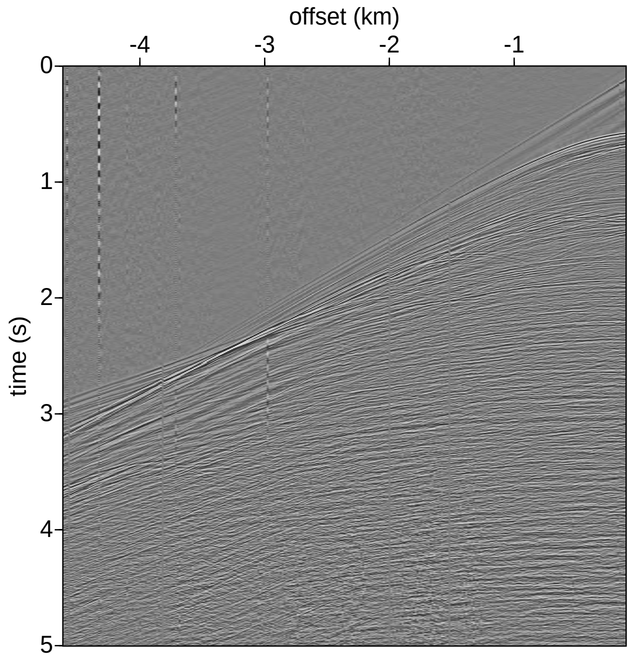
Gathers: distinguished data subsets

Aka “bins”, extracted from data after acquisition.

Characterized by common value of an acquisition parameter

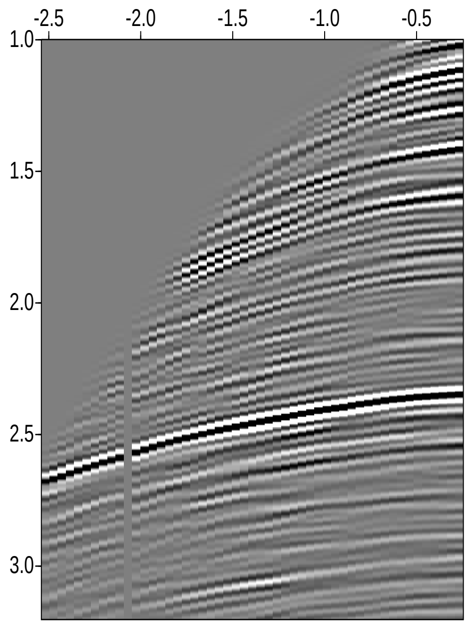
- shot (or common source) gather: traces with same shot location $\mathbf{x}_s$ (previous expls)
- offset (or common offset) gather: traces with same half offset h
- ...

Shot gather, Mississippi Canyon



(thanks: Exxon)

Lightly processed...see the waves!



bandpass filter 4-10-25-40 Hz, mute

A key observation

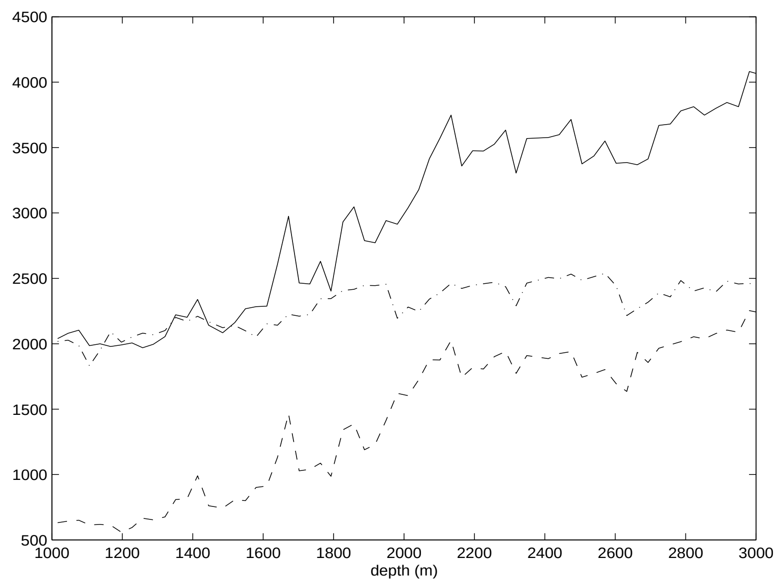
The most striking visual characteristic of seismic reflection data: presence of wave events ("reflections") = coherent space-time structures.

What features in the subsurface structure cause reflections to occur?

Abrupt (wavelength scale) changes in material mechanics act as internal boundaries, causing reflection of waves.

What is the mechanism through which this occurs?

Well logs: a “direct” view of the subsurface



Blocked logs from well in North Sea (thanks: Mobil R & D). Solid: p-wave velocity (m/s), dashed: s-wave velocity (m/s), dash-dot: density (kg/m^3). “Blocked” means “averaged” (over 30 m windows). Original sample rate of log tool < 1 m. **Reflectors** = jumps in velocities, density, **velocity trends**.

The Modeling Task

A useful model of the reflection seismology experiment must

- predict wave motion
- produce reflections from reflectors
- accomodate significant variation of wave velocity, material density,...

A *really good* model will also accomodate

- multiple wave modes, speeds
- material anisotropy
- attenuation, frequency dispersion of waves
- complex source, receiver characteristics

The Acoustic Model

Not *really good*, but good enough for this week and basis of most contemporary processing.

Relates $\rho(\mathbf{x})$= material density, $\lambda(\mathbf{x})$ = bulk modulus, $p(\mathbf{x},t)$= pressure, $\mathbf{v}(\mathbf{x},t)$ = particle velocity, $\mathbf{f}(\mathbf{x},t)$= force density (sound source):

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{f},$$

$$\frac{\partial p}{\partial t} = -\lambda \nabla \cdot \mathbf{v} \ (+\mathrm{i.c.}'\mathrm{s}, \mathrm{b.c.}'\mathrm{s})$$

(compressional) wave speed $c = \sqrt{\frac{\lambda}{\rho}}$

acoustic field potential $u(\mathbf{x},t) = \int_{-\infty}^{t} ds\, p(\mathbf{x},s)$:

$$p = \frac{\partial u}{\partial t},\ \ \mathbf{v} = \frac{1}{\rho}\nabla u$$

Equivalent form: second order wave equation for potential

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla u = \int_{-\infty}^{t} dt\, \nabla \cdot \left(\frac{\mathbf{f}}{\rho}\right) \equiv \frac{f}{\rho}$$

plus initial, boundary conditions.

Theory

Weak solution of Dirichlet problem in $\Omega \subset \mathbf{R}^3$ (similar treatment for other b. c.'s):

$$u \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H_0^1(\Omega))$$

satisfying for any $\phi \in C_0^\infty((0,T) \times \Omega)$,

$$\int_0^T \int_\Omega dt\, dx \left\{ \frac{1}{\rho c^2} \frac{\partial u}{\partial t} \frac{\partial \phi}{\partial t} - \frac{1}{\rho} \nabla u \cdot \nabla \phi + \frac{1}{\rho} f \phi \right\} = 0$$

Theorem (Lions, 1972) Suppose that $\log \rho, \log c \in L^\infty(\Omega)$, $f \in L^2(\Omega \times \mathbf{R})$. Then weak solutions of Dirichlet problem exist, uniquely determined by initial data

$$u(\cdot, 0) \in H_0^1(\Omega),\ \frac{\partial u}{\partial t}(\cdot, 0) \in L^2(\Omega)$$

NB: No hint of waves here...

Further idealizations

- density is constant,
- source force density is *isotropic point radiator with known time dependence* ("source pulse" $w(t)$)

$$f(\mathbf{x}, t; \mathbf{x}_s) = w(t)\delta(\mathbf{x} - \mathbf{x}_s)$$

$\Rightarrow$ acoustic potential, pressure depends on $\mathbf{x}_s$ also.

Forward map $\mathcal{F}[c]$ = time history of pressure for each $\mathbf{x}_s$ at receiver locations $\mathbf{x}_r$ (predicted seismic data), as function of velocity field $c(\mathbf{x})$:

$$\mathcal{F}[c] = \{p(\mathbf{x}_r, t; \mathbf{x}_s)\}$$

Reflection seismic inverse problem

given *observed seismic data* d, find c so that

$$\mathcal{F}[c] \simeq d$$

This inverse problem is

- large scale - up to Tbytes, Pflops
- nonlinear
- yields to no known direct attack

Partial linearization

Almost all useful technology to date relies on partial linearization: write $c = v(1+r)$ and treat r as relative first order perturbation about v, resulting in perturbation of presure field $\delta p = \frac{\partial \delta u}{\partial t} = 0, t \leq 0$, where

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta u = \frac{2r}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Define **linearized forward map** F by

$$F[v]r = \{\delta p(\mathbf{x}_r, t; \mathbf{x}_s)\}$$

Analysis of $F[v]$ is the main content of contemporary reflection seismic theory.

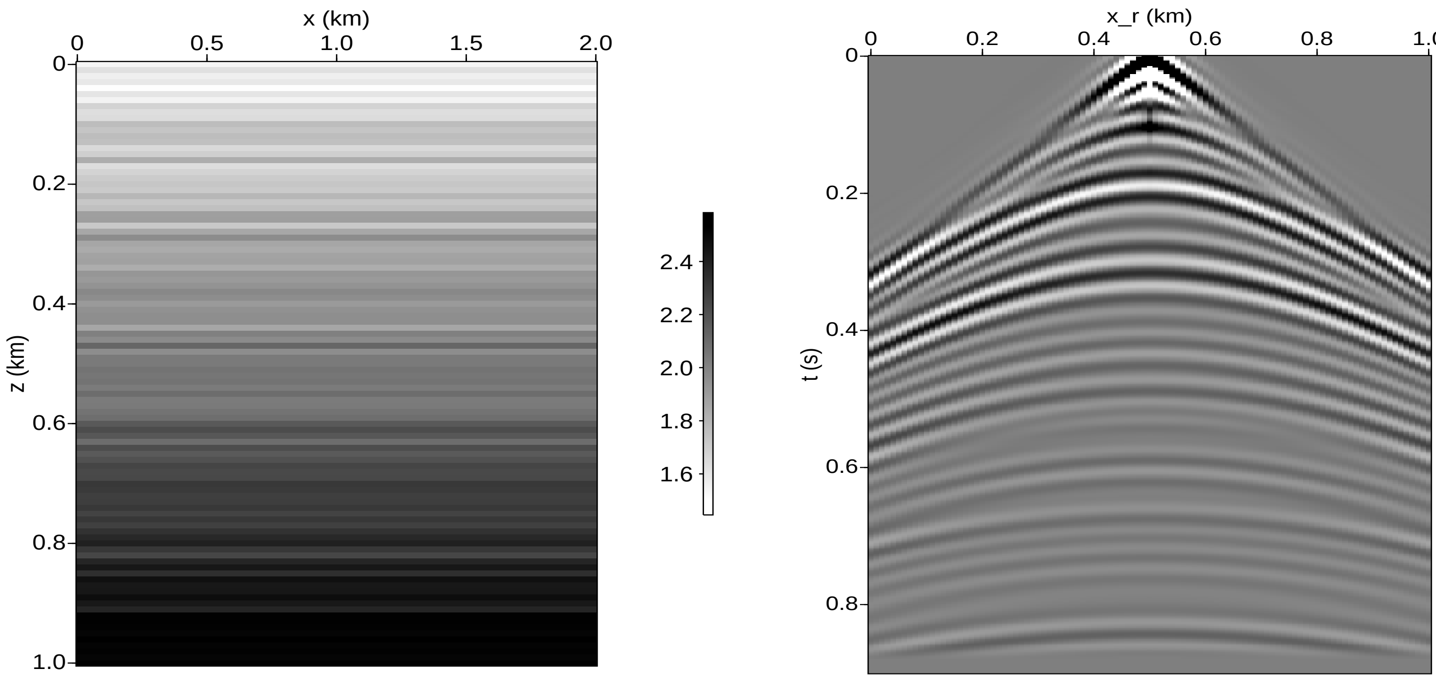
Linearization error

Critical question: If there is any justice $F[v]r$ = directional derivative $D\mathcal{F}[v][vr]$ of $\mathcal{F}$ - but in what sense? Physical intuition, numerical simulation, and not nearly enough mathematics: linearization error

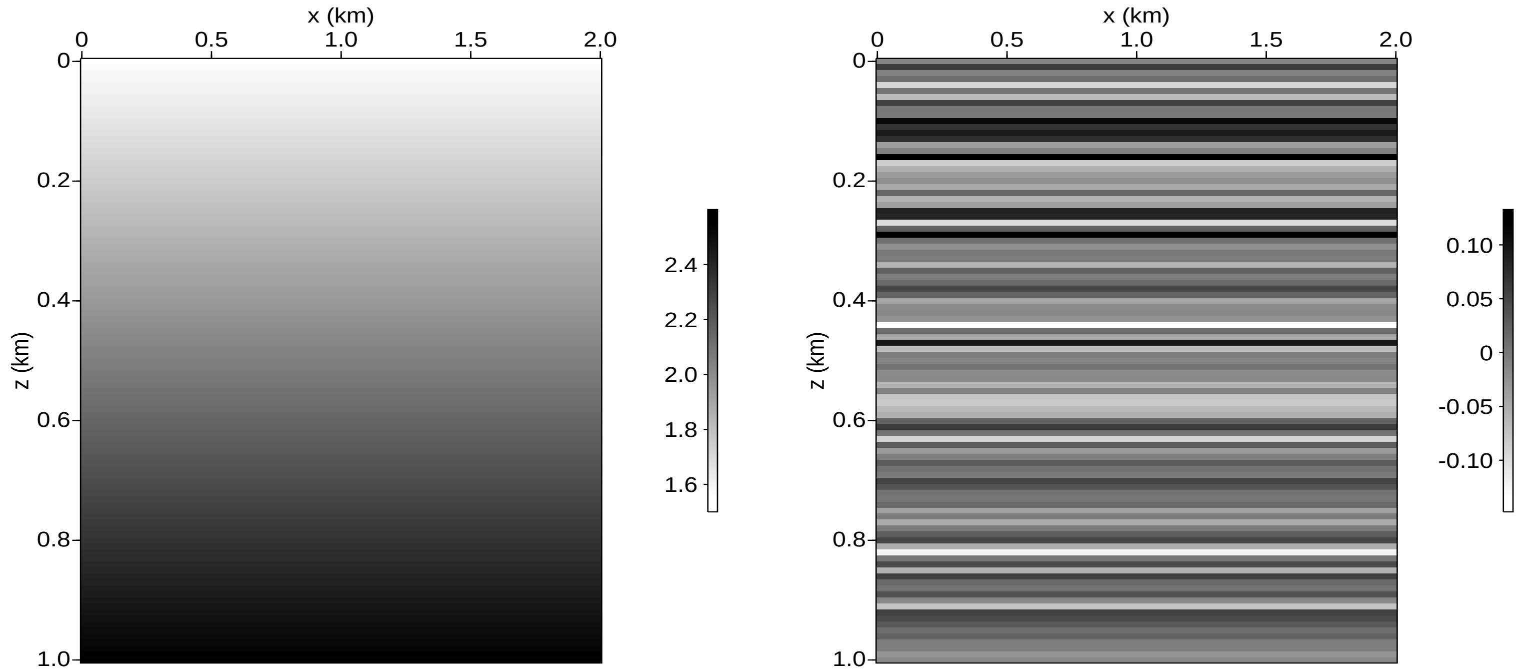
$$\mathcal{F}[v(1+r)] - (\mathcal{F}[v] + F[v]r)$$

- *small* when v smooth, r rough or oscillatory on wavelength scale - well-separated scales
- *large* when v not smooth and/or r not oscillatory - poorly separated scales

2D finite difference simulation: shot gathers with typical marine seismic geometry. Smooth (linear) $v(x,z)$, oscillatory (random) $r(x,z)$ depending only on z("layered medium"). Source wavelet $w(t)$ = bandpass filter.



Left: Total velocity $c = v(1 + r)$ with smooth (linear) background $v(x, z)$, oscillatory (random) $r(x, z)$. Std dev of r = 5%.
Right: Simulated seismic response ($\mathcal{F}[v(1 + r)]$), wavelet = bandpass filter 4-10-30-45 Hz. Simulator is (2,4) finite difference scheme.



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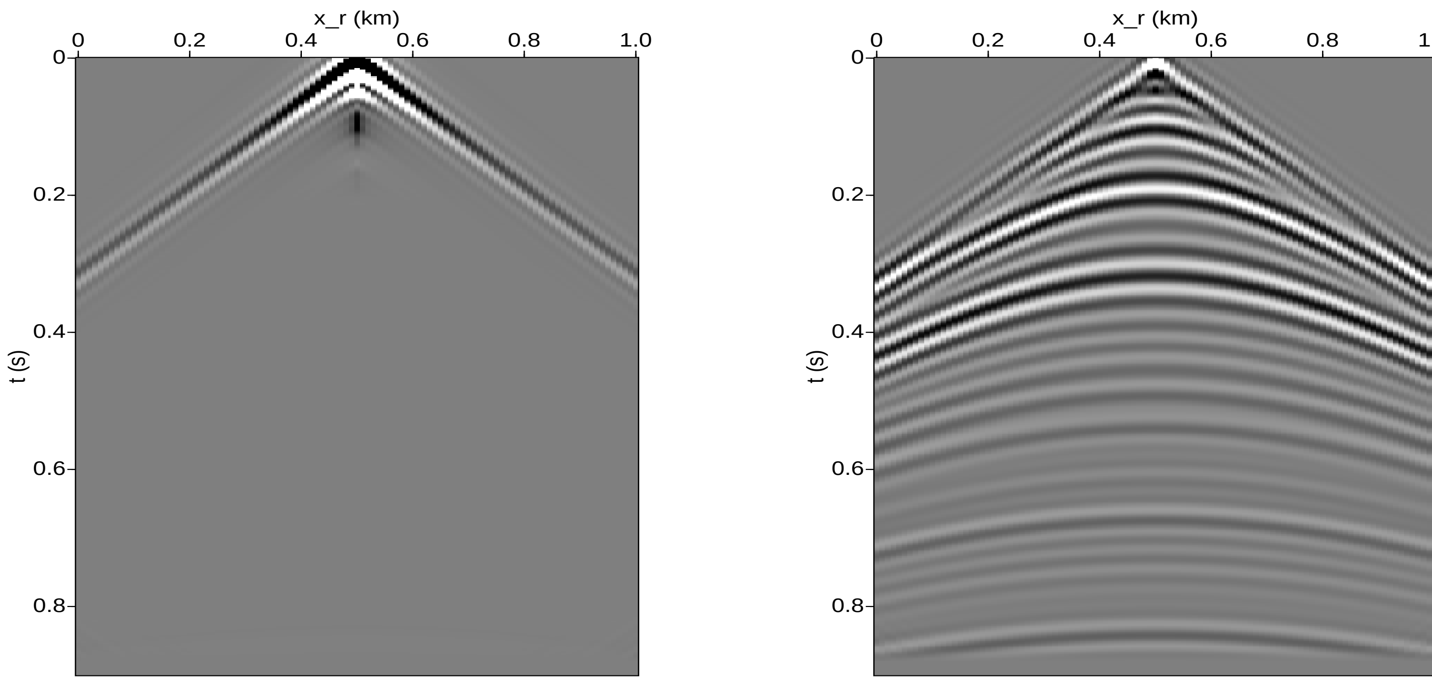
Model in previous slide as smooth background (left, $v(x, z)$) plus rough perturbation (right, $r(x, z)$).



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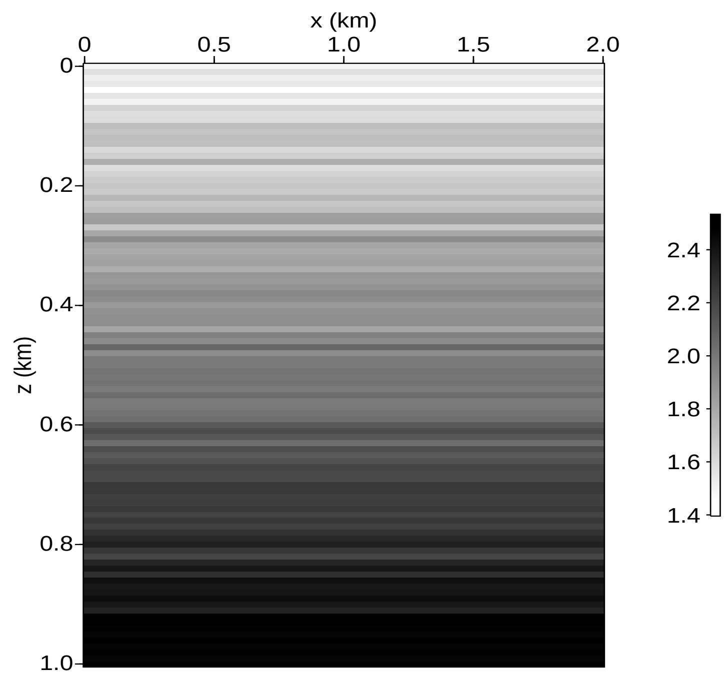
Left: Simulated seismic response of smooth model ($\mathcal{F}[v]$),
Right: Simulated linearized response, rough perturbation of smooth model ($F[v]r$)



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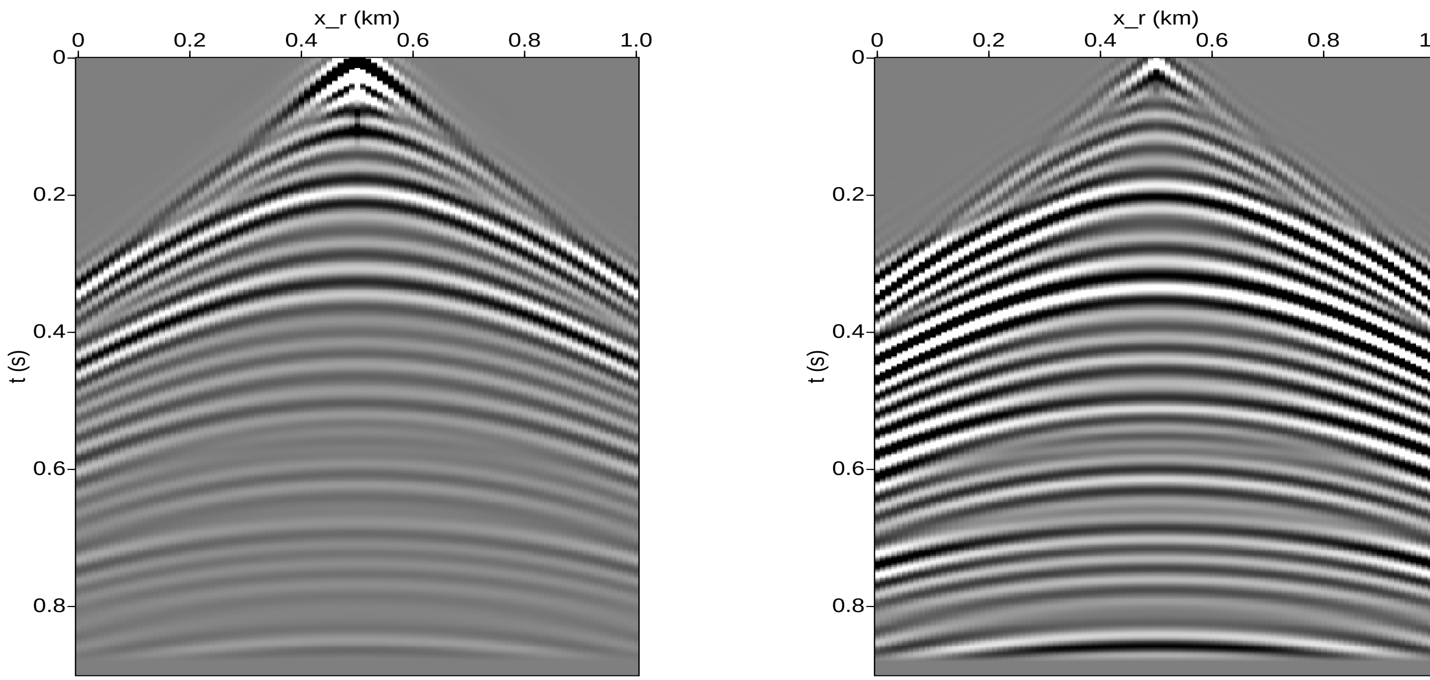
Model in previous slide as rough background (left, $v(x, z)$) plus smooth 5% perturbation ($r(x, z)$).



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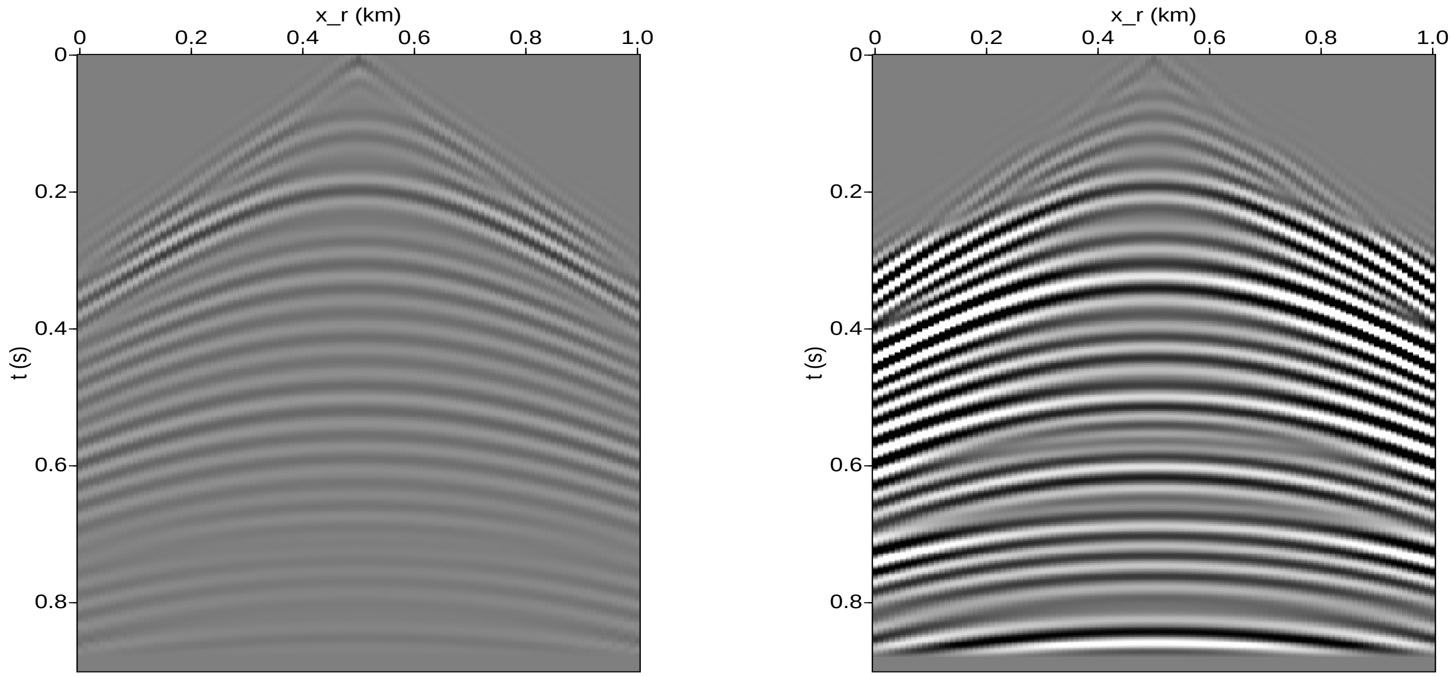
Left: Simulated seismic response of rough model ($\mathcal{F}[v]$),
Right: Simulated linearized response, smooth perturbation of rough model ($F[v]r$)



Left: linearization error ($\mathcal{F}[v(1+r)] - \mathcal{F}[v] - F[v]r$), rough perturbation of smooth background

Right: linearization error, smooth perturbation of rough background (plotted with same grey scale).

Summary

- v smooth, r oscillatory $\Rightarrow$ $F[v]r$ approximates **primary reflection** = result of wave interacting with material heterogeneity only once (single scattering); error consists of **multiple reflections**, which are "not too large" if r is "not too big", and sometimes can be suppressed.
- v nonsmooth, r smooth $\Rightarrow$ error consists of *time shifts* in waves which are very large perturbations as waves are oscillatory.

No mathematical results are known which justify/explain these observations in any rigorous way, except in 1D.

Velocity Analysis and Imaging

Velocity analysis problem = partially linearized inverse problem: given d find v, r so that

$$S[v] + F[v]r \simeq d$$

Imaging problem = linear subproblem: given d and v, find r so that

$$F[v]r \simeq d - S[v]$$

Last 20 years:

- much progress on imaging
- much less on velocity analysis

Aymptotic assumption

Linearization is accurate $\Leftrightarrow$ length scale of $v >> $ length scale of $r \simeq$ wavelength, properties of $F[v]$ dominated by those of $F_\delta[v]$ (= $F[v]$ with $w = \delta$). Implicit in migration concept (eg. Hagedoorn, 1954); explicit use: Cohen & Bleistein, SIAM JAM 1977.

Key idea: **reflectors** (rapid changes in r) emulate *singularities*; **reflections** (rapidly oscillating features in data) also emulate singularities.

NB: "everybody's favorite reflector": the smooth interface across which r jumps. *But* this is an oversimplification - reflectors in the Earth may be complex zones of rapid change, pehaps in all directions. More flexible notion needed!!

Wave Front Sets

Recall characterization of smoothness via Fourier transform: $u \in \mathcal{D}'(\mathbf{R}^n)$ is smooth at $\mathbf{x}_0 \Leftrightarrow$ for some nbhd X of $\mathbf{x}_0$, any $\phi \in \mathcal{E}(X)$ and N, there is $C_N \geq 0$ so that for any $\boldsymbol{\xi} \neq 0$,

$$\left|\mathcal{F}(\phi u)\left(\tau\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)\right| \leq C_N \tau^{-N}$$

Harmonic analysis of singularities, *après* Hörmander: the **wave front set** $WF(u) \subset \mathbf{R}^n \times \mathbf{R}^n - \{0\}$ of $u \in \mathcal{D}'(\mathbf{R}^n)$ - captures orientation as well as position of singularities.

$(\mathbf{x}_0, \boldsymbol{\xi}_0) \notin WF(u) \Leftrightarrow$, there is some open nbhd $X \times \Xi \subset \mathbf{R}^n \times \mathbf{R}^n - \{0\}$ of $(\mathbf{x}_0, \xi_0)$ so that for any $\phi \in \mathcal{E}(X)$, N, there is $C_N \geq 0$ so that for all $\boldsymbol{\xi} \in \Xi$,

$$\left|\mathcal{F}(\phi u)\left(\tau\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)\right| \leq C_N \tau^{-N}$$

Housekeeping chores

(i) note that the nbhds Ξ may naturally be taken to be *cones*;

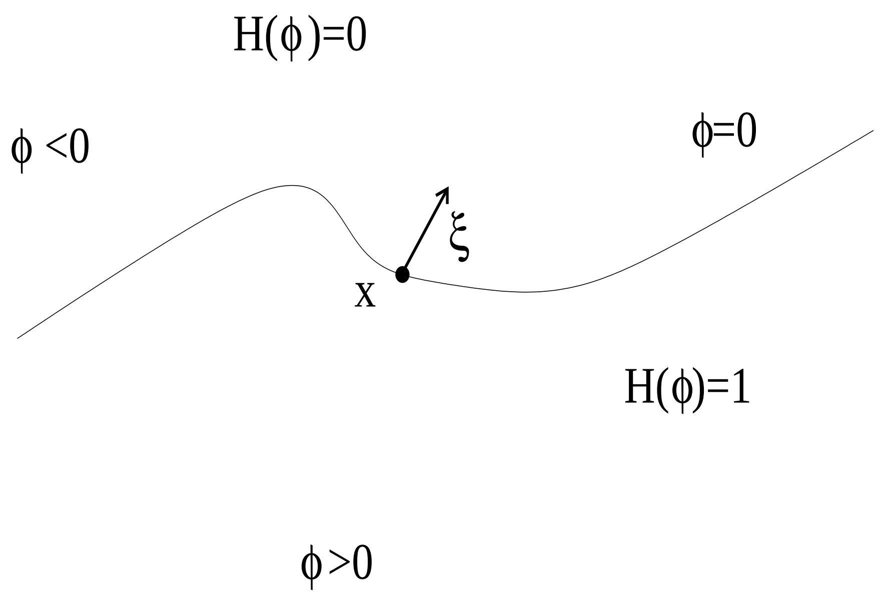
(ii) u is smooth at $\mathbf{x}_0 \Leftrightarrow (\mathbf{x}_0, \boldsymbol{\xi}_0) \notin WF(u)$ for all $\boldsymbol{\xi}_0 \in \mathbf{R}^n - \{0\}$;

(iii) $WF(u)$ is invariant under chg. of coords if it is regarded as a subset of the *cotangent bundle* $T^*(\mathbf{R}^n)$ (i.e. the ξ components transform as covectors).

[Good refs: Duistermaat, 1996; Taylor, 1981; Hörmander, 1983]

The standard example: if u jumps across the interface $f(\mathbf{x}) = 0$, otherwise smooth, then $WF(u) \subset \mathcal{N}_f = \{(\mathbf{x}, \boldsymbol{\xi}) : f(\mathbf{x}) = 0,\ \boldsymbol{\xi} || \nabla f(\mathbf{x})\}$ (*normal bundle* of $f = 0$).

Wavefront set of a jump discontinuity



$$WF(H(\phi)) = \{(\mathbf{x}, \boldsymbol{\xi}) : \phi(\mathbf{x}) = 0,\ \boldsymbol{\xi} || \nabla\phi(\mathbf{x})\}$$

Microlocal property of differential operators

Suppose $u \in \mathcal{D}'(\mathbf{R}^n)$, $(\mathbf{x}_0, \boldsymbol{\xi}_0) \notin WF(u)$, and $P(\mathbf{x}, D)$ is a partial differential operator:

$$P(\mathbf{x}, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

$$D = (D_1, ..., D_n),\ \ D_i = -i\frac{\partial}{\partial x_i}$$

$$\alpha = (\alpha_1, ..., \alpha_n),\ \ |\alpha| = \sum_i \alpha_i,$$

$$D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$$

Then $(\mathbf{x}_0, \xi_0) \notin WF(P(\mathbf{x}, D)u)$ [i.e.: $WF(Pu) \subset WF(u)$].

Proof

Choose $X \times \Xi$ as in the definition, $\phi \in \mathcal{D}(X)$ form the required Fourier transform

$$\int dx\, e^{i\mathbf{x}\cdot(\tau\xi)}\phi(\mathbf{x})P(\mathbf{x}, D)u(\mathbf{x})$$

and start integrating by parts: eventually

$$= \sum_{|\alpha|\leq m} \tau^{|\alpha|}\xi^{\alpha} \int dx\, e^{i\mathbf{x}\cdot(\tau\xi)}\phi_{\alpha}(\mathbf{x})u(\mathbf{x})$$

where $\phi_\alpha \in \mathcal{D}(X)$ is a linear combination of derivatives of ϕ and the a_αs. Since each integral is rapidly decreasing as $\tau \to \infty$ for $\xi \in \Xi$, it remains rapidly decreasing after multiplication by $\tau^{|\alpha|}$, and so does the sum. **Q. E. D.**

Formalizing the reflector concept

Key idea, restated: reflectors (or "reflecting elements") will be points in $WF(r)$. Reflections will be points in $WF(d)$.

These ideas lead to a usable definition of *image*: a reflectivity model $\tilde{r}$ is an image of r if $WF(\tilde{r}) \subset WF(r)$ (the closer to equality, the better the image).

Idealized **migration problem**: given d (hence $WF(d)$) deduce somehow a function which has *the right reflectors*, i.e. a function $\tilde{r}$ with $WF(\tilde{r}) \simeq WF(r)$.

NB: you're going to need v! ("It all depends on v(x,y,z)" - J. Claerbout)

Integral representation of linearized operator

With $w = \delta$, acoustic potential u is same as Causal Green's function $G(\mathbf{x}, t; \mathbf{x}_s)$ = retarded fundamental solution:

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\mathbf{x}, t; \mathbf{x}_s) = \delta(t)\delta(\mathbf{x} - bx_s)$$

and $G \equiv 0, t < 0$. Then ($w = \delta$!) $p = \frac{\partial G}{\partial t}$, $\delta p = \frac{\partial \delta G}{\partial t}$, and

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta G(\mathbf{x}, t; \mathbf{x}_s) = \frac{2}{v^2(\mathbf{x})}\frac{\partial^2 G}{\partial t^2}(\mathbf{x}, t; \mathbf{x}_s) r(\mathbf{x})$$

Simplification: from now on, define $F[v]r = \delta G|_{\mathbf{x}=\mathbf{x}_r}$ - i.e. lose a t-derivative. Duhamel's principle $\Rightarrow$

$$\delta G(\mathbf{x}_r, t; \mathbf{x}_s) = \int dx \, \frac{2r(\mathbf{x})}{v(\mathbf{x})^2} \int ds \, G(\mathbf{x}_r, t - s; \mathbf{x}) \frac{\partial^2 G}{\partial t^2}(\mathbf{x}, s; \mathbf{x}_s)$$

Add geometric optics...

Geometric optics approximation of G should be good, as v is smooth. Summary: if $\mathbf{x}$ "not too far" from $\mathbf{x}_s$, then

$$G(\mathbf{x},t;\mathbf{x}_s) = a(\mathbf{x};\mathbf{x}_s)\delta(t-\tau(\mathbf{x};\mathbf{x}_s)) + R(\mathbf{x},t;\mathbf{x}_s)$$

where the traveltime $\tau(\mathbf{x};\mathbf{x}_s)$ solves the eikonal equation

$$v|\nabla\tau| = 1,\ \ \tau(\mathbf{x};\mathbf{x}_s) \sim \frac{|\mathbf{x}-\mathbf{x}_s|}{v(\mathbf{x}_s)},\ \ \mathbf{x}\rightarrow\mathbf{x}_s$$

and the amplitude $a(\mathbf{x};\mathbf{x}_s)$ solves the transport equation

$$\nabla\cdot(a^2\nabla\tau) = 0, \ldots$$

Refs: Courant & Hilbert, Friedlander *Sound Pulses*, WWS *Foundations* and many refs cited there...

Simple Geometric Optics

"Not too far" means: there should be one and only one ray of geometric optics connecting each $\mathbf{x}_s$ or $\mathbf{x}_r$ to each $\mathbf{x} \in \mathrm{supp} r$.

Will call this the **simple geometric optics** assumption.

Within region satisfying simple geometric optics assumption, τ is smooth ($\mathbf{x} \neq \mathbf{x}_s$) solution of eikonal equation. Effective methods for numerical solution of eikonal, transport equations: ray tracing (Lagrangian), various sorts of upwind finite difference (Eulerian) methods. See eg. Sethian book, WWS 1999 MGSS notes (online) for details.

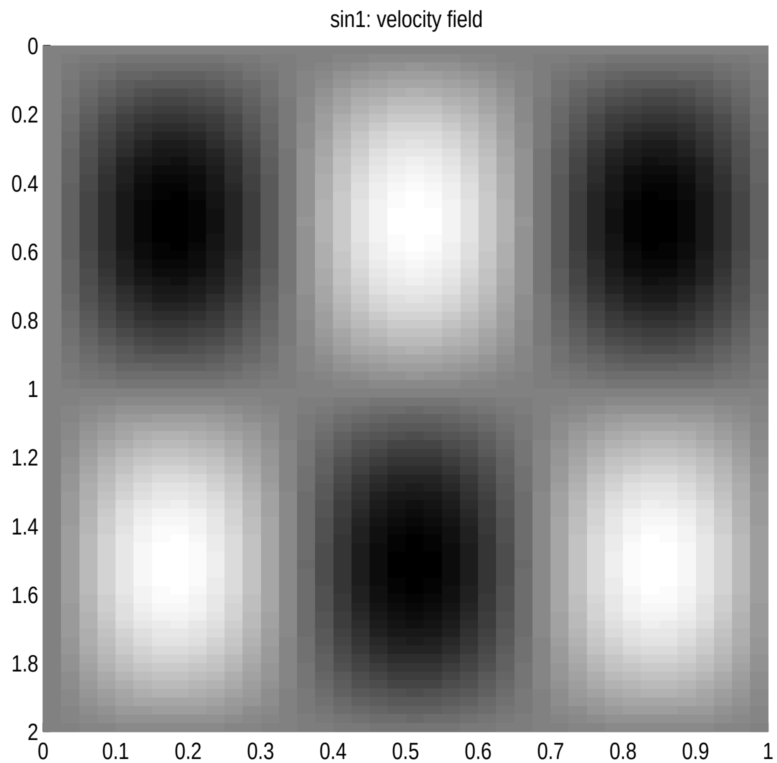
Caution - caustics!

For "random but smooth" $v(\mathbf{x})$ with variance σ, more than one connecting ray occurs as soon as the distance is $O(\sigma^{-2/3})$. Such *multipathing* is invariably accompanied by the formation of a *caustic* = envelope of rays (White, 1982).

Upon caustic formation, the simple geometric optics field description above is no longer correct.

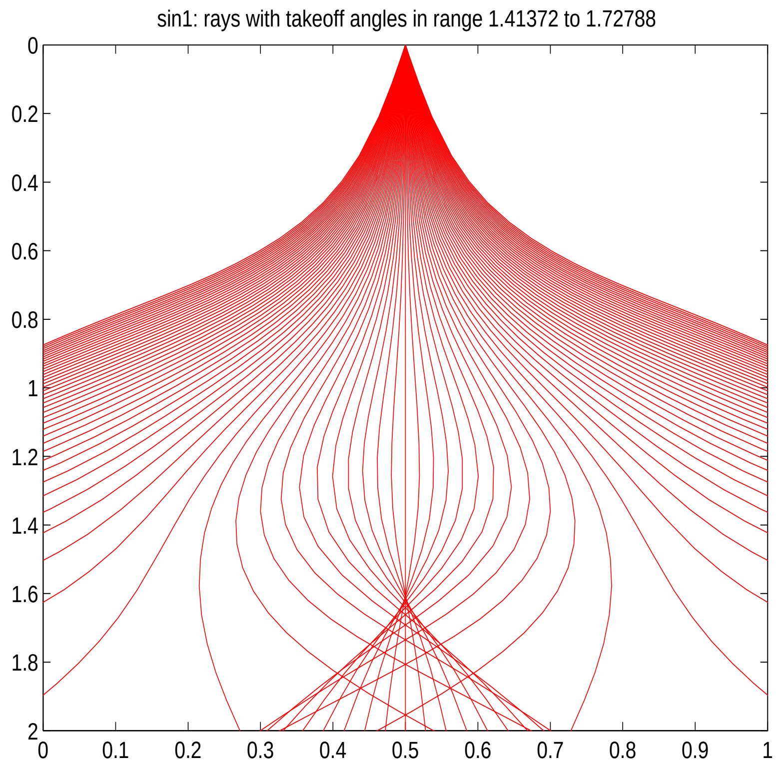
Failure of GO at caustic understood in 19th century. Generalization of GO to regions containing caustics accomplished by Ludwig and Kravtsov, 1966-7, elaborated by Maslov, Hörmander, Duistermaat, many others.

A caustic example (1)



2D Example of strong refraction: Sinusoidal velocity field $v(x, z) = 1+0.2 \sin \frac{\pi z}{2} \sin 3\pi x$

A caustic example (2)



Rays in sinusoidal velocity field, source point = origin. Note formation of caustic, multiple rays to source point in lower center.

An oft-forgotten detail

All of this is meaningful only if the remainder R is small in a suitable sense: energy estimate (**Exercise!**) $\Rightarrow$

$$\int dx \int_0^T dt\,|R(\mathbf{x},t;\mathbf{x}_s)|^2 \leq C\|v\|_{\mathrm{C}^4}$$

(this is an easy, suboptimal estimate - with more work can replace 4 with 2)

If $v \in C^\infty$, can complete the geometric optics approximation of the Green's function so that the difference is C^∞ - then the two sides have the same singularities, ie. the same wavefront set.

Finally, a wave!

The geometric optics approximation to the Green's function

$$G(\mathbf{x}, t; \mathbf{x}_s) \simeq a(\mathbf{x}; \mathbf{x}_s)\delta(t - \tau(\mathbf{x}; \mathbf{x}_s))$$

describes a (singular) quasi-spherical waves [spherical, if $v = $ const., for then $\tau(\mathbf{x}, \mathbf{x}_s) = |\mathbf{x} - \mathbf{x}_s|/v$].

Geometric optics is the the best currently available explanation for waves in heterogeneous media. Note the inadequacy: v must be *smooth*, but the compressional velocity distribution in the Earth varies on all scales!

The linearized operator as Generalized Radon Transform

Assume: $\mathrm{supp}\, r$ contained in simple geometric optics domain (each point reached by unique ray from any source or receiver point).

Then distribution kernel K of $F[v]$ is

$$K(\mathbf{x}_r, t, \mathbf{x}_s; \mathbf{x}) = \int ds\, G(\mathbf{x}_r, t-s; \mathbf{x}) \frac{\partial^2 G}{\partial t^2}(\mathbf{x}, s; \mathbf{x}_s) \frac{2}{v^2(\mathbf{x})}$$

$$\simeq \int ds\, \frac{2a(\mathbf{x}_r, \mathbf{x})a(\mathbf{x}, \mathbf{x}_s)}{v^2(\mathbf{x})} \delta'(t - s - \tau(\mathbf{x}_r, \mathbf{x}))\delta''(s - \tau(\mathbf{x}, \mathbf{x}_s))$$

$$= \frac{2a(\mathbf{x},\mathbf{x}_r)a(\mathbf{x},\mathbf{x}_s)}{v^2(\mathbf{x})}\delta''(t-\tau(\mathbf{x},\mathbf{x}_r)-\tau(\mathbf{x},\mathbf{x}_s))$$

provided that

$$\nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{x}_r)+\nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{x}_s)\neq 0$$

$\Leftrightarrow$ velocity at $\mathbf{x}$ of ray from $\mathbf{x}_s$ **not** negative of velocity of ray from $\mathbf{x}_r$ $\Leftrightarrow$ *no forward scattering*. [Gel'fand and Shilov, 1958 - when is pullback of distribution again a distribution].

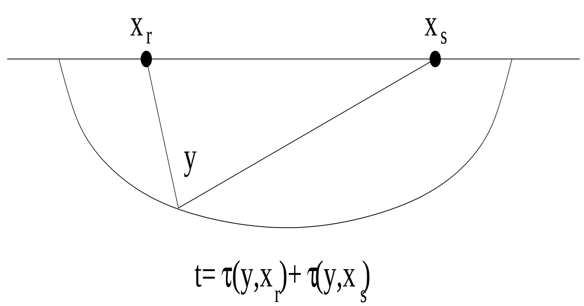
GRT = “Kirchhoff” modeling

So: for r supported in simple geometric optics domain, no forward scattering $\Rightarrow$

$$\delta G(\mathbf{x}_r, t; \mathbf{x}_s) \simeq$$

$$\frac{\partial^2}{\partial t^2} \int dx \, \frac{2r(\mathbf{x})}{v^2(\mathbf{x})} a(\mathbf{x}, \mathbf{x}_r) a(\mathbf{x}, \mathbf{x}_s) \delta(t - \tau(\mathbf{x}, \mathbf{x}_r) - \tau(\mathbf{x}, \mathbf{x}_s))$$

That is: pressure perturbation is sum (integral) of r over *reflection isochron* $\{\mathbf{x} : t = \tau(\mathbf{x}, \mathbf{x}_r) + \tau(\mathbf{x}, \mathbf{x}_s)\}$, w. weighting, filtering. Note: if $v =$const. then isochron is ellipsoid, as $\tau(\mathbf{x}_s, \mathbf{x}) = |\mathbf{x}_s - \mathbf{x}|/v$!



Zero Offset data and the Exploding Reflector

Zero offset data ($\mathbf{x}_s = \mathbf{x}_r$) is seldom actually measured (contrast radar, sonar!), but routinely *approximated* through *NMO-stack* (to be explained later).

Extracting image from zero offset data, rather than from all (100's) of offsets, is tremendous *data reduction* - when approximation is accurate, leads to excellent images.

Imaging basis: the *exploding reflector* model (Claerbout, 1970's).

For zero-offset data, distribution kernel of $F[v]$ is

$$K(\mathbf{x}_s, t, \mathbf{x}_s; \mathbf{x}) = \frac{\partial^2}{\partial t^2} \int ds \, \frac{2}{v^2(\mathbf{x})} G(\mathbf{x}_s, t - s; \mathbf{x}) G(\mathbf{x}, s; \mathbf{x}_s)$$

Under some circumstances (explained below), K (= G time-convolved with itself) is "similar" (also explained) to $\tilde{G}$ = Green's function for $v/2$. Then

$$\delta G(\mathbf{x}_s, t; \mathbf{x}_s) \sim \frac{\partial^2}{\partial t^2} \int dx \, \tilde{G}(\mathbf{x}_s, t, \mathbf{x}) \frac{2r(\mathbf{x})}{v^2(\mathbf{x})}$$

$\sim$ solution w of

$$\left(\frac{4}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) w = \delta(t) \frac{2r}{v^2}$$

Thus reflector "explodes" at time zero, resulting field propagates in "material" with velocity $v/2$.

Explain when the exploding reflector model "works", i.e. when G time-convolved with itself is "similar" to $\tilde{G}$ = Green's function for $v/2$. If supp r lies in simple geometry domain, then

$$K(\mathbf{x}_s, t, \mathbf{x}_s; \mathbf{x}) = \int ds\, \frac{2a^2(\mathbf{x}, \mathbf{x}_s)}{v^2(\mathbf{x})} \delta(t - s - \tau(\mathbf{x}_s, \mathbf{x}))\delta''(s - \tau(\mathbf{x}, \mathbf{x}_s))$$

$$= \frac{2a^2(\mathbf{x}, \mathbf{x}_s)}{v^2(\mathbf{x})} \delta''(t - 2\tau(\mathbf{x}, \mathbf{x}_s))$$

whereas the Green's function $\tilde{G}$ for $v/2$ is

$$\tilde{G}(\mathbf{x}, t; \mathbf{x}_s) = \tilde{a}(\mathbf{x}, \mathbf{x}_s)\delta(t - 2\tau(\mathbf{x}, \mathbf{x}_s))$$

(half velocity = double traveltime, same rays!).

Difference between effects of K, $\tilde{G}$: for each $\mathbf{x}_s$ scale r by smooth fcn - preserves $WF(r)$ hence $WF(F[v]r)$ and relation between them. Also: adjoints have same effect on WF sets.

Upshot: from imaging point of view (i.e. apart from amplitude, derivative (filter)), kernel of $F[v]$ restricted to zero offset is same as Green's function for $v/2$, *provided that simple geometry hypothesis holds:* only one ray connects each source point to each scattering point, ie. *no multipathing*.

See Claerbout, IEI, for examples which demonstrate that multipathing really does invalidate exploding reflector model.

Standard Processing

Inspirational interlude: the sort-of-layered theory =“Standard Processing”

Suppose were v,r functions of $z = x_3$ only, all sources and receivers at $z = 0$. Then the entire system is translation-invariant in $x_1, x_2 \Rightarrow$ Green’s function G its perturbation δG, and the idealized data $\delta G|_{z=0}$ are really only functions of t, z, and *half-offset* $h = |\mathbf{x}_s - \mathbf{x}_r|/2$. There would be *only one seismic experiment*, equivalent to any *common midpoint gather* (“CMP”).

This isn’t really true - *look at the data!!!* However it is *approximately* correct in many places in the world: CMPs change very slowly with midpoint $\mathbf{x}_m = (\mathbf{x}_r + \mathbf{x}_s)/2$.

Standard processing: treat each CMP *as if it were the result of an experiment performed over a layered medium*, but permit the layers to vary with midpoint.

Thus $v = v(z), r = r(z)$ for purposes of analysis, but at the end $v = v(\mathbf{x}_m, z), r = r(\mathbf{x}_m, z)$.

$$F[v]r(\mathbf{x}_r, t; \mathbf{x}_s)$$

$$\simeq \int dx \frac{2r(z)}{v^2(z)} a(\mathbf{x}, x_r) a(\mathbf{x}, x_s) \delta''(t - \tau(\mathbf{x}, x_r) - \tau(\mathbf{x}, x_s))$$

$$= \int dz \, \frac{2r(z)}{v^2(z)} \int d\omega \int dx \omega^2 a(\mathbf{x}, x_r) a(\mathbf{x}, x_s) e^{i\omega(t - \tau(\mathbf{x}, x_r) - \tau(\mathbf{x}, x_s))}$$

Since we have already thrown away smoother (lower frequency) terms, do it again using *stationary phase.* Upshot (see 2000 MGSS notes for details): up to smoother (lower frequency) error,

$$F[v]r(h,t) \simeq A(z(h,t),h)R(z(h,t))$$

Here $z(h,t)$ is the inverse of the 2-way traveltime

$$t(h,z) = 2\tau((h,0,z),(0,0,0))$$

i.e. $z(t(h,z'),h) = z'$. R is (yet another version of) "reflectivity"

$$R(z) = \frac{1}{2}\frac{dr}{dz}(z)$$

That is, $F[v]$ *is a a derivative followed by a change of variable followed by multiplication by a smooth function*. Substitute t_0 (vertical travel time) for z (depth) and you get "Inverse NMO" ($t_0 \rightarrow (t,h)$). Will be sloppy and call $z \rightarrow (t,h)$ INMO.

Anatomy of an adjoint

$$\int dt \int dh\, d(t,h) F[v] r(t,h) = \int dt \int dh\, d(t,h) A(z(t,h),h) R(z(t,h))$$

$$= \int dz\, R(z) \int dh\, \frac{\partial t}{\partial z}(z,h) A(z,h) d(t(z,h),h) = \int dz\, r(z) (F[v]^* d)(z)$$

so $F[v]^* = -\frac{\partial}{\partial z} S M[v] N[v]$, where

- $N[v]$ = **NMO operator** $N[v]d(z,h) = d(t(z,h),h)$
- $M[v]$ = multiplication by $\frac{\partial t}{\partial z} A$
- S = **stacking operator** $Sf(z) = \int dh\, f(z,h)$

Normal Op is PDO $\Rightarrow$ Imaging

$$F[v]^*F[v]r(z) = -\frac{\partial}{\partial z}\left[\int dh\,\frac{dt}{dz}(z,h)A^2(z,h)\right]\frac{\partial}{\partial z}r(z)$$

Microlocal property of PDOs $\Rightarrow WF(F[v]^*F[v]r) \subset WF(r)$ i.e. $F[v]^*$ *is an imaging operator*.

If you leave out the amplitude factor ($M[v]$) and the derivatives, as is commonly done, then you get essentially the same expression - so (NMO, stack) is an imaging operator!

It's even easy to get an (asymptotic) inverse out of this - exercise for the reader.

Now make everything dependent on $\mathbf{x}_m$ and you've got standard processing. (end of layered interlude).

But the Earth is not layered!

In general,

Is $F[v]^*$ an imaging operator?

What sort of thing is $F[v]^*F[v]$??

Stay tuned!