Wave equation techniques for attenuating multiple reflections

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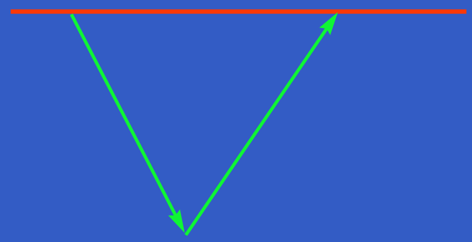
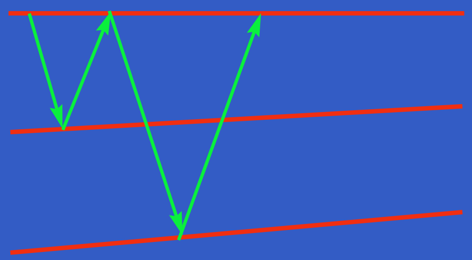
Statement of the problem

Present day imaging algorithms are based on *linear* relationships between data and scattering potential.

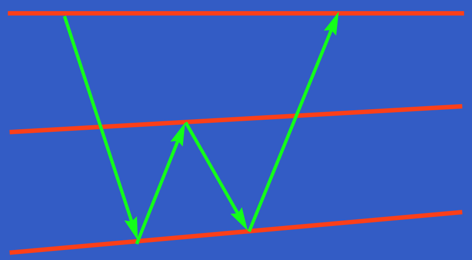


Single scattering

Thereby, they neglect scattering processes of the form



Surface multiples

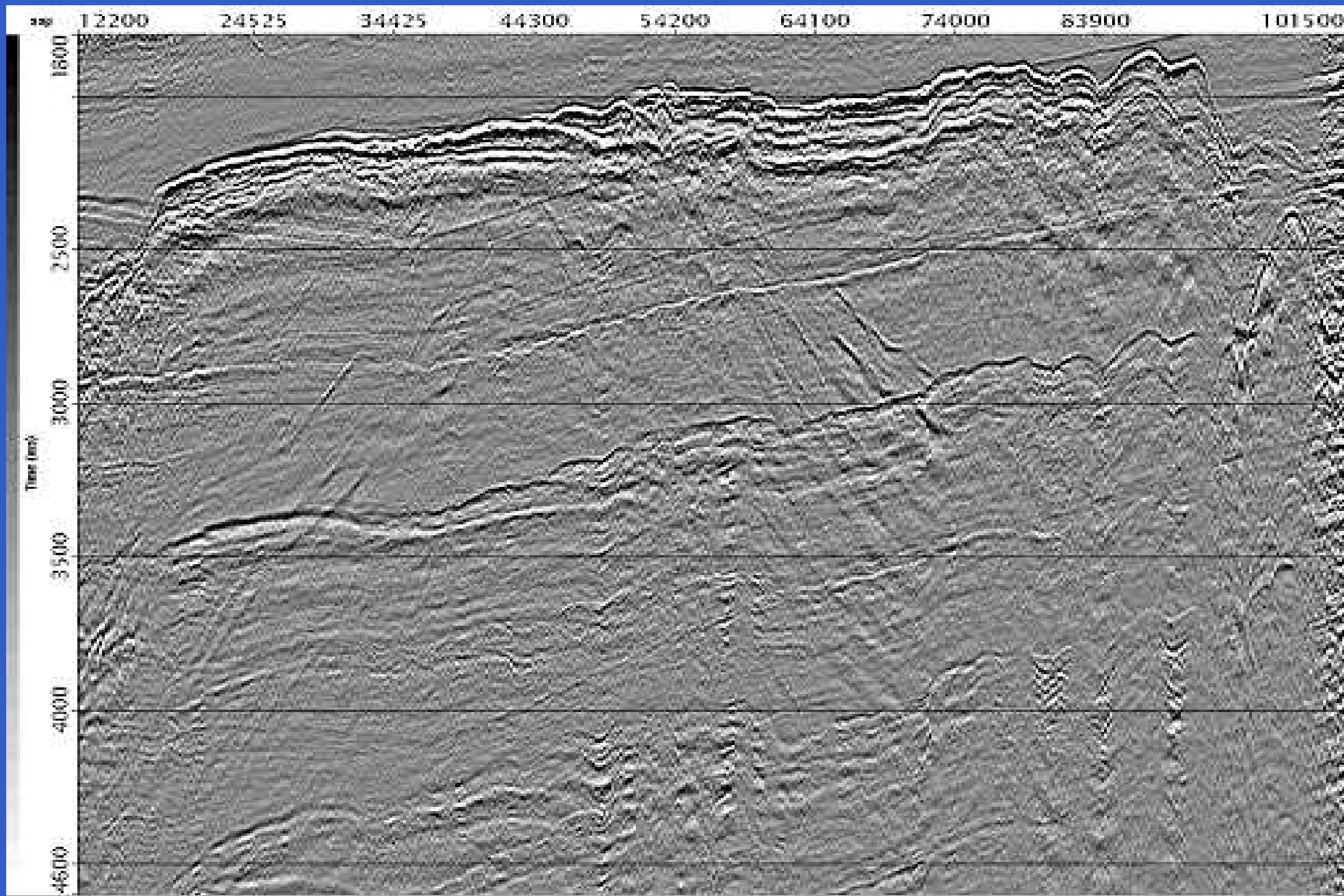


Internal multiples

Multiple reflections show up as artifacts in the images.

Statement of the problem, example

- How to turn this



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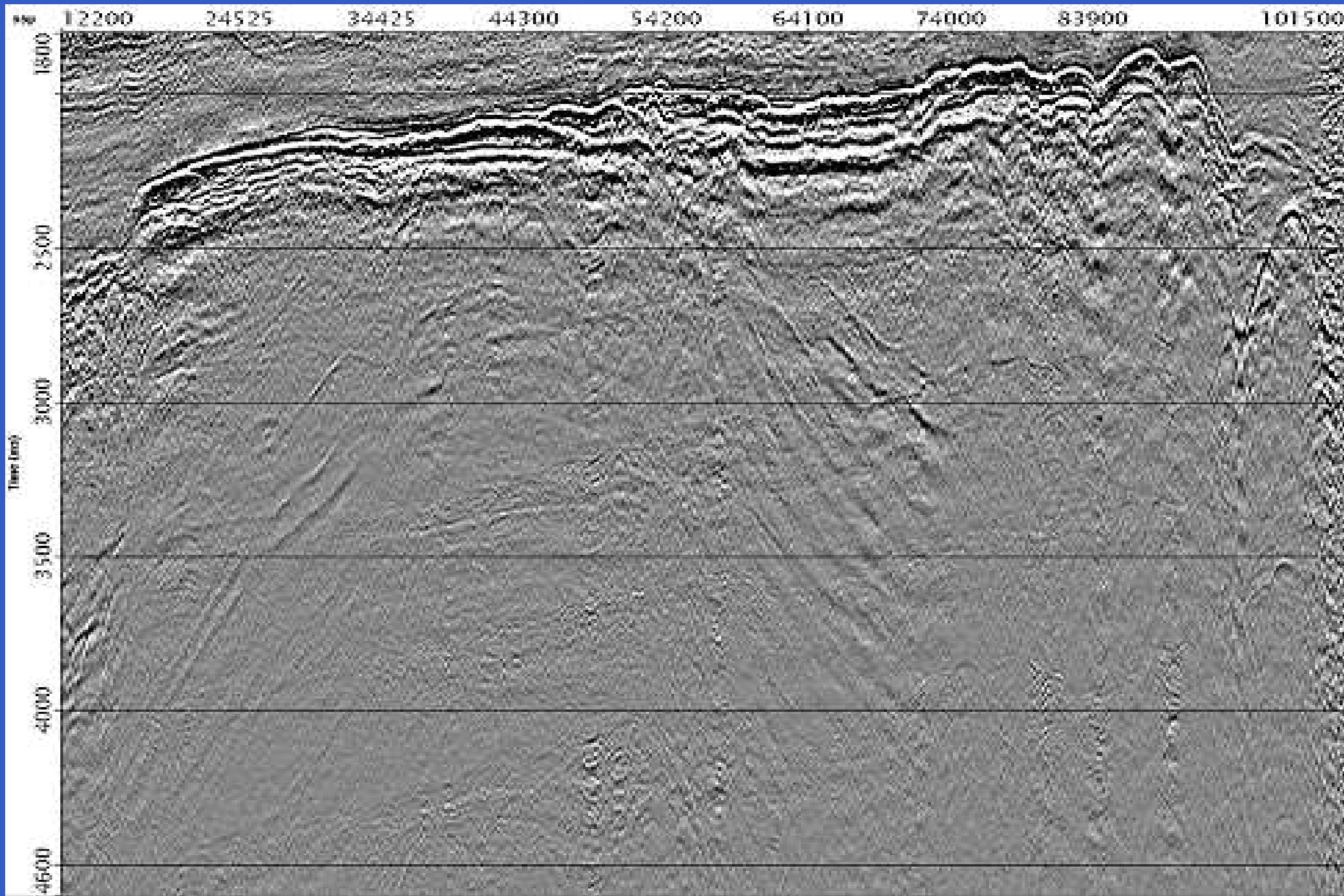
Statement of the problem, example

- into this?



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Basic assumptions and notation

- Earth is a constant density acoustic medium; wave propagation described by

$$\left(\frac{1}{c^2(\vec{x})}\frac{\partial^2}{\partial t^2} - \Delta\right) G(\vec{x}, \vec{x}_s, t) = \delta(\vec{x} - \vec{x}_s)\delta(t).$$

  where $G(\vec{x}, \vec{x}_s, t)$ is Green's function, the response of the medium at location $\vec{x}$ due to an instantaneous point source at location $\vec{x}_s$.

  This equation needs to be supplied with additional initial/boundary/radiation conditions to determine the solution uniquely.

Basic assumptions and notation, continued

- The temporal Fourier transform of G will be denoted by $\hat{G}(\vec{x}, \vec{x}_s, \omega)$.

- The data are obtained from the Green's function by convolving with a (generally unknown) wavelet:

$$\hat{d}(\vec{x}, \vec{x}_s, \omega) = \hat{w}(\omega)\hat{G}(\vec{x}, \vec{x}_s, \omega)$$

Surface multiples, derivation integral equation

Let G^{mf} and G be two solutions for the acoustic wave equation in the region $z > 0$ satisfying the conditions

$$G(\vec{x}, \vec{x}_s, t)\big|_{z=0} = 0,$$

$$G^{mf}(\vec{x}, \vec{x}_s, t) \in O(|\,\vec{x} - \vec{x}_s|^{-1}) \text{ for } |\vec{x} - \vec{x}_s| \to \infty$$

(radiation condition)

The condition for G means vanishing pressure at $z = 0$, as is the case at an air/water interface. G is therefore a wavefield that has reflections against $z = 0$ in it.

G^{mf} on the other hand, is the desired multiple-free solution.

Derivation integral equation, continued

Assume that $z_s, z_r \geq \epsilon > 0$. Consider the domain V bounded by the plane $z = \epsilon$ and the half sphere $x^2 + y^2 + (z-\epsilon)^2 < R^2, z > \epsilon$, which encloses both $\vec{x}_s$ and $\vec{x}_r$.

Using reciprocity, $G(\vec{x}, \vec{y}, t) = G(\vec{y}, \vec{x}, t)$, we find that for all $\vec{x}$ inside V

$$\begin{aligned}\nabla \cdot \Big(\hat{G}(\vec{x}, \vec{x}_r, \omega)\nabla \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega) - \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega)\nabla \hat{G}(\vec{x}, \vec{x}_r, \omega)\Big) \\ = \hat{G}(\vec{x}, \vec{x}_r, \omega)\Delta \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega) - \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega)\Delta \hat{G}(\vec{x}, \vec{x}_r, \omega) \\ = -\hat{G}(\vec{x}, \vec{x}_r, \omega)\delta(\vec{x} - \vec{x}_s) + \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega)\delta(\vec{x} - \vec{x}_r)\end{aligned}$$

Derivation integral equation, continued

Integrating over V, we find (using Gauss' theorem)

$$\hat{G}^{mf}(\vec{x}_s, \vec{x}_r, \omega) - \hat{G}(\vec{x}_s, \vec{x}_r, \omega) = \\ = \int_{\delta V} dS\, \vec{n} \cdot \Big(\hat{G}(\vec{x}, \vec{x}_r, \omega) \nabla \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega) \\ - \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega) \nabla \hat{G}(\vec{x}, \vec{x}_r, \omega) \Big) .$$

Letting $R \to \infty$ and using the radiation condition, we derive

$$\hat{G}^{mf}(\vec{x}_s, \vec{x}_r, \omega) - \hat{G}(\vec{x}_s, \vec{x}_r, \omega) = \\ = - \int_{z=\epsilon} dx dy\, \Big(\hat{G}(\vec{x}, \vec{x}_r, \omega) \partial_z \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega) \\ - \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega) \partial_z \hat{G}(\vec{x}, \vec{x}_r, \omega) \Big) .$$

Derivation integral equation, continued

- Letting $\epsilon \downarrow 0$ and using the vanishing of G at $z = 0$, this becomes

$$\hat{G}^{mf}(\vec{x}_s, \vec{x}_r, \omega) - \hat{G}(\vec{x}_s, \vec{x}_r, \omega) = \int_{z=0} dxdy\, \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega) \partial_z \hat{G}(\vec{x}, \vec{x}_r, \omega).$$

Approximating

$$\begin{aligned} \partial_z \hat{G}(\vec{x}, \vec{x}_r, \omega)\Big|_{z=0} &\cong (\Delta z)^{-1}\, \hat{G}(\vec{x}, \vec{x}_r, \omega)\Big|_{z=\Delta z}, \\ \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega)\Big|_{z=0} &\cong \hat{G}^{mf}(\vec{x}_s, \vec{x}, \omega)\Big|_{z=\Delta z}, \end{aligned}$$

Derivation integral equation, continued

-
- we finally obtain the integral equation we are after
-
-
- $$\hat{G}^{mf}(\vec{x}_s,\vec{x}_r,\omega)-\hat{G}(\vec{x}_s,\vec{x}_r,\omega)$$
$$\cong(\Delta z)^{-1}\int_{z=\Delta z}dxdy\,\hat{G}^{mf}(\vec{x}_s,\vec{x},\omega)\hat{G}(\vec{x},\vec{x}_r,\omega).$$

Using the (unknown) wavelet, we can rewrite the integral equation as

$$\hat{d}^{mf}(\vec{x}_s,\vec{x}_r,\omega)-\hat{d}(\vec{x}_s,\vec{x}_r,\omega)$$
$$\cong(\hat{w}(\omega)\Delta z)^{-1}\int_{z=\Delta z}dxdy\,\hat{d}^{mf}(\vec{x}_s,\vec{x},\omega)\hat{d}(\vec{x},\vec{x}_r,\omega).$$

Since the data is non-singular at $\vec{x}=\vec{x}_r$ or $\vec{x}=\vec{x}_s$ we can and will assume from now on that $\Delta z=z_r=z_s\equiv z_{acq}$.

Power series solution

Define the linear integral operator

$$\left(M \cdot \hat{d}^{mf}\right)(\vec{x}_s, \vec{x}_r, \omega)$$

$$:= (\hat{w}(\omega) z_{acq})^{-1} \int_{z=z_{acq}} d\vec{x}\, \hat{d}^{mf}(\vec{x}_s, \vec{x}, \omega) \hat{d}(\vec{x}, \vec{x}_r, \omega).$$

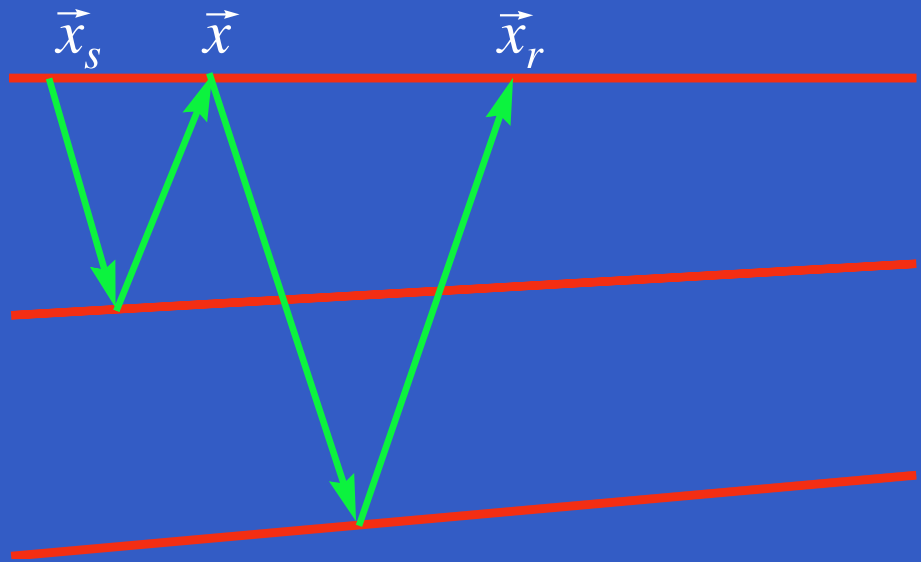
Then the integral equation can be rewritten as

$$(1 - M) \cdot \hat{d}^{mf} = \hat{d}.$$

The formal solution expressing the multiple free data in the measured data is

$$\hat{d}^{mf} = \hat{d} + M \cdot \hat{d} + M^2 \cdot \hat{d} + \cdots .$$

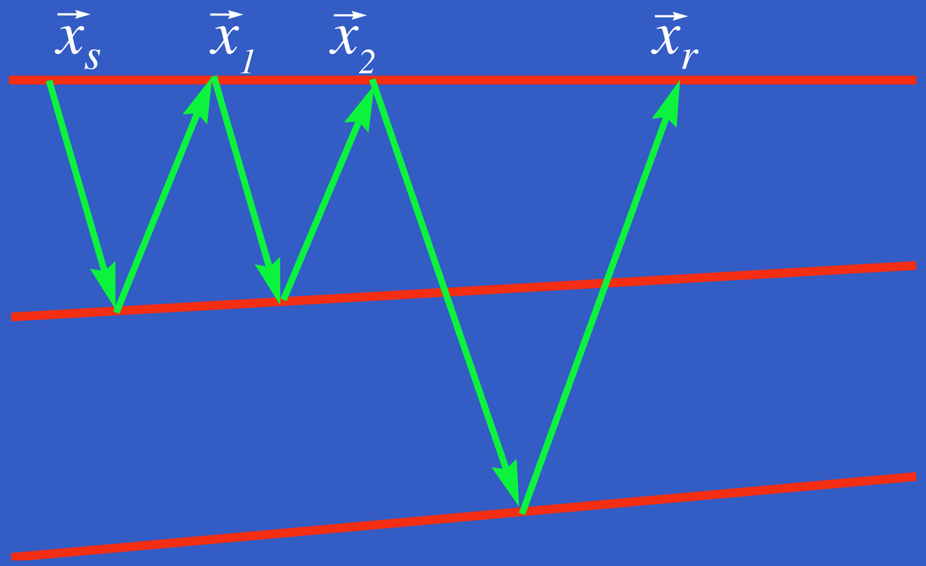
Power series solution, first order term



$$M \cdot \hat{d} \sim \int_{z=z_{acq}} d\vec{x}\, \hat{d}(\vec{x}_s, \vec{x}, \omega)\hat{d}(\vec{x}, \vec{x}_r, \omega)$$

The integral over $\vec{x}$ is *stationary* when Snell's law is obeyed in the picture above. Note that no velocity information is required to calculate the multiples.

Power series solution, second order term



$$M^2 \cdot \hat{d} \sim \int_{z_1 = z_2 = z_{acq}} d\vec{x}_1 d\vec{x}_2 \, \hat{d}(\vec{x}_s, \vec{x}_1, \omega) \hat{d}(\vec{x}_1, \vec{x}_2, \omega) \hat{d}(\vec{x}_2, \vec{x}_r, \omega)$$

The integrals over $\vec{x}_1$ and $\vec{x}_2$ are stationary when Snell's law is obeyed in the picture above. Again, no velocity information is required.

Practical implementation

- To attenuate e.g. the first order surface multiple, one first predicts the kinematics by evaluating the first order term

$$(\hat{w}(\omega) z_{acq})^{-1} \int_{z=z_{acq}} d\vec{x}\, \hat{d}(\vec{x}_s, \vec{x}, \omega) \hat{d}(\vec{x}, \vec{x}_r, \omega).$$

Since we do not know the wavelet, we cannot directly add this to the data. Instead, we use an ad hoc energy minimization criterion to attenuate the first order multiple:

$$\min_{\hat{f}} \int d\omega \left| \hat{d}(\vec{x}_s, \vec{x}_r, \omega) + \hat{f}(\omega) \int_{z=z_{acq}} d\vec{x}\, \hat{d}(\vec{x}_s, \vec{x}, \omega) \hat{d}(\vec{x}, \vec{x}_r, \omega) \right|^2 .$$

Solution by deconvolution, principle

- The integral equation relating multiple free data and measured data can also be solved by multi-dimensional deconvolution. To this end, rewrite the integral equation in the form

$$(1+D)\cdot\hat{d} = \hat{d}^{mf},$$

where D is defined as the integral operator

$$\begin{aligned}\left(D\cdot\hat{d}\right)(\vec{x}_s,\vec{x}_r,\omega) &:= (z_{acq})^{-1}\int d\vec{x}\,\hat{G}^{mf}(\vec{x}_s,\vec{x},\omega)\hat{d}(\vec{x},\vec{x}_r,\omega)\\ &= \int d\omega\, e^{i\omega t}\int d\vec{x}d\tau\, F(\vec{x}_s,\vec{x},\tau)d(\vec{x},\vec{x}_r,t-\tau),\end{aligned}$$

with

$$F(\vec{x}_s,\vec{x},t) := (z_{acq})^{-1}\,G^{mf}(\vec{x}_s,\vec{x},t).$$

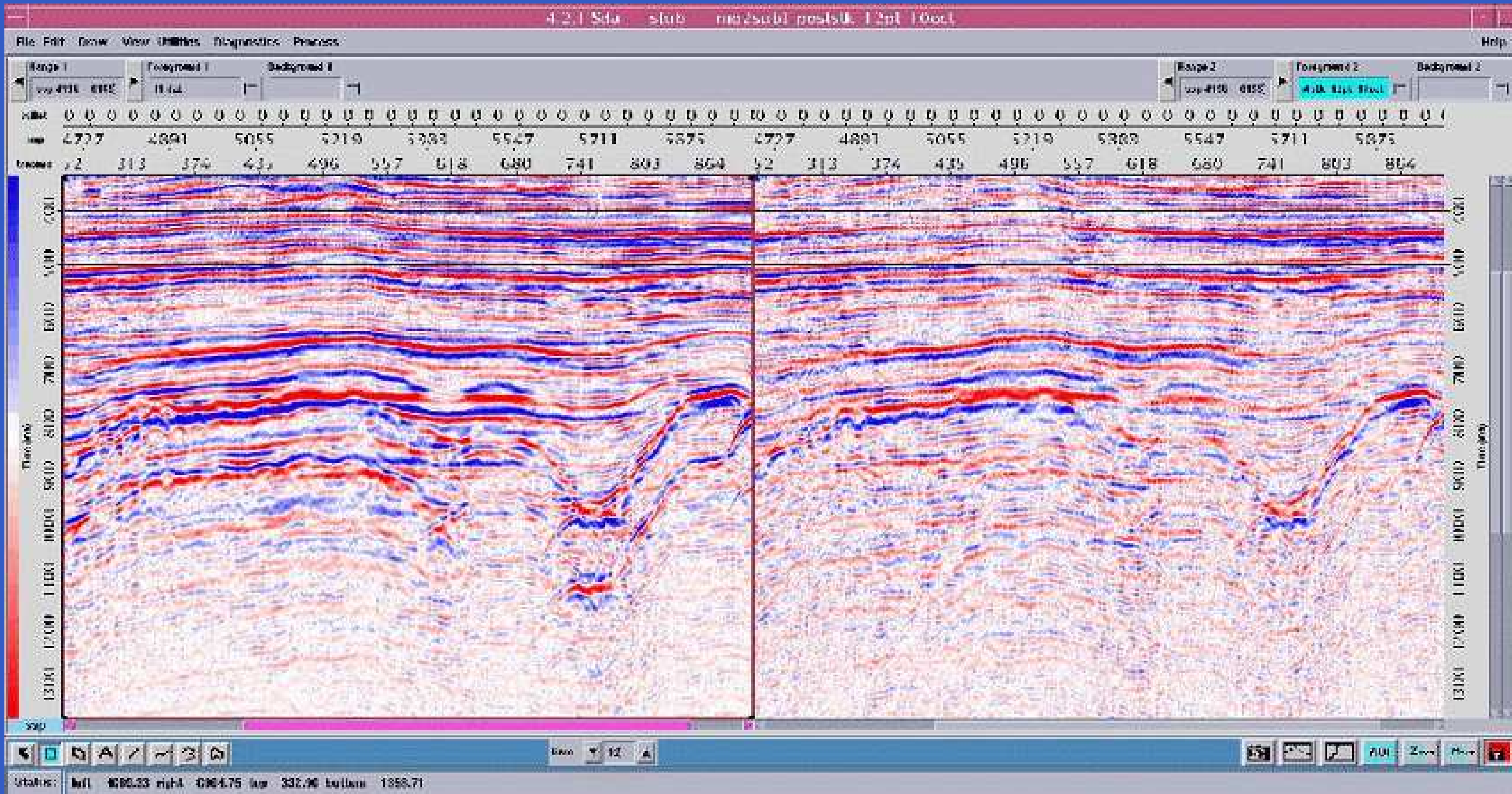
Solution by deconvolution, principle

We now try to find a multi dimensional deconvolution filter $F(\vec{x}_s, \vec{x}, t)$ which minimizes the energy in $(1 + D) \cdot \hat{d}$,

$$\int d\vec{x}_s d\vec{x}_r dt \left| d(\vec{x}_s, \vec{x}_r, t) + \int d\vec{x} \int_{T_{gap}}^{Tmax} d\tau \, F(\vec{x}_s, \vec{x}, \tau) d(\vec{x}, \vec{x}_r, t - \tau) \right|^2 .$$

The time T_{gap} is introduced to avoid the trivial solution $F(\vec{x}_s, \vec{x}, t) = -\delta(\vec{x}_s - \vec{x})\delta(t)$ for which the energy would be zero.

Solution by deconvolution, example



Left: input stack, right: stack after 2D deconvolution

(Courtesy of Shell Geoscience Services)

Contents

Forward scattering

Let $c_0(\vec{x})$ be a smooth approximation of the true velocity model $c(\vec{x})$, G_0 the surface multiple free Green's function associated to the model c_0, G the surface multiple free Green's function for the true velocity model.

Thus, G_0 and G are determined by

$$\begin{aligned}\left(\frac{1}{c_0^2(\vec{x})}\frac{\partial^2}{\partial t^2} - \Delta\right) G_0(\vec{x}, \vec{x}_s, t) &= \delta(\vec{x} - \vec{x}_s)\delta(t), \\ \left(\frac{1}{c^2(\vec{x})}\frac{\partial^2}{\partial t^2} - \Delta\right) G(\vec{x}, \vec{x}_s, t) &= \delta(\vec{x} - \vec{x}_s)\delta(t),\end{aligned}$$

plus radiation conditions for $|\vec{x}| \rightarrow \infty$.

Forward scattering, continued

Subtracting these two equations, we get

$$\left(\frac{1}{c_0^2(\vec{x})}\frac{\partial^2}{\partial t^2} - \Delta\right)\Big[G(\vec{x},\vec{x}_s,t) - G_0(\vec{x},\vec{x}_s,t)\Big] =$$
$$\left(c_0^{-2}(\vec{x}) - c^{-2}(\vec{x})\right)\frac{\partial^2 G}{\partial t^2}.$$

This can be solved as

$$G(\vec{x}_s,\vec{x}_r,t) - G_0(\vec{x}_s,\vec{x}_r,t) =$$
$$\int d\vec{x}d\tau G_0(\vec{x}_s,\vec{x},\tau)V(\vec{x})\frac{\partial^2}{\partial t^2}G(\vec{x},\vec{x}_r,t-\tau),$$

where we have introduced the scattering potential

$$V(\vec{x}) := c_0^{-2}(\vec{x}) - c^{-2}(\vec{x}).$$

Forward scattering, continued

This is called the *Lipmann-Schwinger equation*. In the frequency domain it reads

$$\hat{G}(\vec{x}_s, \vec{x}_r, \omega) - \hat{G}_0(\vec{x}_s, \vec{x}_r, \omega) = -\omega^2 \int d\vec{x} \hat{G}_0(\vec{x}_s, \vec{x}, \omega) V(\vec{x}) \hat{G}(\vec{x}, \vec{x}_r, \omega).$$

Introducing the integral operator

$$\left(C \cdot \hat{G} \right) (\vec{x}_s, \vec{x}_r, \omega) := -\omega^2 \int d\vec{x} \hat{G}_0(\vec{x}_s, \vec{x}, \omega) V(\vec{x}) \hat{G}(\vec{x}, \vec{x}_r, \omega),$$

we rewrite this as

$$(1 - C) \cdot \hat{G} = \hat{G}_0.$$

Forward scattering, continued

- Formally inverting, we obtain the *forward scattering series*

$$
\begin{aligned}
\hat{G}(\vec{x}_s, \vec{x}_r, \omega) &= \sum_{k \geq 0} \left(C^k \cdot \hat{G}_0 \right) (\vec{x}_s, \vec{x}_r, \omega) \\
&= -\omega^2 \int d\vec{x} \hat{G}_0(\vec{x}_s, \vec{x}, \omega) V(\vec{x}) \hat{G}_0(\vec{x}, \vec{x}_r, \omega) \\
&\quad + \omega^4 \int d\vec{x}_1 d\vec{x}_2 \hat{G}_0(\vec{x}_s, \vec{x}_1, \omega) V(\vec{x}_1) \hat{G}_0(\vec{x}_1, \vec{x}_2, \omega) \\
&\qquad\qquad V(\vec{x}_2) \hat{G}_0(\vec{x}_2, \vec{x}_r, \omega) \\
&\quad - \omega^6 \int d\vec{x}_1 d\vec{x}_2 d\vec{x}_3 \, \hat{G}_0(\vec{x}_s, \vec{x}_1, \omega) V(\vec{x}_1) \hat{G}_0(\vec{x}_1, \vec{x}_2, \omega) V(\vec{x}_2) \\
&\qquad\qquad \hat{G}_0(\vec{x}_2, \vec{x}_3, \omega) V(\vec{x}_3) \hat{G}_0(\vec{x}_3, \vec{x}_r, \omega) \\
&\quad + \cdots
\end{aligned}
$$

1st order term, Born modelling and inversion

Assuming that $\hat{G}_0$ vanishes at the acquisition level (i.e. that the background medium is reflection free), and neglecting all terms after the first one, we find the Born approximation for the scattered field

$$\hat{G}(\vec{x}_s, \vec{x}_r, \omega) \cong -\omega^2 \int d\vec{x} \hat{G}_0(\vec{x}_s, \vec{x}, \omega) V(\vec{x}) \hat{G}_0(\vec{x}, \vec{x}_r, \omega).$$

This is a linear relation between deconvolved data $\hat{G}$ and the scattering potential V. To invert it, we use high frequency asymptotics

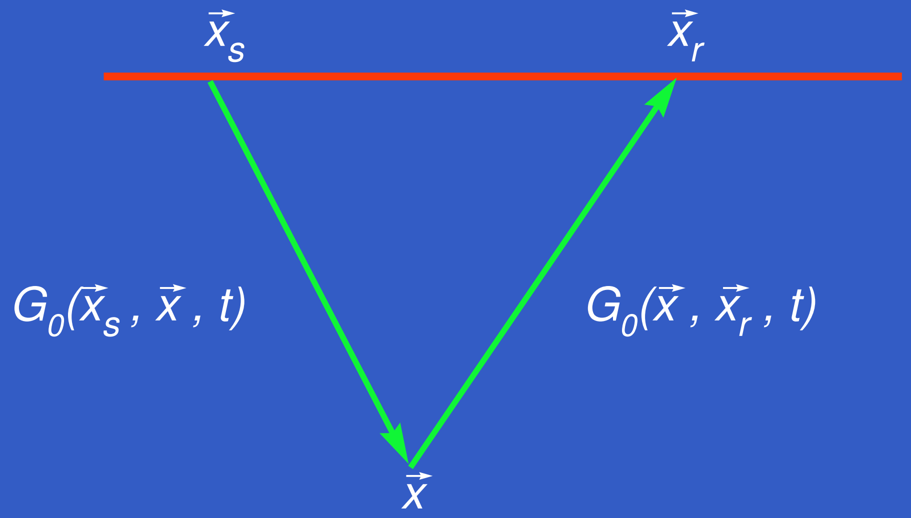
$$\hat{G}_0(\vec{x}, \vec{y}, \omega) \sim A(\vec{x}, \vec{y}) e^{i\omega\phi(\vec{x}, \vec{y})},$$

where A is an amplitude and $\phi(\vec{x}, \vec{y})$ is the traveltime for a ray from $\vec{x}$ to $\vec{y}$ in the velocity model $c_0(\vec{x})$.

Born modelling and inversion, continued

- Inserting this, the Born approximation becomes

$$\hat{G}(\vec{x}_s, \vec{x}_r, \omega) \sim -\omega^2 \int d\vec{x} A(\vec{x}_s, \vec{x}) A(\vec{x}, \vec{x}_r) V(\vec{x}) e^{i\omega(\phi(\vec{x}_s,\vec{x})+\phi(\vec{x},\vec{x}_r))}.$$



Because of the high frequency approximation this is a relation between singularities in V and singularities in $\hat{G}$.

Born modelling and inversion, continued

Now fix the shot coordinates x_s and y_s, i.e., consider only the data from a single shot. In order to invert the resulting forward operator $F(\vec{x}_s)$, one requires ideal illumination and the absence of caustics. The result is

$$V(\vec{x}) \sim \int_{z_r=0} dx_r dy_r W(\vec{x}_s, \vec{x}, \vec{x}_r) G(\vec{x}_s, \vec{x}_r, t = \phi(\vec{x}_s, \vec{x}) + \phi(\vec{x}, \vec{x}_r)).$$

Here W is a weight, which we will not specify further. This formula is usually referred to as *common shot migration*.

Note that the left hand side does not depend on the shot coordinate. In other words: the image of the earth does not depend on the shot used. For this to be true, $c_0(\vec{x})$ needs to be a good approximation of $c(\vec{x})$.

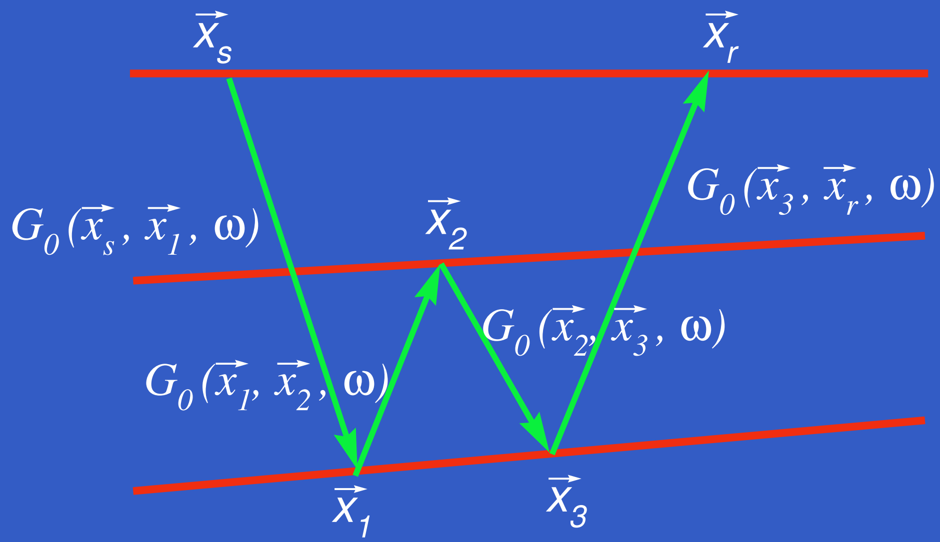
3rd order term: internal multiples

The third order term,

$$\omega^6 \int d\vec{x}_1 d\vec{x}_2 d\vec{x}_3 \, \hat{G}_0(\vec{x}_s, \vec{x}_1, \omega) V(\vec{x}_1) \hat{G}_0(\vec{x}_1, \vec{x}_2, \omega) V(\vec{x}_2)$$

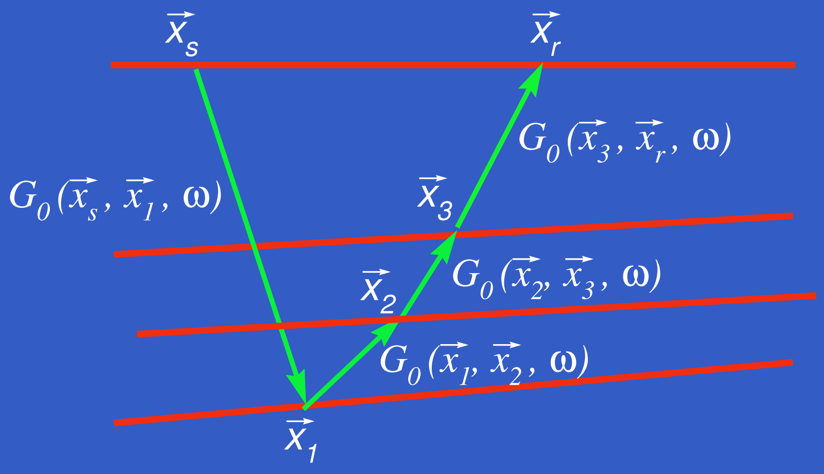
$$\hat{G}_0(\vec{x}_2, \vec{x}_3, \omega) V(\vec{x}_3) \hat{G}_0(\vec{x}_3, \vec{x}_r, \omega),$$

contains contributions, which can easily be identified as internal multiples. This is explained in the following figure.

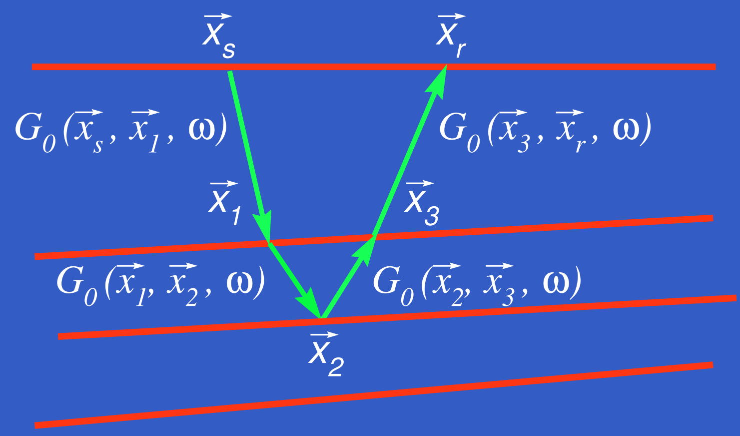


3rd order term, other contributions

The third order term also gets contributions of the form



$$z_1 > z_2 > z_3$$



$$z_2 > z_1, z_2 > z_3$$

To avoid these, we restrict the integration domain to $z_1 > z_2, z_3 > z_2$.

Internal multiples, velocity dependent prediction

The resulting formula,

$$\hat{d}^{IM}(\vec{x}_s,\vec{x}_r,\omega) = \hat{w}(\omega)\omega^6 \int_{\substack{z_1>z_2\\ z_3>z_2}} d\vec{x}_1 d\vec{x}_2 d\vec{x}_3\, \hat{G}_0(\vec{x}_s,\vec{x}_1,\omega)V(\vec{x}_1)$$
$$\hat{G}_0(\vec{x}_1,\vec{x}_2,\omega)V(\vec{x}_2)\hat{G}_0(\vec{x}_2,\vec{x}_3,\omega)V(\vec{x}_3)\hat{G}_0(\vec{x}_3,\vec{x}_r,\omega),$$

models interbed multiples, but the prediction clearly requires the velocity model $c_0(\vec{x})$.

Kinematics can be predicted independent of velocity

Proposition 1 *Under suitable assumptions, the velocity dependent formula is asymptotically equivalent to*

$$\hat{d}^{IM}(\vec{x}_s, \vec{x}_r, \omega) = \hat{w}(\omega)^{-2} \int_{\substack{t_1 > t_2 \\ t_3 > t_2}} d\vec{r}_1 d\vec{r}_2 dt_1 dt_2 dt_3 B \tilde{d}(\vec{x}_s, \vec{x}_{r_1}, t_1) \tilde{d}(\vec{x}_{r_1}, \vec{x}_{r_2}, t_2) d(\vec{x}_{r_2}, \vec{x}_r, t_3) e^{i\omega(t_1 - t_2 + t_3)}.$$

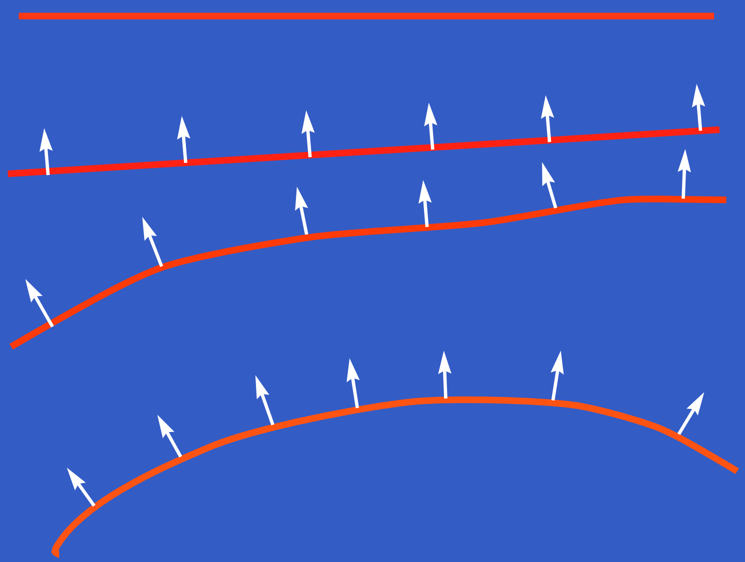
Here

$$\tilde{d}(\vec{x}_s, \vec{x}_r, t) := \frac{1}{8\pi^3} \int d\vec{y}_r d\vec{k}_r dt' d\omega \sqrt{\omega^2 / c(\vec{x}_r)^2 - k_r^2} \, d(\vec{x}_s, \vec{y}_r, t') e^{-i\vec{k}_r \cdot (\vec{x}_r - \vec{y}_r) - i\omega(t - t')}$$

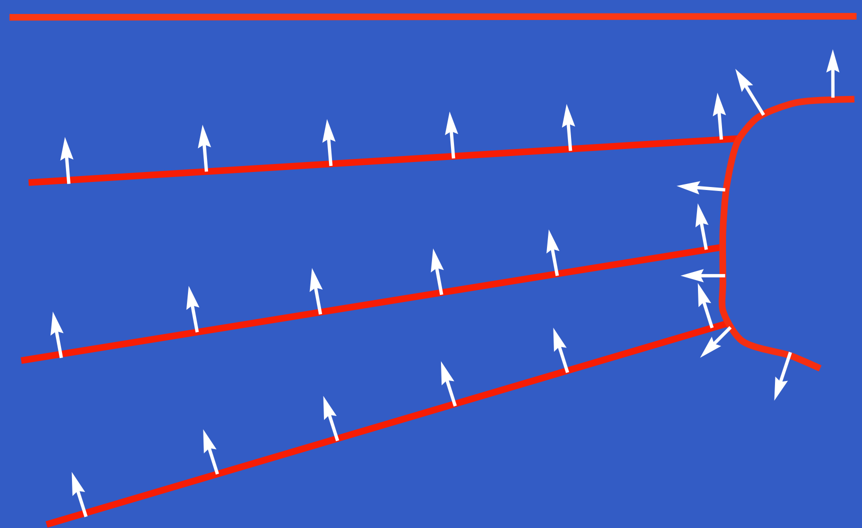
and $B = B[c_0]$ is a velocity dependent amplitude, which reduces to 1 for a constant velocity background medium.

Velocity free prediction, assumptions

Assumption 1 *(Conormality assumption)*
The reflectivity of the earth is *conormal*, i.e., there is a smooth map $\vec{x} \mapsto \vec{n}(\vec{x})$ from $\mathbf{R}^3$ to the unit sphere such that the function $V(\vec{x})$ is singular in the direction $\vec{n}(\vec{x})$ only.



conormal reflectivity distribution

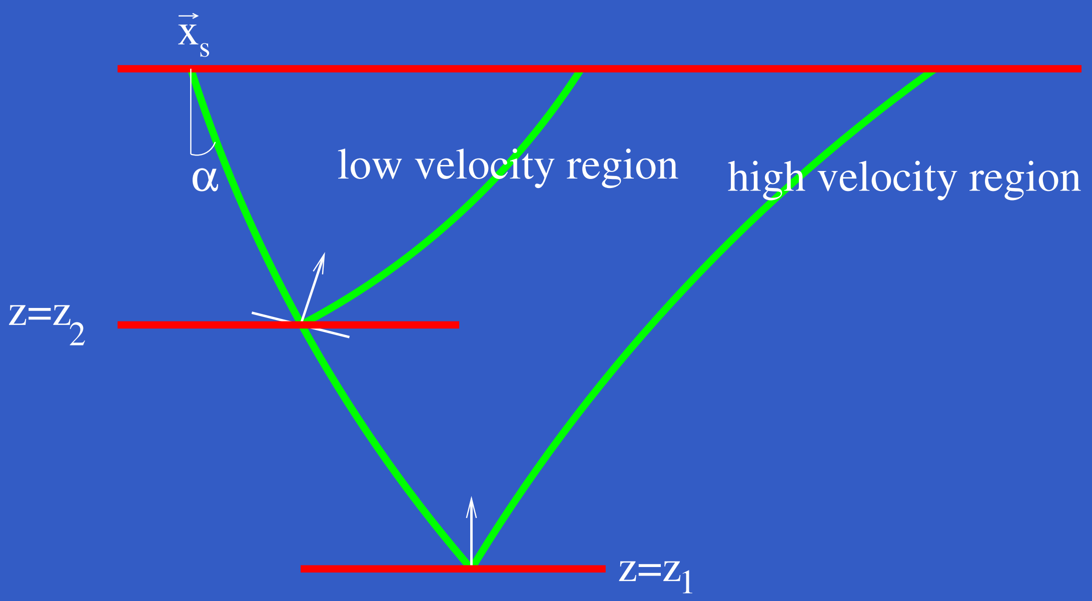


non-conormal reflectivity distribution

Velocity free prediction, assumptions

Assumption 2 *(Traveltime Monotonicity Condition)*
Let $\tau(\vec{x}_s;\alpha,z)$ be the traveltime of a ray taking off at a source location $\vec{x}_s$ at angle α with the normal, reflecting at depth z according to Snell's law with respect to the local normal and travelling back to the surface. Then $\forall \vec{x}_s, \alpha$

$$\tau(\vec{x}_s;\alpha,z_1) > \tau(\vec{x}_s;\alpha,z_2) \Longleftrightarrow z_1 > z_2$$



Possible violation of Assumption 2

Sketch of the proof

- Replacing the scattering potentials occurring in the velocity dependent formula by common shot migrated data, we get

$$\hat{d}^{IM}(\vec{x}_s, \vec{x}_r, \omega) = \hat{w}(\omega)^{-2} \int_{\substack{z_1 > z_2 \\ z_3 > z_2}} dt_1 dt_2 dt_3 d\vec{r}_1 d\vec{r}_2 \Big(d\vec{r}_3 d\vec{x}_1 d\vec{x}_2 d\vec{x}_3$$

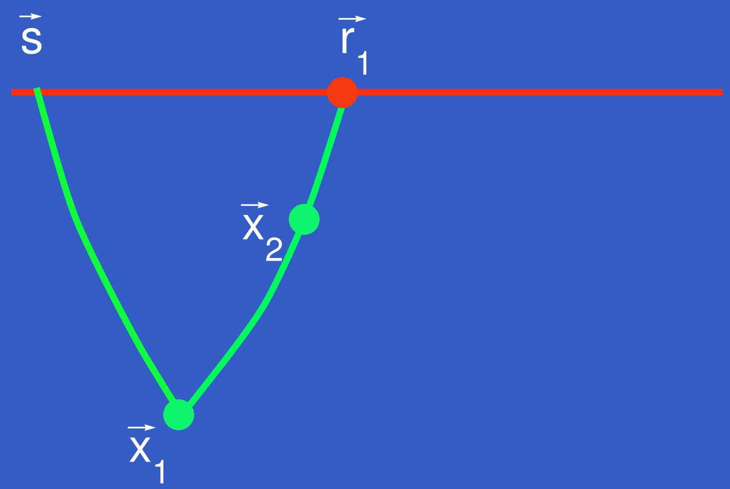
$$d\omega_1 d\omega_2 d\omega_3 \Big) B d(\vec{x}_s, \vec{x}_{r_1}, t_1) d(\vec{x}_{r_1}, \vec{x}_{r_2}, t_2) d(\vec{x}_{r_2}, \vec{x}_{r_3}, t_3) e^{i\psi},$$

where $\vec{r}_i = (x_{r_i}, y_{r_i})$, B is a product of forward amplitudes A and migration weights W and the phase ψ is given by

$$\begin{aligned} \psi \quad := \quad & \omega \left[\phi(\vec{x}_s, \vec{x}_1) + \phi(\vec{x}_1, \vec{x}_2) + \phi(\vec{x}_2, \vec{x}_3) + \phi(\vec{x}_3, \vec{x}_r) \right] \\ & - \omega_1 \left[\phi(\vec{x}_s, \vec{x}_1) + \phi(\vec{x}_1, \vec{x}_{r_1}) - t_1 \right] \\ & - \omega_2 \left[\phi(\vec{x}_{r_1}, \vec{x}_2) + \phi(\vec{x}_2, \vec{x}_{r_2}) - t_2 \right] \\ & - \omega_3 \left[\phi(\vec{x}_{r_2}, \vec{x}_3) + \phi(\vec{x}_3, \vec{x}_{r_3}) - t_3 \right] \end{aligned}$$

Sketch of the proof, continued

Step 1: Integration with respect to $(\vec{x}_1, \omega_1)$. Relative part ψ:
$\omega\left[\phi(\vec{x}_s,\vec{x}_1)+\phi(\vec{x}_1,\vec{x}_2)\right]-\omega_1\left[\phi(\vec{x}_s,\vec{x}_1)+\phi(\vec{x}_1,\vec{x}_{r_1})-t_1\right]$

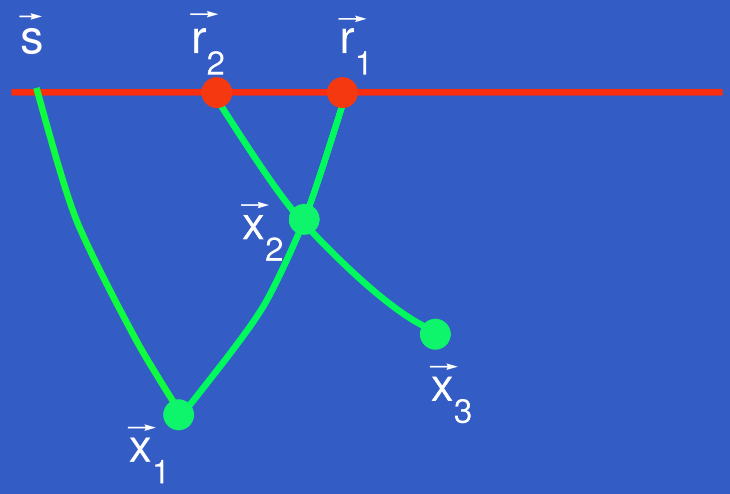


The result of this integration is

1. $\vec{x}_1$ is on the ray connecting $\vec{x}_2$ and $\vec{r}_1$,
2. $\phi(\vec{x}_s,\vec{x}_1)+\phi(\vec{x}_1,\vec{x}_{r_1})=t_1$.

Sketch of the proof, continued

Step 2: Integration with respect to $(\vec{x}_2, \omega_2)$. Relative part ψ:
$\omega\left[-\phi(\vec{x}_{r_1}, \vec{x}_2) + \phi(\vec{x}_2, \vec{x}_3)\right] - \omega_2\left[\phi(\vec{x}_{r_1}, \vec{x}_2) + \phi(\vec{x}_2, \vec{x}_{r_2}) - t_2\right]$

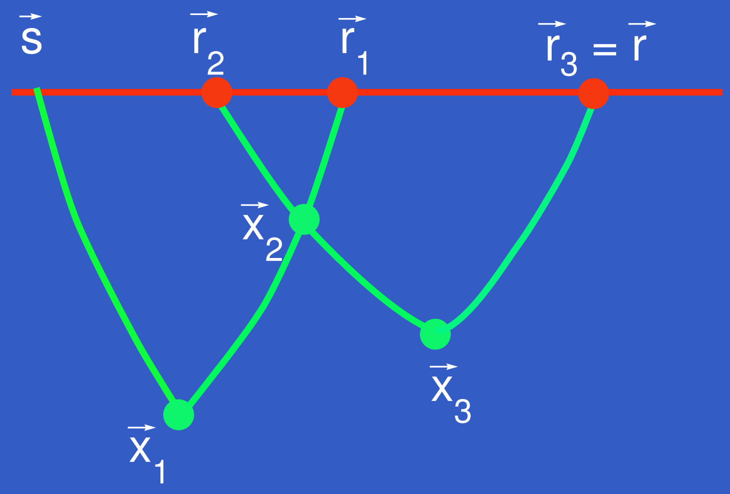


The result of this integration is

1. $\vec{x}_2$ must be on the ray connecting $\vec{r}_2$ and $\vec{x}_3$,
2. $\phi(\vec{x}_{r_1}, \vec{x}_2) + \phi(\vec{x}_2, \vec{x}_{r_2}) = t_2$.

Sketch of the proof, continued

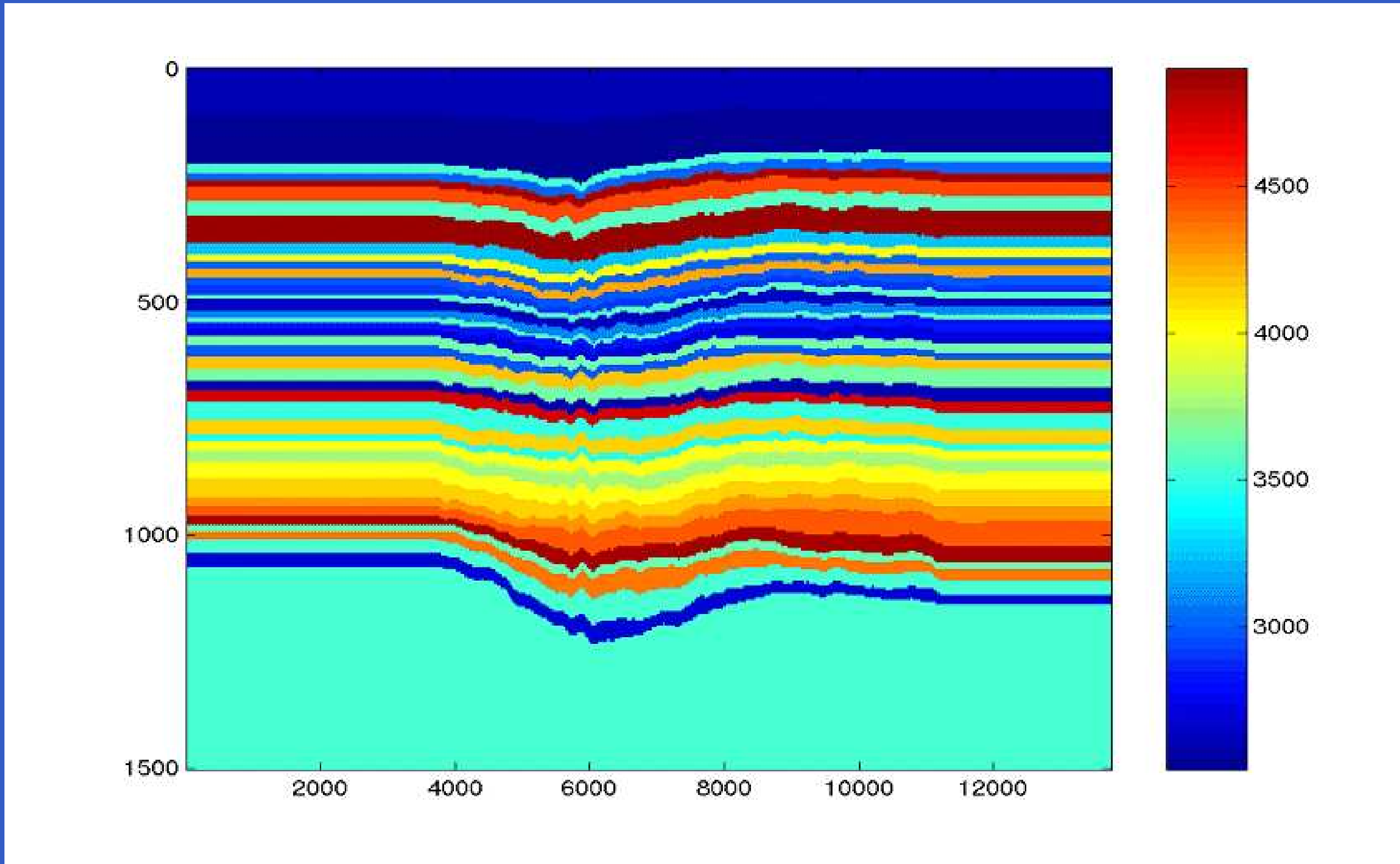
Step 3: Integration with respect to $(\vec{x}_3, \omega_3, \vec{r}_3)$. Relative part ψ: $\omega\left[\phi(\vec{x}_{r_2}, \vec{x}_3) + \phi(\vec{x}_3, \vec{x}_r)\right] - \omega_3\left[\phi(\vec{x}_{r_2}, \vec{x}_3) + \phi(\vec{x}_3, \vec{x}_{r_3}) - t_3\right]$



The result of this integration is

1. $\vec{r}_3 = \vec{r}$,
2. $\phi(\vec{x}_{r_2}, \vec{x}_3) + \phi(\vec{x}_3, \vec{x}_{r_3}) = t_3$.

Velocity model



Velocity model used in synthetic interbed multiple study

Images

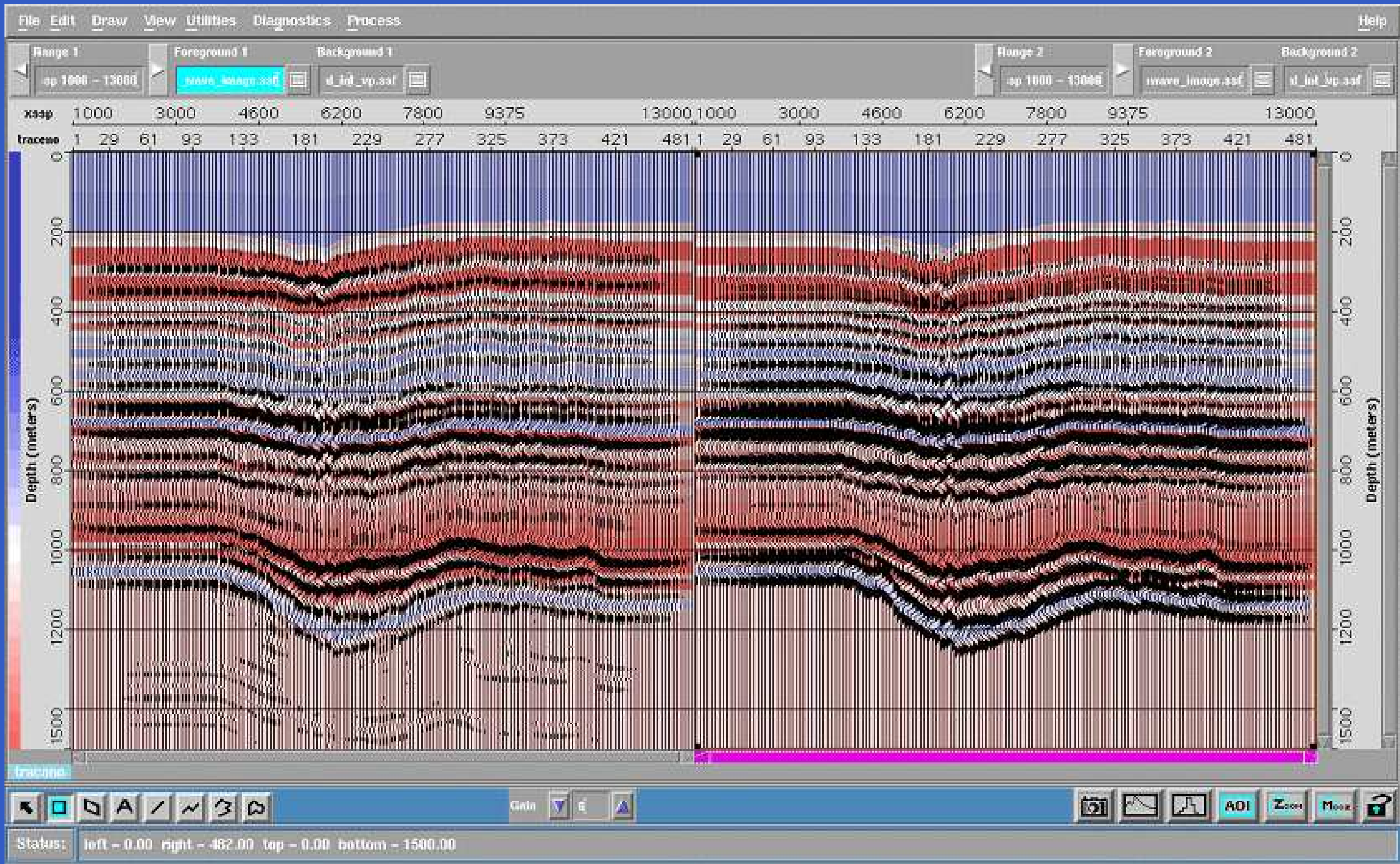


Image of the data with (left) and without (right) interbed multiples

Predictions

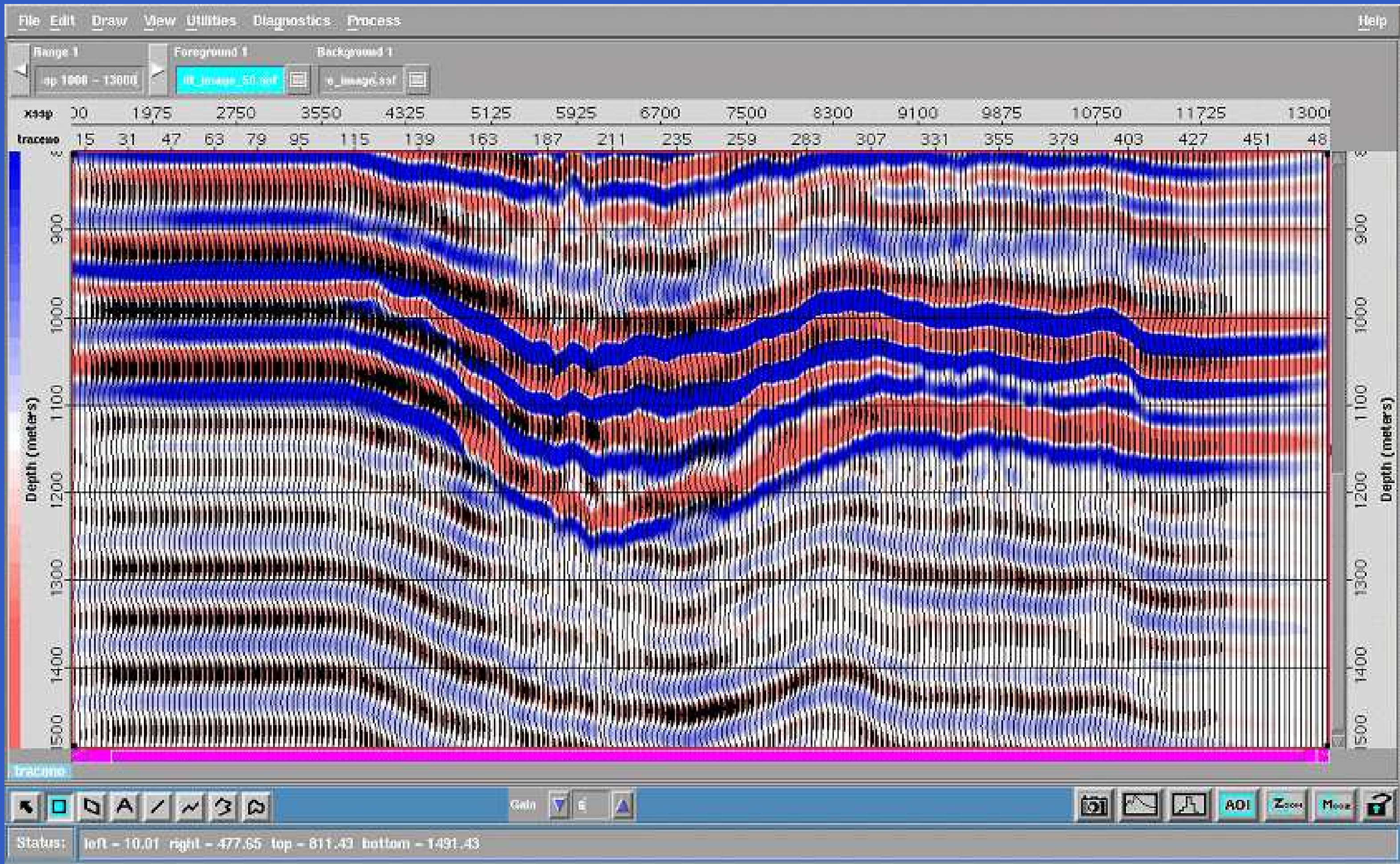


Image of predicted multiples (black) on top of image of the data (color)

Cleaned images

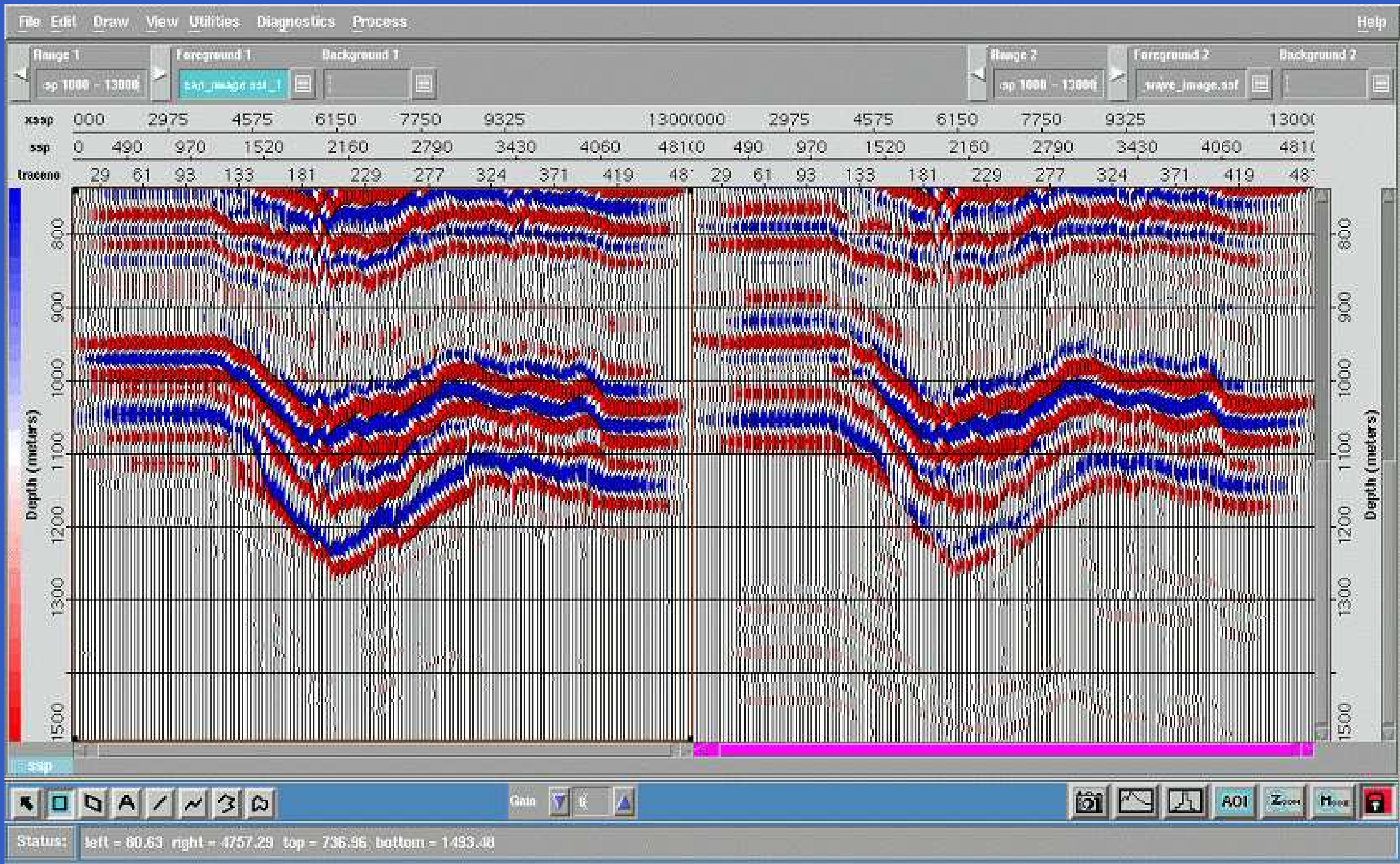


Image of the data (right) vs image of the data after adaptive multiple subtraction (right)