Seismic Inverse Scattering

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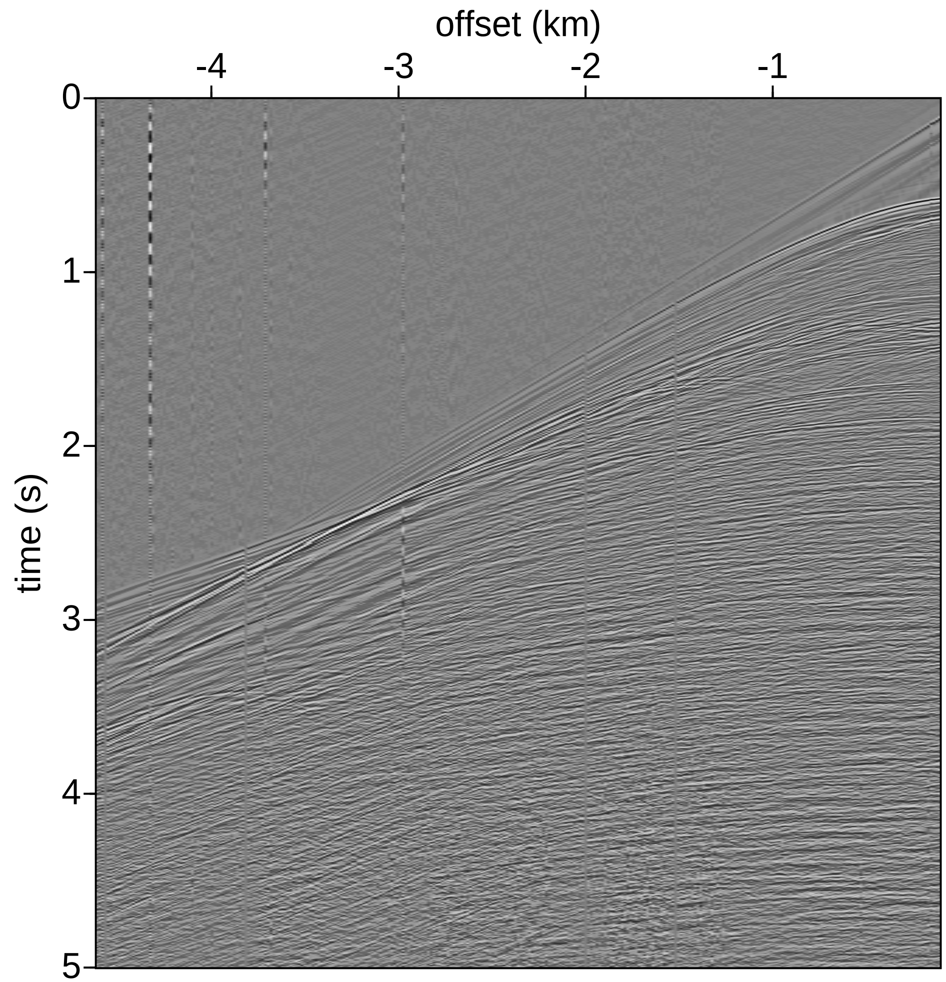
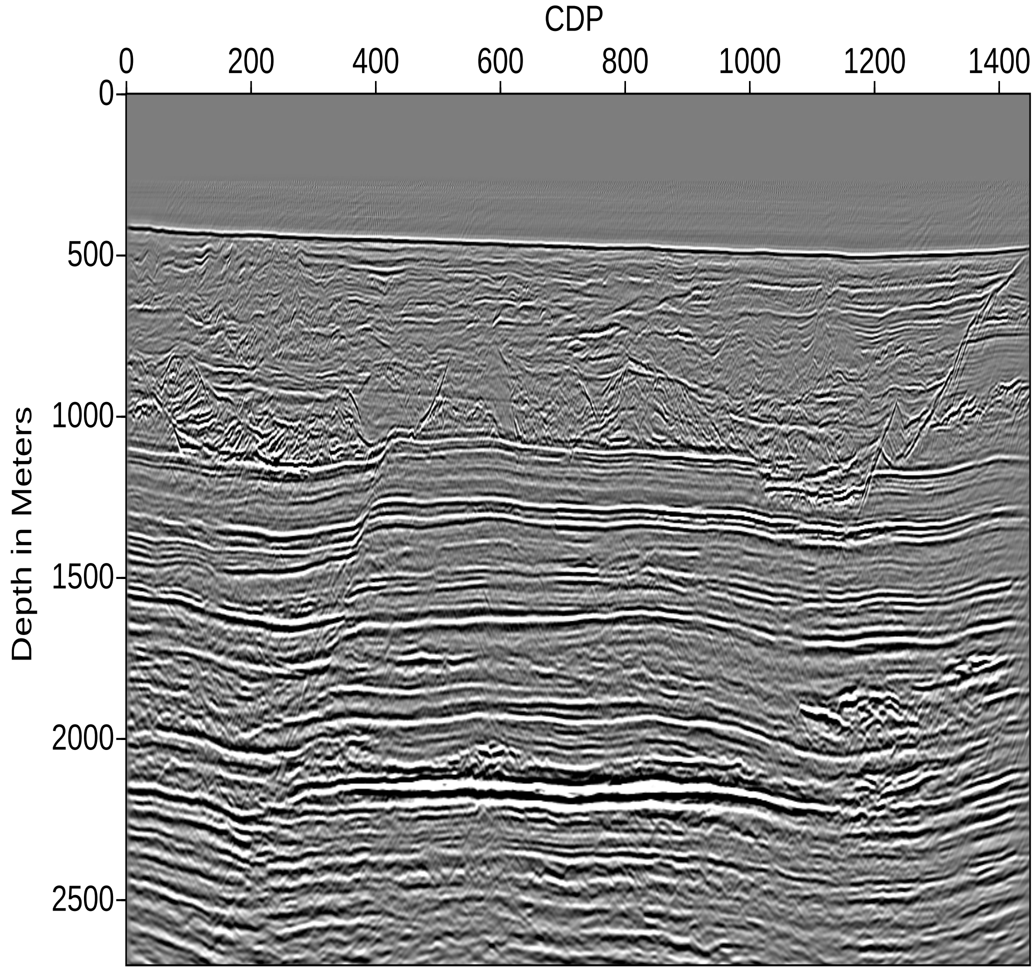


How do you turn lots of this... (field seismogram from the Gulf of Mexico - thanks: Exxon.)



into this (a fair rendition of subsurface structure)?

Some other questions, inspired by Claerbout and Biondi:

- Why all the emphasis on linear operators? Isn't the seismic inverse problem nonlinear?
- Why are adjoints so important (usually adjoint + \$3.95 = coffee at Starbuck's)
- What are really the connections between NMO, stacking, migration, velocity analysis? Why does the exploding reflector model work?
- Do downward continuation and Kirchhoff migration do the same thing?
- How is 2D different from 3D?

A mathematical view of reflection seismic processing:

- an inverse problem, based on a model of seismic wave propagation

- contemporary practice relies on *partial linearization* and high-frequency asymptotics

- recent progress in understanding capabilities, limitations of methods based on linearization/asymptotics in presence of *strong refraction*: applications of *microlocal analysis* with implications for practice

- limitations of linearization lead to many open problems

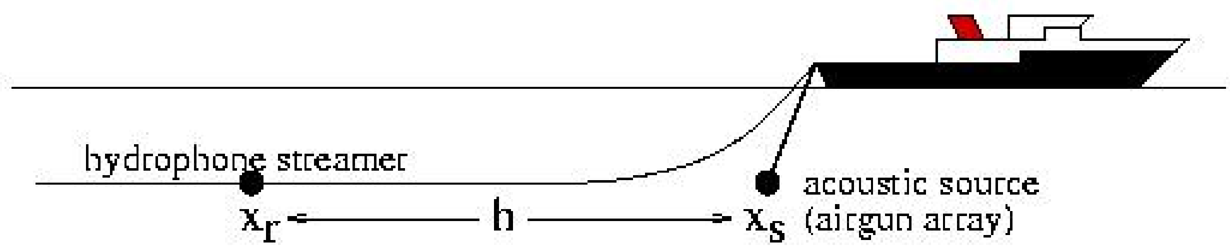
Agenda

1. Seismic inverse problem in the acoustic model: linearization, reflectors and reflections, naive geometric optics, Kirchhoff modeling and migration, why adjoints are so important (WWS)
2. Beylkin inversion, two versions of wave equation migration. Towards multipathing: Rakesh's construction (WWS)
3. Not-so-naive geometric optics, migration and asymptotic inversion w/ caustics and multipathing (FtK)
4. Beyond linearization, part I: suppressing multiple reflections (FtK)
5. Beyond linearization, part II: velocity analysis, the artifact issue, and prestack migration a la Claerbout (WWS)

1.1 The Acoustic Model

Marine reflection seismology apparatus:

- source (airgun array, explosives,...)
- receivers (hydrophone streamer, ocean bottom cable,...)
- recording and onboard processing



Land acquisition similar, but acquisition and processing are more complex. Vast bulk (90%+) of data acquired each year is marine.

Data parameters: time t, source location $\mathbf{x}_s$, and receiver location $\mathbf{x}_r$ or *half offset* $\mathbf{h} = \frac{\mathbf{x}_r - \mathbf{x}_s}{2}$, $h = |\mathbf{h}|$.

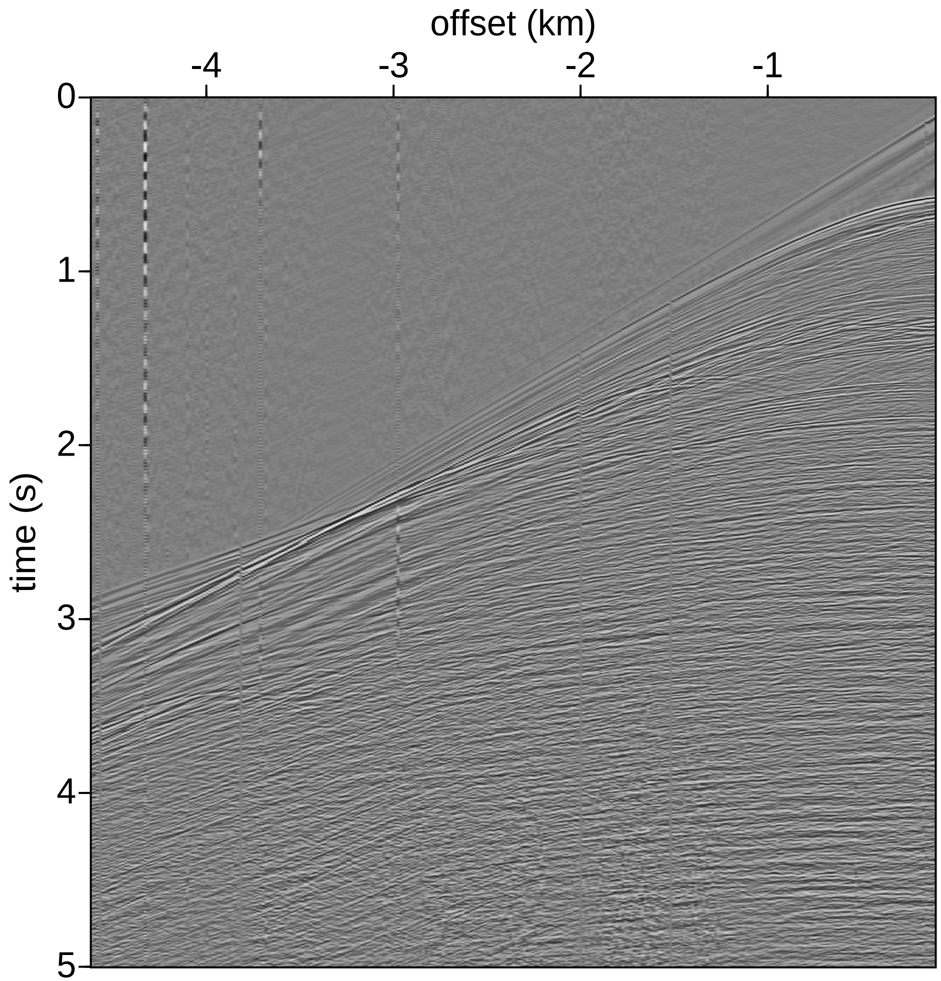
Idealized marine "streamer" geometry: $\mathbf{x}_s$ and $\mathbf{x}_r$ lie roughly on constant depth plane, source-receiver lines are parallel $\rightarrow$ 3 spatial degrees of freedom (eg. $\mathbf{x}_s, h$): *codimension 1*. [Other geometries are interesting, eg. ocean bottom cables, but streamer surveys still prevalent.]

How much data? Contemporary surveys may feature

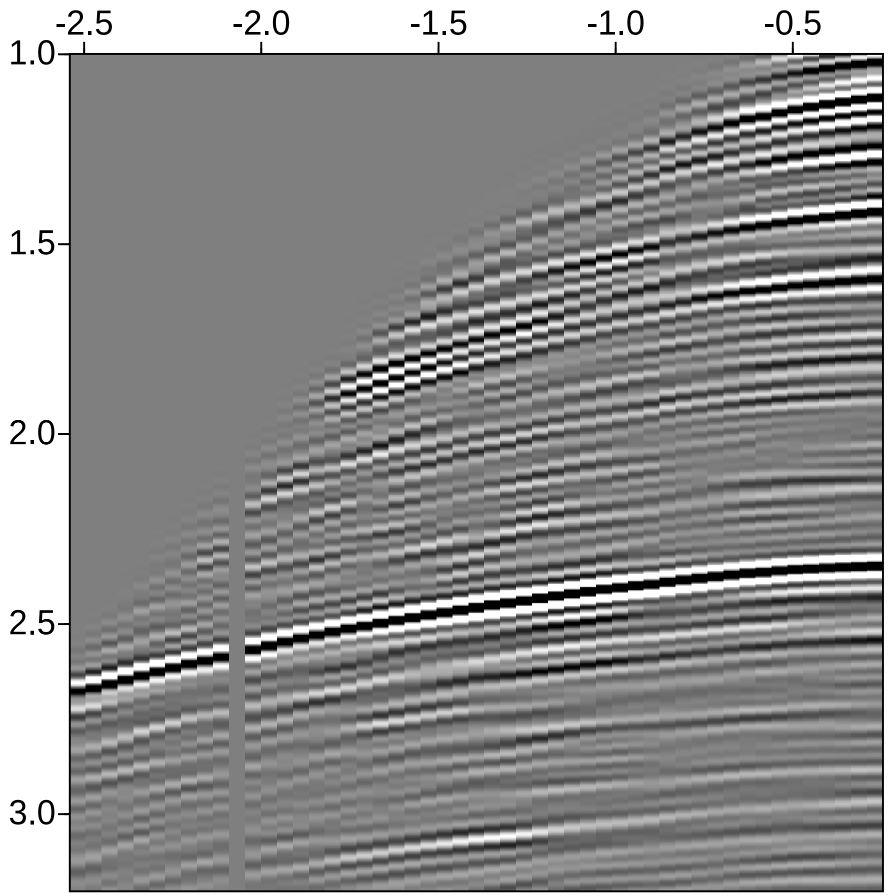
- Simultaneous recording by multiple streamers (up to 12!)
- Many (roughly) parallel ship tracks ("lines"), areal coverage

- single line ("2D") $\sim$ Gbyte; multiple lines ("3D") $\sim$ Tbyte

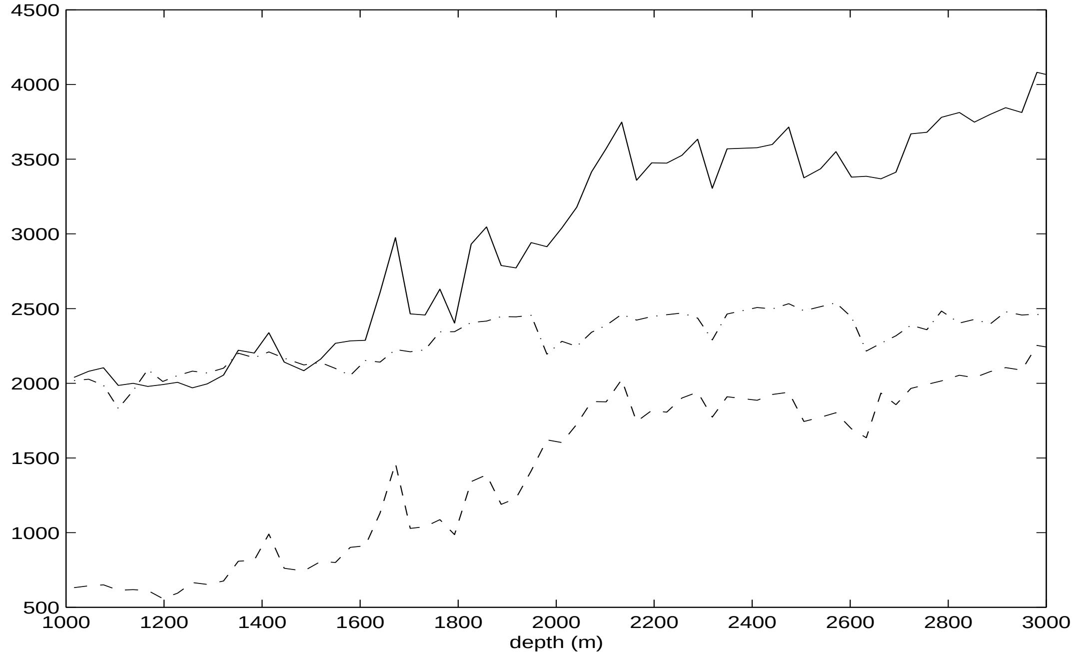
Main characteristic of data: *wave nature*, presence of **reflections** = amplitude coherence along trajectories in space-time.



Seismogram extracted from survey, Gulf of Mexico (thanks: Exxon)



Lightly processed version of data displayed in previous slide - bandpass filtered (in t), truncated ("muted").



Blocked logs from well in North Sea (thanks: Mobil R & D). Solid: p-wave velocity (m/s), dashed: s-wave velocity (m/s), dash-dot: density (kg/m^3). "Blocked" means "averaged" (over 30 m windows). Original sample rate of log tool < 1 m. **Reflectors** = jumps in velocities, density, **velocity trends**.

The Modeling Task: any model of the reflection seismogram must

- predict wave motion
- produce reflections from reflectors
- accomodate significant variation of wave velocity, material density,...

A *really good* model will also accomodate

- multiple wave modes, speeds
- material anisotropy
- attenuation, frequency dispersion of waves
- complex source, receiver characteristics

Acoustic Model (only compressional waves)

Not *really good*, but good enough for today and basis of most contemporary processing.

Relates $\rho(\mathbf{x})$= material density, $\lambda(\mathbf{x})$ = bulk modulus, $p(\mathbf{x},t)$= pressure, $\mathbf{v}(\mathbf{x},t)$ = particle velocity, $\mathbf{f}(\mathbf{x},t)$= force density (sound source):

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{f},$$

$$\frac{\partial p}{\partial t} = -\lambda \nabla \cdot \mathbf{v} \;(+\,\mathrm{i.c.'s, b.c.'s})$$

(compressional) wave speed $c = \sqrt{\frac{\lambda}{\rho}}$

acoustic field potential $u(\mathbf{x},t) = \int_{-\infty}^{t} ds\, p(\mathbf{x},s)$:

$$p = \frac{\partial u}{\partial t},\ \mathbf{v} = \frac{1}{\rho}\nabla u$$

Equivalent form: second order wave equation for potential

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla u = \int_{-\infty}^{t} dt\, \nabla \cdot \left(\frac{\mathbf{f}}{\rho}\right) \equiv \frac{f}{\rho}$$

plus initial, boundary conditions.

Weak solution of Dirichlet problem in $\Omega \subset \mathbf{R}^3$ (similar treatment for other b. c.'s):

$$u \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H^1_0(\Omega))$$

satisfying for any $\phi \in C_0^\infty((0,T) \times \Omega)$,

$$\int_0^T \int_\Omega dt\, dx \left\{ \frac{1}{\rho c^2} \frac{\partial u}{\partial t} \frac{\partial \phi}{\partial t} - \frac{1}{\rho} \nabla u \cdot \nabla \phi + \frac{1}{\rho} f \phi \right\} = 0$$

Theorem (Lions, 1972) Suppose that $\log \rho, \log c \in L^\infty(\Omega), f \in L^2(\Omega \times \mathbf{R})$. Then weak solutions of Dirichlet problem exist; initial data

$$u(\cdot, 0) \in H^1_0(\Omega),\ \frac{\partial u}{\partial t}(\cdot, 0) \in L^2(\Omega)$$

uniquely determine them.

Further idealizations: (i) density is constant, (ii) source force density is *isotropic point radiator with known time dependence* ("source pulse" $w(t)$)

$$f(\mathbf{x}, t; \mathbf{x}_s) = w(t)\delta(\mathbf{x} - \mathbf{x}_s)$$

$\Rightarrow$ acoustic potential, pressure depends on $\mathbf{x}_s$ also.

Forward map $S =$ time history of pressure for each $\mathbf{x}_s$ at receiver locations $\mathbf{x}_r$ (predicted seismic data), depends on velocity field $c(\mathbf{x})$:

$$S[c] = \{p(\mathbf{x}_r, t; \mathbf{x}_s)\}$$

Reflection seismic inverse problem: given *observed seismic data* S^{obs}, find c so that

$$S[c] \simeq S^{\text{obs}}$$

This inverse problem is

- large scale - up to Tbytes, Pflops
- nonlinear

Almost all useful technology to date relies on partial linearization: write $c = v(1+r)$ and treat r as relative first order perturbation about v, resulting in perturbation of presure field $\delta p = \frac{\partial \delta u}{\partial t} = 0, t \leq 0$, where

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta u = \frac{2r}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Define **linearized forward map** F by

$$F[v]r = \{\delta p(\mathbf{x}_r, t; \mathbf{x}_s)\}$$

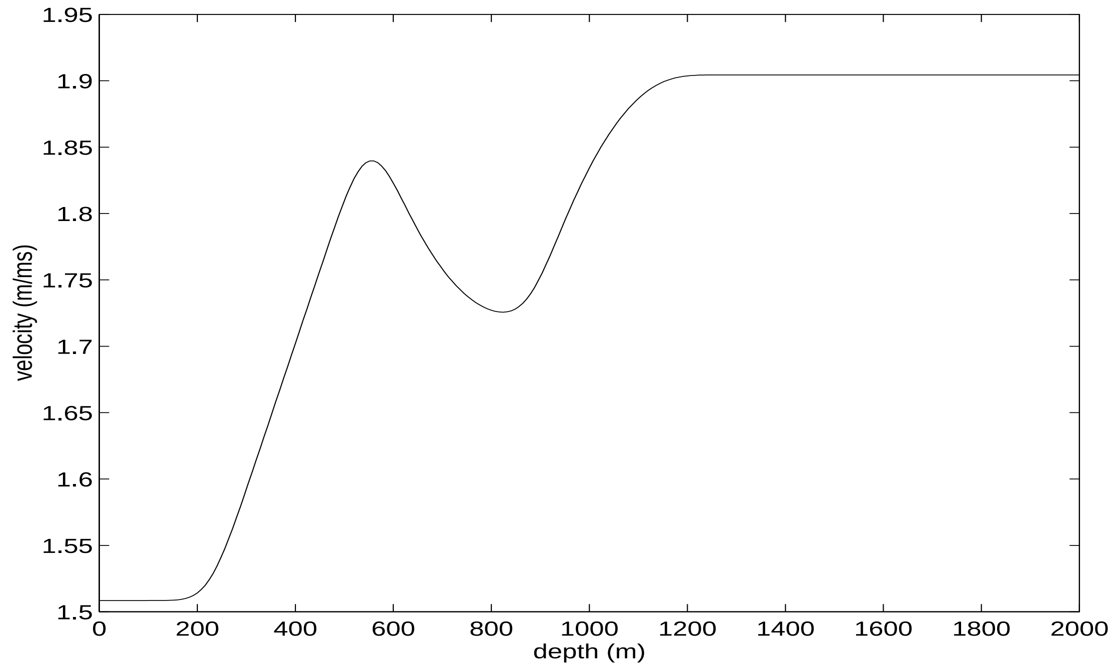
Claim: $F[v]$ is the main actor in the Claerbout, Biondi lectures.

Critical question: If there is any justice $F[v]r =$ derivative $DS[v][vr]$ of S - but in what sense? Physical intuition, numerical simulation, and not nearly enough mathematics: linearization error

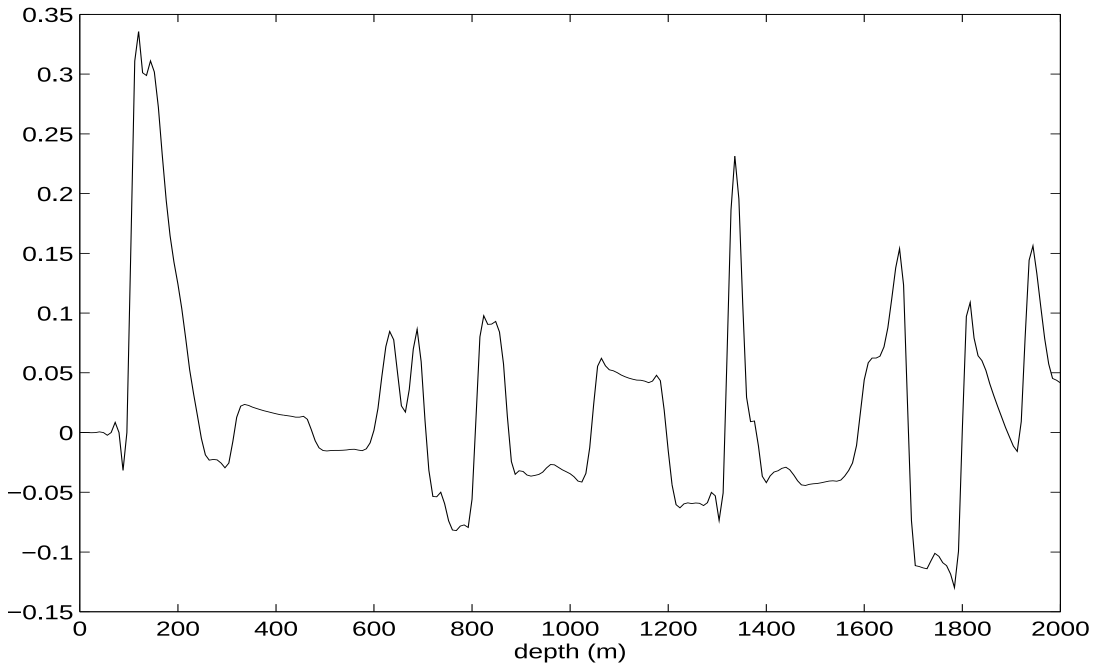
$$S[v(1+r)] - (S[v] + F[v]r)$$

- *small* when v smooth, r rough or oscillatory on wavelength scale - well-separated scales
- *large* when v not smooth and/or r not oscillatory - poorly separated scales

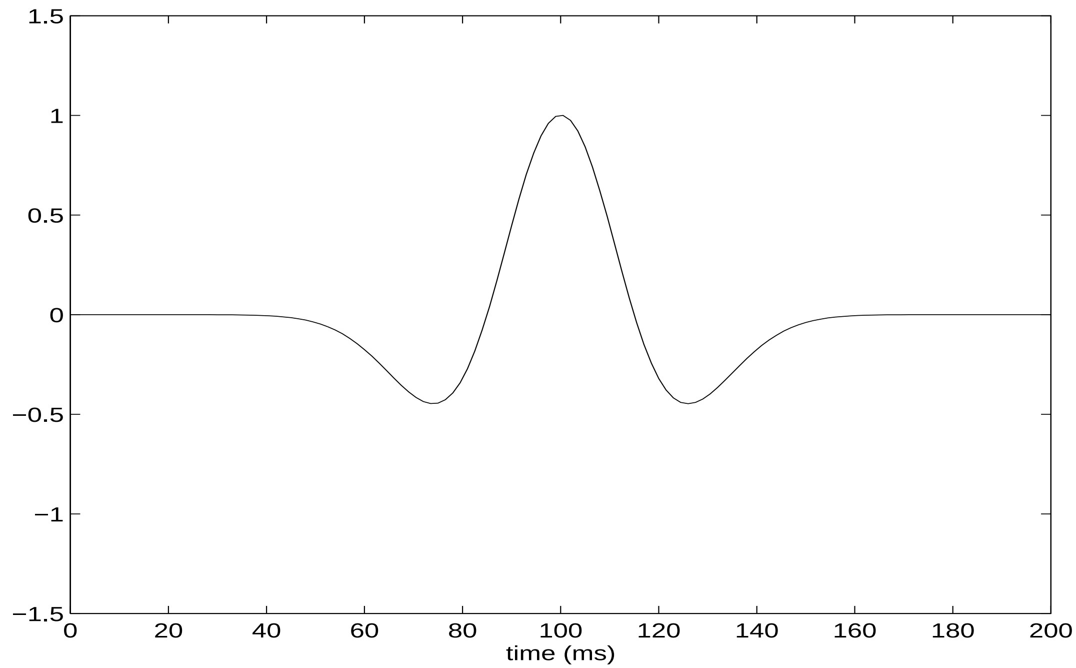
2D finite difference simulation: shot gathers with typical marine seismic geometry. Smooth $v(z)$, oscillatory $r(z)$ ("layered medium") extracted from "Marmousi" synthetic data set (Grau & Versteeg, 1994)



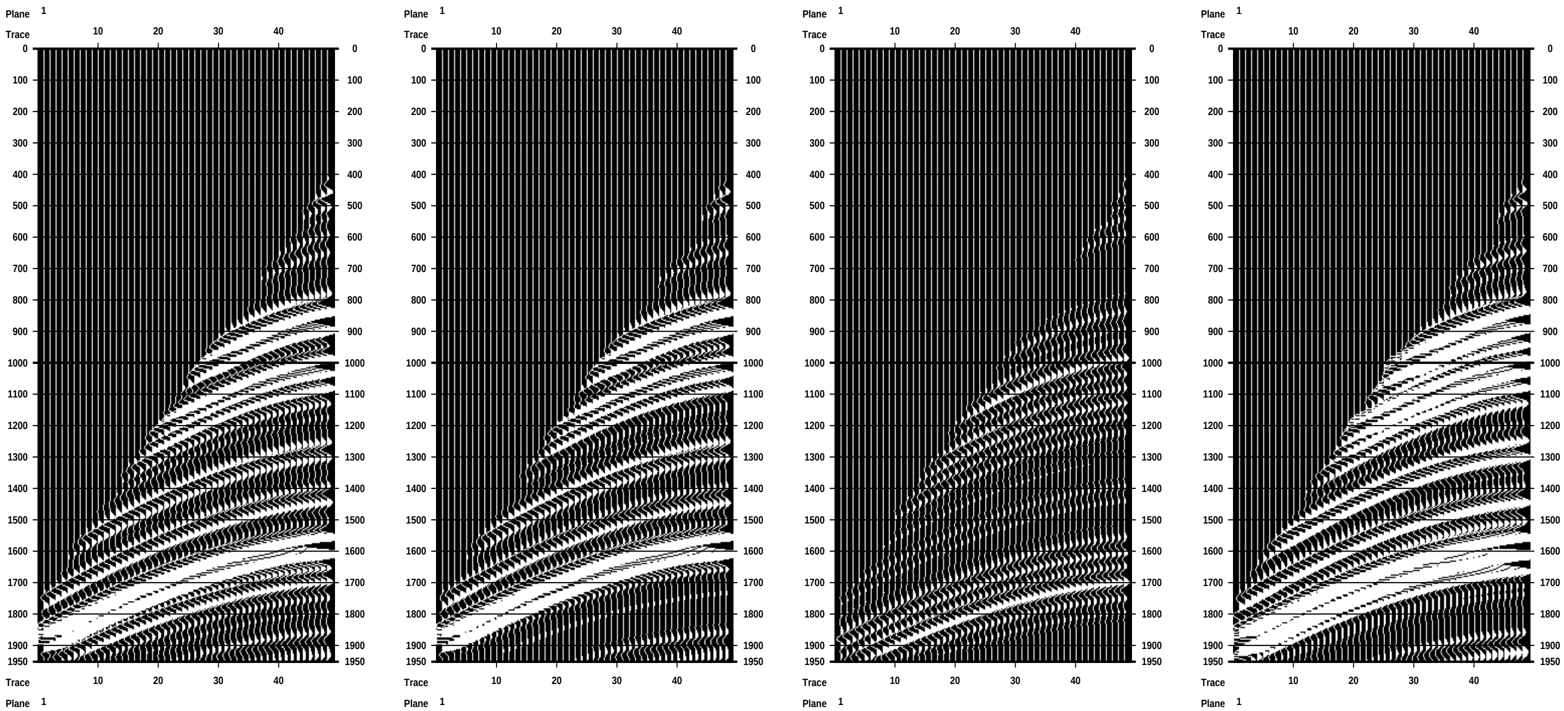
Velocity $v(x_1)$ (function of depth only) used in numerical linearization study.



Reflectivity function $r(x_1)$ (function of depth only) used in numerical linearization study.



Source pulse $w(t)$ used in numerical linearization study.



(a) $S[v(1+r)]$, (b) $S[v] + F[v]r$, (c) $S[v(1+r)] - S[v] - F[v]r$,
(d) $S[v(1+r) + 0.02v] - S[v(1+r)] + F[v(1+r)](0.02v)$

Implications:

- Some geologies have well-separated scales - cf. sonic logs - linearization-based methods work well there. Other geologies do not - expect trouble!

- v smooth, r oscillatory $\Rightarrow F[v]r$ approximates **primary reflection** $=$ result of wave interacting with material heterogeneity only once (single scattering); error consists of **multiple reflections**, which are "not too large" if r is "not too big", and sometimes can be suppressed (lecture 4).

- v nonsmooth, r smooth $\Rightarrow$ error consists of *time shifts* in waves which are very large perturbations as waves are oscillatory.

No mathematical results are known which justify/explain these observations in any rigorous way.

Partially linearized inverse problem = **velocity analysis problem**: given S^{obs} find v, r so that

$$S[v] + F[v]r \simeq S^{\text{obs}}$$

Linear subproblem = **imaging problem**: given S^{obs} and v, find r so that

$$F[v]r \simeq S^{\text{obs}} - S[v]$$

Last 20 years:

- much progress on imaging problem
- much less on velocity analysis problem.

1.2 Reflectors and Reflections

Importance of *high frequency asymptotics*: when linearization is accurate, properties of $F[v]$ dominated by those of $F_\delta[v]$ ($= F[v]$ with $w = \delta$). Implicit in migration concept (eg. Hagedoorn, 1954); explicit use: Cohen & Bleistein, SIAM JAM 1977.

Key idea: **reflectors** (rapid changes in r) emulate *singularities*; **reflections** (rapidly oscillating features in data) also emulate singularities.

NB: "everybody's favorite reflector" : the smooth interface across which r jumps. *But* this is an oversimplification - reflectors in the Earth may be complex zones of rapid change, pehaps in all directions (cf Claerbout). More flexible notion needed!!

Paley-Wiener characterization of smoothness: $u \in \mathcal{D}'(\mathbf{R}^n)$ is smooth at $\mathbf{x}_0 \Leftrightarrow$ for some nbhd X of $\mathbf{x}_0$, any $\phi \in \mathcal{E}(X)$ and N, there is $C_N \geq 0$ so that for any $\xi \neq 0$,

$$|\mathcal{F}(\phi u)(\tau\xi)| \leq C_N(\tau|\xi|)^{-N}$$

Harmonic analysis of singularities, *après* Hörmander: the **wave front set** $WF(u) \subset \mathbf{R}^n \times \mathbf{R}^n - 0$ of $u \in \mathcal{D}'(\mathbf{R}^n)$ - captures orientation as well as position of singularities.

$(\mathbf{x}_0, \xi_0) \notin WF(u) \Leftrightarrow$, there is some open nbhd $X \times \Xi \subset \mathbf{R}^n \times \mathbf{R}^n - 0$ of $(\mathbf{x}_0, \xi_0)$ so that for any $\phi \in \mathcal{E}(X)$, N, there is $C_N \geq 0$ so that for all $\xi \in \Xi$,

$$|\mathcal{F}(\phi u)(\tau\xi)| \leq C_N(\tau|\xi|)^{-N}$$

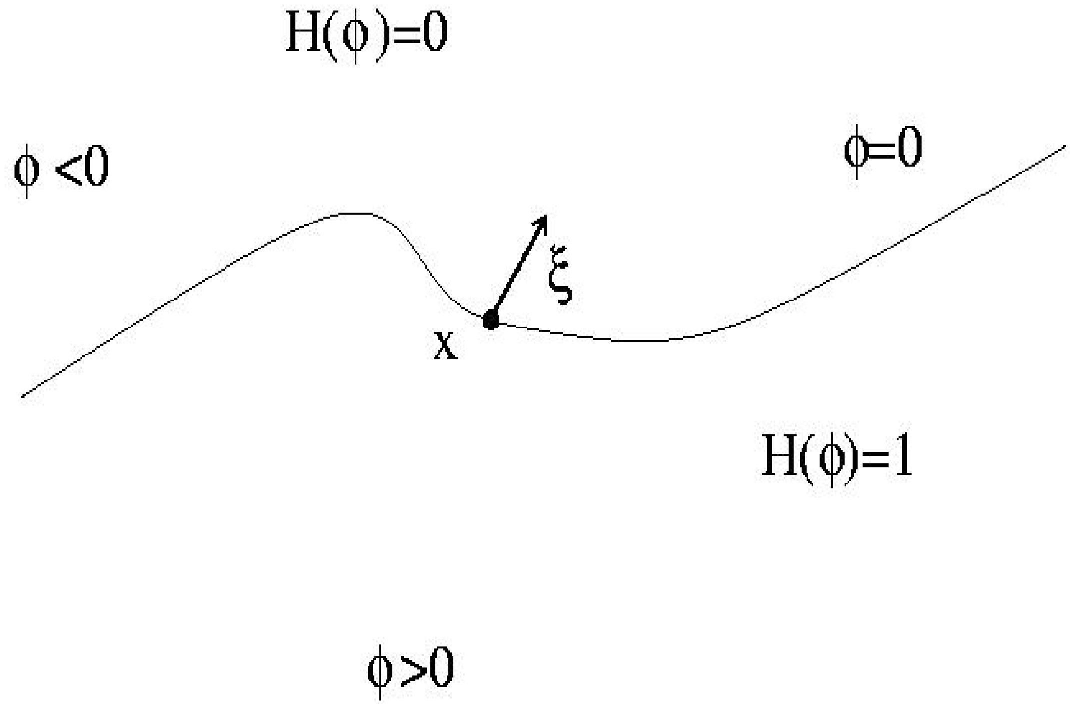
Housekeeping chores:

(i) note that the nbhds Ξ may naturally be taken to be *cones*

(ii) $WF(u)$ is invariant under chg. of coords if it is regarded as a subset of the *cotangent bundle* $T^*(\mathbf{R}^n)$ (i.e. the ξ components transform as covectors).

[Good refs: Duistermaat, 1996; Taylor, 1981; Hörmander, 1983]

The standard example: if u jumps across the interface $f(\mathbf{x}) = 0$, otherwise smooth, then $WF(u) \subset \mathcal{N}_f = \{(\mathbf{x}, \xi) : f(\mathbf{x}) = 0, \xi || \nabla f(\mathbf{x})\}$ (*normal bundle* of $f = 0$).



$$WF(H(f)) = \{(\mathbf{x}, \xi) : f(x) = 0,\, \xi || \nabla f(x)\}$$

Fact ("microlocal property of differential operators"):

Suppose $u \in \mathcal{D}'(\mathbf{R}^n)$, $(\mathbf{x}_0, \xi_0) \notin WF(u)$, and $P(\mathbf{x}, D)$ is a partial differential operator:

$$P(\mathbf{x}, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

$$D = (D_1, ..., D_n), \; D_i = -i \frac{\partial}{\partial x_i}$$

$$\alpha = (\alpha_1, ..., \alpha_n), \; |\alpha| = \sum_i \alpha_i,$$

$$D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$$

Then $(\mathbf{x}_0, \xi_0) \notin WF(P(\mathbf{x}, D)u)$ [i.e.: $WF(Pu) \subset WF(u)$].

Proof: Choose $X \times \Xi$ as in the definition, $\phi \in \mathcal{D}(X)$ form the required Fourier transform

$$\int dx\, e^{i\mathbf{x}\cdot(\tau\xi)}\phi(\mathbf{x})P(\mathbf{x}, D)u(\mathbf{x})$$

and start integrating by parts: eventually

$$= \sum_{|\alpha|\le m} \tau^{|\alpha|}\xi^{\alpha} \int dx\, e^{i\mathbf{x}\cdot(\tau\xi)}\phi_\alpha(\mathbf{x})u(\mathbf{x})$$

where $\phi_\alpha \in \mathcal{D}(X)$ is a linear combination of derivatives of ϕ and the a_αs. Since each integral is rapidly decreasing as $\tau \to \infty$ for $\xi \in \Xi$, it remains rapidly decreasing after multiplication by $\tau^{|\alpha|}$, and so does the sum. **Q. E. D.**

Key idea, restated: reflectors (or "reflecting elements") will be points in $WF(r)$. Reflections will be points in $WF(d)$.

These ideas lead to a usable definition of *image*: a reflectivity model $\tilde{r}$ is an image of r if $WF(\tilde{r}) \subset WF(r)$ (the closer to equality, the better the image).

Idealized **migration problem**: given d (hence $WF(d)$) deduce somehow a function which has *the right reflectors*, i.e. a function $\tilde{r}$ with $WF(\tilde{r}) \simeq WF(r)$.

NB: you're going to need v! ("It all depends on v(x,y,z)" - J. Claerbout)

1.3 Kirchhoff Migration

With $w = \delta$, acoustic potential u is same as Causal Green's function $G(\mathbf{x}, t; \mathbf{x}_s)$ = retarded fundamental solution:

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\mathbf{x}, t; \mathbf{x}_s) = \delta(t)\delta(\mathbf{x} - bx_s)$$

and $G \equiv 0, t < 0$. Then ($w = \delta$!) $p = \frac{\partial G}{\partial t}$, $\delta p = \frac{\partial \delta G}{\partial t}$, and

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta G(\mathbf{x}, t; \mathbf{x}_s) = \frac{2}{v^2(\mathbf{x})}\frac{\partial^2 G}{\partial t^2}(\mathbf{x}, t; \mathbf{x}_s) r(\mathbf{x})$$

Simplification: from now on, define $F[v]r = \delta G|_{\mathbf{x}=\mathbf{x}_r}$ - i.e. lose a t-derivative. Duhamel's principle $\Rightarrow$

$$\delta G(\mathbf{x}_r, t; \mathbf{x}_s) = \int dx \, \frac{2r(\mathbf{x})}{v(\mathbf{x})^2} \int ds \, G(\mathbf{x}_r, t - s; \mathbf{x}) \frac{\partial^2 G}{\partial t^2}(\mathbf{x}, s; \mathbf{x}_s)$$

Digression: where the exploding reflector model comes from

For zero-offset data ($\mathbf{x}_s = \mathbf{x}_r$), distribution kernel of $F[v]$ is

$$K(\mathbf{x}_s, t, \mathbf{x}_s; \mathbf{x}) = \frac{\partial^2}{\partial t^2} \int ds \, \frac{2}{v^2(\mathbf{x})} G(\mathbf{x}_s, t - s; \mathbf{x}) G(\mathbf{x}, s; \mathbf{x}_s)$$

Under some circumstances (explained below), K ($= G$ time-convolved with itself) is "similar" (also explained) to $\tilde{G} =$ Green's function for $v/2$. Then

$$\delta G(\mathbf{x}_s, t; \mathbf{x}_s) \sim \frac{\partial^2}{\partial t^2} \int dx \, \tilde{G}(\mathbf{x}_s, t, \mathbf{x}) \frac{2r(\mathbf{x})}{v^2(\mathbf{x})}$$

$\sim$ solution w of

$$\left(\frac{4}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) w = \delta(t) \frac{2r}{v^2}$$

Will have to say what "similar" means...

Geometric optics approximation of G should be good, as v is smooth. Local version: if $\mathbf{x}$ "not too far" from $\mathbf{x}_s$, then

$$G(\mathbf{x},t;\mathbf{x}_s) = a(\mathbf{x};\mathbf{x}_s)\delta(t-\tau(\mathbf{x};\mathbf{x}_s)) + R(\mathbf{x},t;\mathbf{x}_s)$$

where the traveltime $\tau(\mathbf{x};\mathbf{x}_s)$ solves the eikonal equation

$$v|\nabla\tau| = 1$$

$$\tau(\mathbf{x};\mathbf{x}_s) \sim \frac{|\mathbf{x}-\mathbf{x}_s|}{v(\mathbf{x}_s)},\ \mathbf{x}\rightarrow\mathbf{x}_s$$

and the amplitude $a(\mathbf{x};\mathbf{x}_s)$ solves the transport equation

$$\nabla\cdot(a^2\nabla\tau) = 0$$

All of this is meaningful only if the remainder R is small in a suitable sense: energy estimate (**Exercise!**) $\Rightarrow$

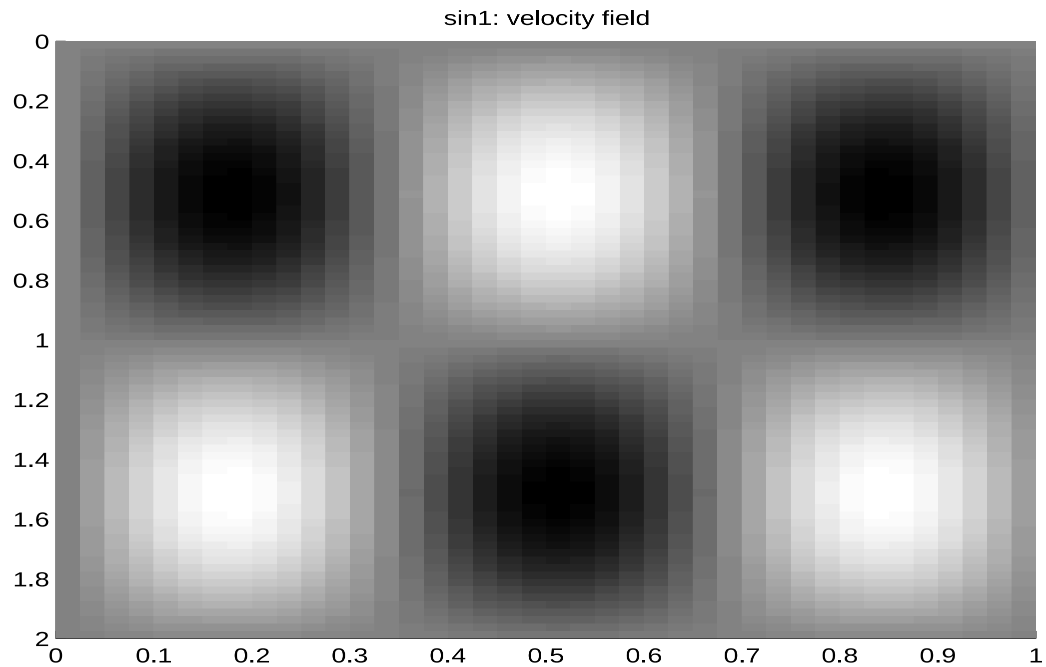
$$\int dx \int_0^T dt\,|R(\mathbf{x},t;\mathbf{x}_s)|^2 \le C\|v\|_{C^4}$$

Numerical solution of eikonal, transport: ray tracing (Lagrangian), various sorts of upwind finite difference (Eulerian) methods. See Sethian lectures, WWS 1999 MGSS notes (online) for details.

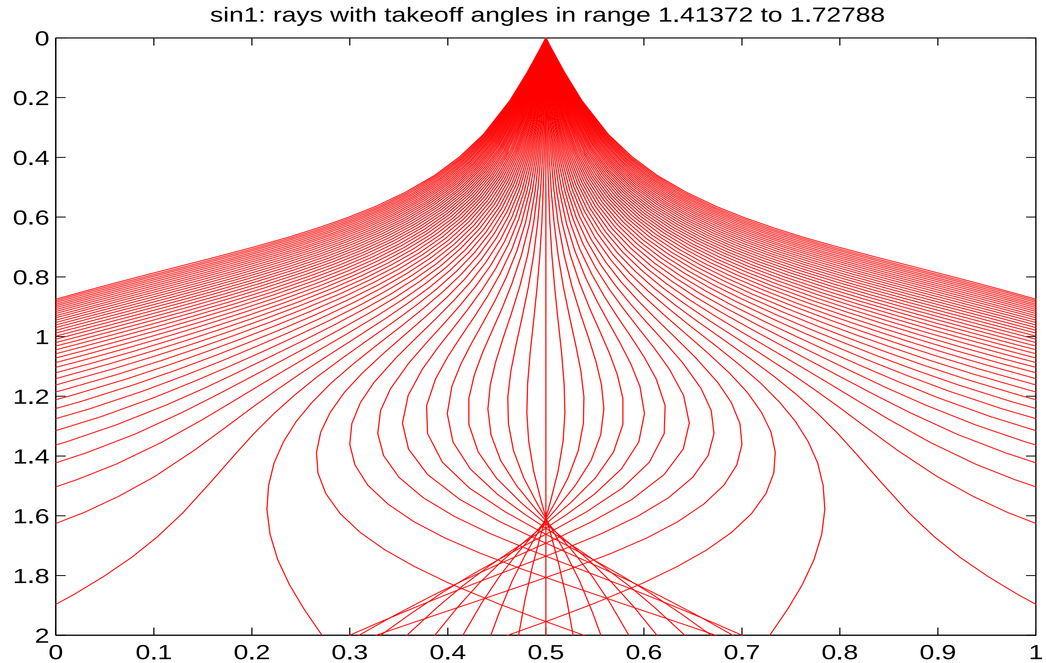
"Not too far" means: there should be one and only one ray of geometric optics connecting each $\mathbf{x}_s$ or $\mathbf{x}_r$ to each $\mathbf{x} \in \mathrm{supp} r$.

For "random but smooth" $v(\mathbf{x})$ with variance σ, more than one connecting ray occurs as soon as the distance is $O(\sigma^{-2/3})$. Such *multipathing* is invariably accompanied by the formation of a *caustic* (White, 1982).

Upon caustic formation, the simple geometric optics field description above is no longer correct (Ludwig, 1966).



2D Example of strong refraction: Sinusoidal velocity field $v(x,z) = 1 + 0.2 \sin \frac{\pi z}{2} \sin 3\pi x$



Rays in sinusoidal velocity field, source point = origin. Note formation of caustic, multiple rays to source point in lower center.

Assume: $\mathrm{supp}\, r$ contained in simple geometric optics domain (each point reached by unique ray from any source point $\mathbf{x}_s$).

Then distribution kernel K of $F[v]$ is

$$K(\mathbf{x}_r, t, \mathbf{x}_s; \mathbf{x}) = \int ds\, G(\mathbf{x}_r, t-s; \mathbf{x}) \frac{\partial^2 G}{\partial t^2}(\mathbf{x}, s; \mathbf{x}_s) \frac{2}{v^2(\mathbf{x})}$$

$$\simeq \int ds\, \frac{2a(\mathbf{x}_r, \mathbf{x})a(\mathbf{x}, \mathbf{x}_s)}{v^2(\mathbf{x})} \delta'(t - s - \tau(\mathbf{x}_r, \mathbf{x}))\delta''(s - \tau(\mathbf{x}, \mathbf{x}_s))$$

$$= \frac{2a(\mathbf{x}, \mathbf{x}_r)a(\mathbf{x}, \mathbf{x}_s)}{v^2(\mathbf{x})} \delta''(t - \tau(\mathbf{x}, \mathbf{x}_r) - \tau(\mathbf{x}, \mathbf{x}_s))$$

provided that $\nabla_{\mathbf{x}} \tau(\mathbf{x}, \mathbf{x}_r) + \nabla_{\mathbf{x}} \tau(\mathbf{x}, \mathbf{x}_s) \neq 0$ $\Leftrightarrow$ velocity at $\mathbf{x}$ of ray from $\mathbf{x}_s$ **not** negative of velocity of ray from $\mathbf{x}_r$ $\Leftrightarrow$ *no forward scattering*. [Gel'fand and Shilov, 1958 - when is pullback of distribution again a distribution].

Q: What does $\simeq$ mean?

A: It means "differs by something smoother".

In theory, can complete the geometric optics approximation of the Green's function so that the difference is C^∞ - then the two sides have the same singularities, ie. the same wavefront set.

In practice, it's sufficient to make the difference just a bit s-moother, so the first term of the geometric optics approximation (displayed above) suffices (can formalize this with modification of wavefront set defn).

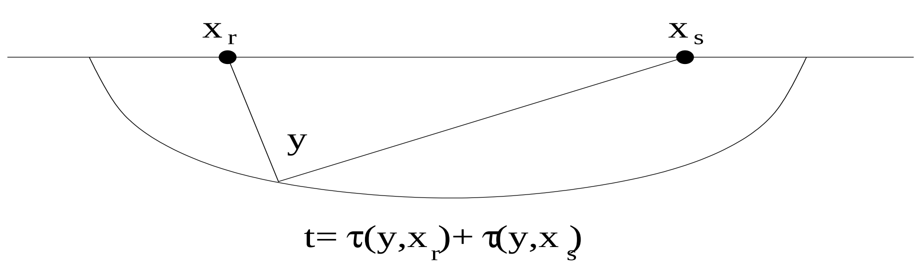
These lectures will ignore the distinction.

So: for r supported in simple geometric optics domain, no forward scattering $\Rightarrow$

$$\delta G(\mathbf{x}_r, t; \mathbf{x}_s) \simeq$$

$$\frac{\partial^2}{\partial t^2} \int dx \, \frac{2r(\mathbf{x})}{v^2(\mathbf{x})} a(\mathbf{x}, \mathbf{x}_r) a(\mathbf{x}, \mathbf{x}_s) \delta(t - \tau(\mathbf{x}, \mathbf{x}_r) - \tau(\mathbf{x}, \mathbf{x}_s))$$

You've seen this before (except that this is 3D): pressure perturbation is sum (integral) of r over *reflection isochron* $\{\mathbf{x} : t = \tau(\mathbf{x}, \mathbf{x}_r) + \tau(\mathbf{x}, \mathbf{x}_s)\}$, w. weighting, filtering. Note: if v =const. then isochron is ellipsoid, as $\tau(\mathbf{x}_s, \mathbf{x}) = |\mathbf{x}_s - \mathbf{x}|/v$!



Now can explain: when the exploding reflector model "works", i.e. when G time-convolved with itself is "similar" to $\tilde{G}$ = Green's function for $v/2$. If supp r lies in simple geometry domain, then

$$K(\mathbf{x}_s, t, \mathbf{x}_s; \mathbf{x}) = \int ds \, \frac{2a^2(\mathbf{x}, \mathbf{x}_s)}{v^2(\mathbf{x})} \delta(t - s - \tau(\mathbf{x}_s, \mathbf{x})) \delta''(s - \tau(\mathbf{x}, \mathbf{x}_s))$$

$$= \frac{2a^2(\mathbf{x}, \mathbf{x}_s)}{v^2(\mathbf{x})} \delta''(t - 2\tau(\mathbf{x}, \mathbf{x}_s))$$

whereas the Green's function $\tilde{G}$ for $v/2$ is

$$\tilde{G}(\mathbf{x}, t; \mathbf{x}_s) = \tilde{a}(\mathbf{x}, \mathbf{x}_s) \delta(t - 2\tau(\mathbf{x}, \mathbf{x}_s))$$

(half velocity = double traveltime, same rays!).

Difference between effects of K, $\tilde{G}$: for each $\mathbf{x}_s$ scale r by smooth fcn - preserves $WF(r)$ hence $WF(F[v]r)$ and relation between them. Also: adjoints have same effect on WF sets.

Upshot: from imaging point of view (i.e. apart from amplitude, derivative (filter)), kernel of $F[v]$ restricted to zero offset is same as Green's function for $v/2$, *provided that simple geometry hypothesis holds:* only one ray connects each source point to each scattering point, ie. *no multipathing*.

See Claerbout, BEI, for examples which demonstrate that multipathing really does invalidate exploding reflector model.

Inspirational interlude: the sort-of-layered theory = "Standard Processing"

Suppose were v,r functions of $z = x_3$ only, all sources and receivers at $z = 0$. Then the entire system is translation-invariant in $x_1, x_2 \Rightarrow$ Green's function G its perturbation δG, and the idealized data $\delta G|_{z=0}$ are really only functions of t and *half-offset* $h = |\mathbf{x}_s - \mathbf{x}_r|/2$. There would be *only one seismic experiment*, equivalent to any *common midpoint gather* ("CMP").

This isn't really true - *look at the data!!!* However it is *approximately* correct in many places in the world: CMPs change very slowly with midpoint $\mathbf{x}_m = (\mathbf{x}_r + \mathbf{x}_s)/2$.

Standard processing: treat each CMP *as if it were the result of an experiment performed over a layered medium*, but permit the layers to vary with midpoint.

Thus $v = v(z), r = r(z)$ for purposes of analysis, but at the end $v = v(\mathbf{x}_m, z), r = r(\mathbf{x}_m, z)$.

$$F[v]r(\mathbf{x}_r, t; \mathbf{x}_s)$$

$$\simeq \int dx \frac{2r(z)}{v^2(z)} a(\mathbf{x}, x_r) a(\mathbf{x}, x_s) \delta''(t - \tau(\mathbf{x}, x_r) - \tau(\mathbf{x}, x_s))$$

$$= \int dz \, \frac{2r(z)}{v^2(z)} \int d\omega \int dx \omega^2 a(\mathbf{x}, x_r) a(\mathbf{x}, x_s) e^{i\omega(t - \tau(\mathbf{x}, x_r) - \tau(\mathbf{x}, x_s))}$$

Since we have already thrown away smoother (lower frequency) terms, do it again using *stationary phase.* Upshot (see 2000 MGSS notes for details): up to smoother (lower frequency) error,

$$F[v]r(h,t) \simeq A(z(h,t),h)R(z(h,t))$$

Here $z(h,t)$ is the inverse of the 2-way traveltime

$$t(h,z) = 2\tau((h,0,z),(0,0,0))$$

i.e. $z(t(h,z'),h) = z'$. R is (yet another version of) "reflectivity"

$$R(z) = \frac{1}{2}\frac{dr}{dz}(z)$$

That is, $F[v]$ *is a a derivative followed by a change of variable followed by multiplication by a smooth function*. Substitute t_0 (vertical travel time) for z (depth) and you get "Inverse NMO" ($t_0 \to (t,h)$). Will be sloppy and call $z \to (t,h)$ INMO.

Anatomy of an adjoint:

$$\int dt \int dh\, d(t,h) F[v] r(t,h) = \int dt \int dh\, d(t,h) A(z(t,h),h) R(z(t,h))$$

$$= \int dz\, R(z) \int dh\, \frac{\partial t}{\partial z}(z,h) A(z,h) d(t(z,h),h) = \int dz\, r(z) (F[v]^* d)(z)$$

so $F[v]^* = -\frac{\partial}{\partial z} S M[v] N[v]$,

$N[v]$ = **NMO operator** $N[v]d(z,h) = d(t(z,h),h)$

$M[v]$ = multiplication by $\frac{\partial t}{\partial z} A$

S = **stacking operator**

$$Sf(z) = \int dh\, f(z,h)$$

So

$$F[v]^*F[v]r(z) = -\frac{\partial}{\partial z}\left[\int dh\,\frac{dt}{dz}(z,h)A^2(z,h)\right]\frac{\partial}{\partial z}r(z)$$

Microlocal property of PDOs $\Rightarrow WF(F[v]^*F[v]r) \subset WF(r)$ i.e. $F[v]^*$ is an imaging operator.

If you leave out the amplitude factor ($M[v]$) and the derivatives, as is commonly done, then you get essentially the same expression - so (NMO, stack) is an imaging operator!

It's even easy to get an inverse out of this - exercise for the reader.

Now make everything dependent on $\mathbf{x}_m$ and you've got standard processing. (end of layered interlude).

Rehabilitation of the adjoint: if $d = F[v]r$, then

$$F[v]^* d = F[v]^* F[v] r$$

In the layered case, $F[v]^* F[v]$ is an operator which preserves wave front sets. *Whenever* $F[v]^* F[v]$ *preserves wave front sets,* $F[v]^*$ *is an imaging operator.*

Beylkin, JMP 1985: for r supported in simple geometric optics domain,

- $WF(F_\delta[v]^* F_\delta[v] r) \subset WF(r)$
- if $S^{\text{obs}} = S[v] + F_\delta[v] r$ (data consistent with linearized model), then $F_\delta[v]^*(S^{\text{obs}} - S[v])$ is an image of r
- an operator $F_\delta[v]^\dagger$ exists for which $F_\delta[v]^\dagger(S^{\text{obs}} - S[v]) - r$ is *smoother* than r, under some constraints on r - an *inverse modulo smoothing operators* or *parametrix*.

Outline of proof: (i) express $F[v]^*F[v]$ as "Kirchhoff modeling" followed by "Kirchhoff migration"; (ii) introduce Fourier transform; (iii) approximate for large wavenumbers using stationary phase, leads to representation of $F[v]^*F[v]$ modulo smoothing error as *pseudodifferential operator* ("ΨDO"):

$$F[v]^*F[v]r(\mathbf{x}) \simeq p(\mathbf{x}, D)r(\mathbf{x}) \equiv \int d\xi \, p(\mathbf{x}, \xi)e^{i\mathbf{x}\cdot\xi}\hat{r}(\xi)$$

in which $p \in C^\infty$, and for some m (the *order* of p), all multiindices α, β, and all compact $K \subset \mathbf{R}^n$, there exist constants $C_{\alpha,\beta,K} \geq 0$ for which

$$|D_{\mathbf{x}}^\alpha D_\xi^\beta p(\mathbf{x}, \xi)| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\beta|}, \ \mathbf{x} \in K$$

Explicit computation of **symbol** p - for details, see 1998 MGSS notes.

Imaging property of Kirchhoff migration follows from *microlocal property of* Ψ*DOs*:

> if $p(x,D)$ is a ΨDO, $u \in \mathcal{E}'(\mathbf{R}^n)$ then $WF(p(x,D)u) \subset WF(u)$.

Will prove this. First, a few other properties:

- differential operators are ΨDOs (easy - exercise)
- ΨDOs of order m form a module over $C^\infty(\mathbf{R}^n)$ (also easy)
- product of ΨDO order m, ΨDO order l = ΨDO order $\leq m+l$; adjoint of ΨDO order m is ΨDO order m (much harder)

Complete accounts of theory, many apps: books of Duistermaat, Taylor, Nirenberg, Treves, Hörmander.

Proof of microlocal property: suppose $(\mathbf{x}_0, \xi_0) \notin WF(u)$, choose neighborhoods X, Ξ as in defn, with Ξ conic. Need to choose analogous nbhds for $P(x, D)u$. Pick $\delta > 0$ so that $B_{3\delta}(\mathbf{x}_0) \subset X$, set $X' = B_\delta(\mathbf{x}_0)$.

Similarly pick $0 < \epsilon < 1/3$ so that $B_{3\epsilon}(\xi_0/|\xi_0|) \subset \Xi$, and chose $\Xi' = \{\tau\xi : \xi \in B_\epsilon(\xi_0/|\xi_0|), \tau > 0\}$.

Need to choose $\phi \in \mathcal{E}'(X')$, estimate $\mathcal{F}(\phi P(\mathbf{x}, D)u)$. Choose $\psi \in \mathcal{E}(X)$ so that $\psi \equiv 1$ on $B_{2\delta}(\mathbf{x}_0)$.

NB: this implies that if $\mathbf{x} \in X'$, $\psi(\mathbf{y}) \neq 1$ then $|\mathbf{x} - \mathbf{y}| \geq \delta$.

Write $u = (1-\psi)u + \psi u$. Claim: $\phi P(\mathbf{x}, D)((1-\psi)u)$ is smooth.

$$\phi(\mathbf{x}) P(\mathbf{x}, D)((1-\psi)u))(\mathbf{x})$$

$$= \phi(\mathbf{x}) \int d\xi \, P(\mathbf{x}, \xi) e^{i\mathbf{x}\cdot\xi} \int dy \, (1-\psi(\mathbf{y}))u(\mathbf{y}) e^{-i\mathbf{y}\cdot\xi}$$

$$= \int d\xi \int dy \, P(\mathbf{x}, \xi) \phi(\mathbf{x})(1-\psi(\mathbf{y})) e^{i(\mathbf{x}-\mathbf{y})\cdot\xi} u(\mathbf{y})$$

$$= \int d\xi \int dy \, (-\nabla_\xi^2)^M P(\mathbf{x}, \xi) \phi(\mathbf{x})(1-\psi(\mathbf{y})) |\mathbf{x}-\mathbf{y}|^{-2M} e^{i(\mathbf{x}-\mathbf{y})\cdot\xi} u(\mathbf{y})$$

using the identity

$$e^{i(\mathbf{x}-\mathbf{y})\cdot\xi} = |\mathbf{x}-\mathbf{y}|^{-2} \left[-\nabla_\xi^2 e^{i(\mathbf{x}-\mathbf{y})\cdot\xi} \right]$$

and integrating by parts $2M$ times in ξ. This is permissible because $\phi(\mathbf{x})(1-\psi(\mathbf{y})) \neq 0 \Rightarrow |\mathbf{x}-\mathbf{y}| > \delta$.

According to the definition of ΨDO,

$$|(-\nabla_\xi^2)^M P(\mathbf{x},\xi)| \leq C|\xi|^{m-2M}$$

For any K, the integral thus becomes absolutely convergent after K differentiations of the integrand, provided M is chosen large enough. Q.E.D. Claim.

This leaves us with $\phi P(\mathbf{x},D)(\psi u)$. Pick $\eta \in \Xi'$ and w.l.o.g. scale $|\eta| = 1$. Fourier transform:

$$\mathcal{F}(\phi P(\mathbf{x},D)(\psi u))(\tau\eta) = \int dx \int d\xi\, P(\mathbf{x},\xi)\phi(\mathbf{x})\widehat{\psi u}(\xi)e^{i\mathbf{x}\cdot(\xi-\tau\eta)}$$

Introduce $\tau\theta = \xi$, and rewrite this as

$$= \tau^n \int dx \int d\theta\, P(\mathbf{x},\tau\theta)\phi(\mathbf{x})\widehat{\psi u}(\tau\theta)e^{i\tau\mathbf{x}\cdot(\theta-\eta)}$$

Divide the domain of the inner integral into $\{\theta : |\theta - \eta| > \epsilon\}$ and its complement. Use

$$-\nabla_x^2 e^{i\tau \mathbf{x}\cdot(\theta-\eta)} = \tau^2|\theta-\eta|^2 e^{i\tau \mathbf{x}\cdot(\theta-\eta)}$$

and integration by parts $2M$ times to estimate the first integral:

$$\tau^{n-2M}\left|\int dx \int_{|\theta-\eta|>\epsilon} d\theta \, (-\nabla_x^2)^M [P(\mathbf{x},\tau\theta)\phi(\mathbf{x})]\widehat{\psi u}(\tau\theta)\right.$$
$$\left. \times \, |\theta-\eta|^{-2M} e^{i\tau\mathbf{x}\cdot(\theta-\eta)}\right|$$
$$\leq C\tau^{n+m-2M}$$

m being the order of P. Thus the first integral is rapidly decreasing in τ.

For the second integral, note that $|\theta - \eta| \le \epsilon \Rightarrow \theta \in \Xi$, per the defn of Ξ'. Since $X \times \Xi$ is disjoint from the wavefront set of u, for a sequence of constants C_N, $|\widehat{\psi u}(\tau\theta)| \le C_N \tau^{-N}$ uniformly for θ in the (compact) domain of integration, whence the second integral is also rapidly decreasing in τ. **Q. E. D.**

And that's why Kirchhoff migration works, at least in the simple geometric optics regime.

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