Asymptotic linearized inversion in the presence of caustics

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Statement of the problem

- Beylkin’s work on high frequency linearized inversion uses a simple ray path assumption, which excludes the occurrence of caustics. The question is how to image in genuinely complex models, such as the ones below, in which wavefields do develop caustics.

Left: complicated overburden due to faulting; right: salt dome
Example: Marmousi
Wave fronts generated by shot in Marmousi model
Rays associated to shot from previous slide
Marmousi: seismogram of shot
Wave fronts generated by shot in salt dome model
Rays associated to shot from previous slide
Basic assumptions and notation

- Earth is a constant density acoustic medium; wave propagation described by

\[
\left( \frac{1}{c^2(\vec{x})} \frac{\partial^2}{\partial t^2} - \Delta \right) G(\vec{x}, \vec{x}_s, t) = \delta(\vec{x} - \vec{x}_s) \delta(t).
\]

where \( G(\vec{x}, \vec{x}_s, t) \) is Green’s function, the response of the medium at location \( \vec{x} \) due to an instantaneous point source at location \( \vec{x}_s \).

This equation needs to be supplied with additional initial/boundary/radiation conditions to determine the solution uniquely.
Basic assumptions and notation, continued

- The temporal Fourier transform of $G$ will be denoted by $\hat{G}(\vec{x}, \vec{x}_s, \omega)$.

- We assume we have acquired an idealized surface seismic data set, where sources and receivers vary over the surface of the earth, i.e., $\vec{x}_s = (x_s, y_s, 0)$, $\vec{x}_r = (x_r, y_r, 0)$, $x_s, y_s, x_r, y_r \in \mathbb{R}$.

- Model space $X$ is the set

$$\{ \vec{x} = (x, y, z) \mid z > 0 \}.$$

- Data space $Y$ is the set

$$\{ \vec{y} = (\vec{x}_s, \vec{x}_r, t) \mid \vec{x}_s = (x_s, y_s, 0), \vec{x}_r = (x_r, y_r, 0), t > 0 \}.$$
The seismic inverse problem

- Solving the wave equation defines a functional $S$, which maps the coefficient $c(\vec{x})$ to the Green’s function,

$$S : c(\vec{x}) \mapsto G(\vec{x}_r, \vec{x}_s, t).$$

The seismic inverse problem is to determine $c(\vec{x})$ from the data $G(\vec{x}_r, \vec{x}_s, t)$, i.e., to construct $S^{-1}$.

The construction of $S^{-1}$ is difficult because $S$ is highly nonlinear. Therefore, we linearize against a background velocity model $c_0(\vec{x})$. The linearized inverse problem is to invert the formal derivative map

$$DS[c_0] : \delta c(\vec{x}) \mapsto \delta G(\vec{x}_r, \vec{x}_s, t).$$

Linearization

Let $G_0$ and $G$ be the Green’s functions associated to the background model $c_0$ and the true model $c$, respectively. Formally differentiating the wave equation

$$\left( \frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \Delta \right) G(x, x_s, t) = \delta(x - x_s) \delta(t)$$

with respect to the velocity $c$ at $c = c_0$, we get

$$- \frac{2\delta c(x)}{c_0^3(x)} \frac{\partial^2}{\partial t^2} G_0(x, x_s, t) + \left( \frac{1}{c_0^2(x)} \frac{\partial^2}{\partial t^2} - \Delta \right) \delta G(x, x_s, t) = 0.$$ 

This can be solved as

$$\delta G(x_r, x_s, t) = \int d\vec{x}' d\tau \ G_0(x_r, \vec{x}', \tau) V(\vec{x}') \frac{\partial^2}{\partial t^2} G_0(\vec{x}', x_s, t - \tau),$$
where we have introduced the *scattering potential* or *reflectivity function* $V(\vec{x})$ by

$$V(\vec{x}) := \frac{2\delta c(\vec{x})}{c_0^3(\vec{x})}. $$

This equation is called the *Born approximation* to the scattered wave field. Using the Fourier transform of $G_0$ it can be rewritten as

$$\delta G(\vec{x}_r, \vec{x}_s, t) = -\frac{1}{2\pi} \int d\vec{x} d\omega \omega^2 e^{-i\omega t} \hat{G}_0(\vec{x}_r, \vec{x}, \omega) V(\vec{x}) \hat{G}_0(\vec{x}, \vec{x}_s, \omega). $$
High frequency asymptotics

Replacing $\hat{G}_0$ by a high-frequency approximation $\hat{G}^{hf}_0$, we obtain a map $F : C^\infty_0(X) \to D'(Y)$ defined by

$$F(V)(\vec{x}_r, \vec{x}_s, t) := \frac{1}{2\pi} \int d\omega d\vec{x}(i\omega)^2 e^{-i\omega t} \hat{G}^{hf}_0(\vec{x}_r, \vec{x}, \omega)V(\vec{x})\hat{G}^{hf}_0(\vec{x}, \vec{x}_s, \omega).$$

Replacing $DS[c_0](V)$ by $F(V)$, one makes an error in the low-frequency part of this integral. This error is smooth. Therefore, $F(V)$ correctly represents the singularities in $\delta G$, corresponding to the wavefront of the data.

Asymptotic linearized inversion is to construct $F^{-1}$ instead of $DS[c_0]^{-1}$. $F^{-1}(\delta G)$ recovers the singularities in the reflectivity function, corresponding to the reflectors.
Asymptotic linearized forward map is an FIO

- **Theorem 1** *(Rakesh, 1988)* If
  
  - there are no grazing rays (i.e. rays tangent to the source or receiver manifold),
  
  - there are no rays scattering over \( \pi \) (i.e. travelling from a source to a receiver via the support of \( \delta_c(\vec{x}) \)),

  then \( F \) is a Fourier Integral Operator (FIO).

For caustic-free media, this had been shown by G. Beylkin in the early eighties.
Consider an integral operator $F : C^\infty_0(X) \rightarrow D'(Y)$ of the form

$$F = \int d\tilde{\theta} A(\vec{x}, \vec{y}, \tilde{\theta}) e^{i\Phi(\vec{x}, \vec{y}, \tilde{\theta})}$$

Recall that the stationary set of $F$ is the set

$$S_\Phi := \{ (\vec{x}, \vec{y}, \tilde{\theta}) \in X \times Y \times \mathbb{R}_{\tilde{\theta}} | \nabla_{\tilde{\theta}} \Phi(\vec{x}, \vec{y}, \tilde{\theta}) = 0 \}.$$  

Then $F$ is called a Fourier integral operator if

- The phase function $\Phi$ is homogeneous in $\theta$, i.e.,

$$\Phi(\vec{x}, \vec{y}, \lambda \tilde{\theta}) = \lambda \Phi(\vec{x}, \vec{y}, \tilde{\theta}).$$
Definition FIO, continued

- The map \( i_\Phi : S_\Phi \to T^*X \times T^*Y \) defined by
  \[
i_\Phi : (\vec{x}, \vec{y}, \vec{\theta}) \in S_\Phi \mapsto (\vec{x}, \nabla_{\vec{x}} \Phi, \vec{y}, \nabla_{\vec{y}} \Phi)\]
  is an immersion, i.e., its derivative \( Di_\Phi \) is injective.

- \( \forall \vec{\theta} \neq 0 : \ i_\Phi(\vec{x}, \vec{y}, \vec{\theta}) \in T^*X \setminus 0 \times T^*Y \setminus 0.\)

- The amplitude \( A \) satisfies certain Sobolev estimates.

The image \( i_\Phi(S_\Phi) \) is called the wavefront relation of the FIO \( F \) and will be denoted by \( \Lambda_\Phi \).
An example of FIOs, pseudo-differential operators

A simple example of an FIO is the pseudo-differential operator

$$F = \int dp^* A(\vec{x}, \vec{y}, \vec{p}) e^{i\vec{p} \cdot (\vec{x} - \vec{y})}.$$ 

Here $X = Y$ and $i_{\Phi}$ is the map

$$i_{\Phi} : (\vec{x}, \vec{x}, \vec{p}) \mapsto (\vec{x}, \vec{p}, \vec{x}, -\vec{p}) \} ,$$

which is clearly immersive. The wavefront relation is the diagonal $\{ (\vec{x}, \vec{p}, \vec{x}, -\vec{p}) \}$ in $T^* X \times T^* X$.

In general, any FIO with a wavefront relation like this is called a pseudo-differential operator.
Sketch of the proof, Maslov’s representation

- To construct a high frequency approximation for Green’s function at a source point $\vec{x}_0$, which remains valid in the presence of caustics, we consider Hamilton’s equations

\[
\frac{d\vec{x}}{dt} = c^2(\vec{x})\vec{p}, \quad \frac{d\vec{p}}{dt} = -c^{-1}(\vec{x}) \nabla c,
\]

with initial conditions

\[
\vec{x}(0) = \vec{x}_0, \quad \vec{p}(0) = \frac{1}{c(\vec{x}_0)} (\sin \alpha_1 \cos \alpha_2, \sin \alpha_1 \sin \alpha_2, \cos \alpha_1)^t.
\]

Denote the solution manifold by $\Lambda_{\vec{x}_0}$,

\[
\Lambda_{\vec{x}_0} := \{ (\vec{x}(\alpha_1, \alpha_2, t), \vec{p}(\alpha_1, \alpha_2, t), \alpha_1 \in [0, \pi], \alpha_2 \in [0, 2\pi], t > 0) \}.
\]
Maslov’s representation, back to the no-caustics case

- The manifold $\Lambda_{\vec{x}_0}$ of all rays emerging from $\vec{x}_0$ is an $n$-dimensional manifold. In the absence of caustics, there is a global traveltime function $\phi(\vec{x}_0, \vec{x})$ and $\Lambda_{\vec{x}_0}$ can also be parameterized as

$$\Lambda_{\vec{x}_0} := \{(\vec{x}, \vec{p}) | \vec{p} = \nabla_{\vec{x}} \phi\}.$$  

Alternatively, $\Lambda_{\vec{x}_0}$ can be described as the stationary set of the phase function

$$\psi(\vec{x}, \vec{p}) := \frac{1}{2} \sum_{i=1}^{3} \left( p_i - \frac{\partial \phi}{\partial x_i} \right)^2.$$  

In the case of caustics, this breaks down, since $\Lambda_{\vec{x}_0}$ will have vertical directions. Therefore, a more general approach is needed.
Maslov’s representation, continued

- It can be shown that $\Lambda_{\bar{x}_0}$ is a *Lagrangian* submanifold, i.e. that the restriction of the natural symplectic form on phase space $T^* X$ to $\Lambda_{\bar{x}_0}$ vanishes. From this fundamental fact one derives that $\Lambda_{\bar{x}_0}$ is locally described by the stationarity of non-degenerate phase functions $\psi(\bar{x}_0; \bar{x}, \bar{p})$, i.e.,

$$ (\bar{x}, \bar{p}) \in \Lambda_{\bar{x}_0} \iff \nabla_{\bar{p}} \psi(\bar{x}_0; \bar{x}, \bar{p}) = 0. $$

Non-degeneracy of the phase function $\psi$ means that

$$ \text{rank } \nabla_{(\bar{x}, \bar{p})} \nabla_{\bar{p}} \psi = 3. $$

The implicit function theorem states that, under this condition the stationarity conditions indeed define a 3-dimensional manifold.
This way one obtains Maslov’s representation for the Green’s function, which remains valid if the wave field develops caustics.

\[
\hat{G}_{0}^{hf}(\vec{x}, \vec{x}_s, \omega) \sim (-i\omega)^{3/2} \int_{\mathbb{R}^3} d\vec{p}_s a(\vec{x}; \vec{p}_s, \vec{x}_s) e^{i\omega \psi(\vec{x}; \vec{p}_s, \vec{x}_s)}.
\]

The amplitude function \(a(\vec{x}; \vec{p}_s, \vec{x}_s)\) is constructed by solving a transport equation on the manifold \(\Lambda_{\vec{x}}\).

If \(\det (d^2_{\vec{p}_s} \psi) \neq 0\), the equation \(\nabla_{\vec{p}_s} \psi(\vec{x}; \vec{p}_s, \vec{x}_s) = 0\) can be solved to yield \(\vec{p}_s = \vec{p}_s^{(i)}(\vec{x}; \vec{x}_s), i = 1, \ldots N\). Using this, we find

\[
\hat{G}_{0}^{hf}(\vec{x}, \vec{x}_s, \omega) \sim \sum_{i=1}^{N} A^{(i)}(\vec{x}, \vec{x}_s) e^{i\omega \text{sgn}(d^2_{\vec{p}_s} \psi) e^{i\omega \psi^{(i)}(\vec{x}, \vec{x}_s)}}.
\]
Maslov’s representation, continued

where

$$\phi^{(i)}(\vec{x}, \vec{x}_s) := \psi(\vec{x}; p^{(i)}_s(\vec{x}; \vec{x}_s), \vec{x}_s),$$

$$A^{(i)}(\vec{x}, \vec{x}_s) := (2\pi)^{3/2} \frac{a(\vec{x}; p^{(i)}_s(\vec{x}; \vec{x}_s), \vec{x}_s)}{\det \left( d^2_{\vec{p}_s} \psi \right)}.$$

This represents a superposition of $N$ wavefronts, except in so-called *caustic* points,

$$\{ \vec{x}_s \mid \exists \vec{p}_s : \det \left( d^2_{\vec{p}_s} \psi(\vec{x}; \vec{x}_s, \vec{p}_s) \right) = 0 \}.$$

In those points one needs to retain Maslov’s integral representation.
Sketch of the proof, continued

Substituting Maslov’s representation in the Born integral yields

$$
\delta G(\vec{x}_r, \vec{x}_s, t) \sim \frac{1}{2\pi} \int (-i\omega)^{-1} d\omega d(\omega \vec{p}_r) d(\omega \vec{p}_s) d\vec{x} \ V(\vec{x}) \\
a(\vec{x}; \vec{x}_r, \vec{p}_r) a(\vec{x}; \vec{x}_s, \vec{p}_s) e^{i\omega(\psi(\vec{x}; \vec{x}_r, \vec{p}_r) + \psi(\vec{x}; \vec{x}_s, \vec{p}_s) - t)}.
$$

To get this in the form $\delta G \sim F(V)$, with $F = \int d\vec{\theta} A e^{i\Phi}$, we take $\vec{\theta} := (\omega \vec{p}_s, \omega \vec{p}_r, \omega)$, $\vec{y} := (\vec{x}_r, \vec{y}_r, t)$ and set

$$
\Phi(\vec{x}, \vec{y}, \vec{\theta}) := \omega \left( \psi(\vec{x}_r, \vec{p}_r; \vec{x}) + \psi(\vec{x}_s, \vec{p}_s; \vec{x}) - t \right),
$$

$$
A(\vec{x}, \vec{y}, \vec{\theta}) := (2\pi)^{-1} (-i\omega)^{-1} a(\vec{x}_r, \vec{p}_r; x) a(\vec{x}_s, \vec{p}_s; \vec{x}).
$$
Sketch of the proof, continued

- The conditions on $\Phi$ in the theorem are now easily verified.
- $\Phi(\vec{x}, \vec{y}, \vec{\theta}) = \omega(\psi(\vec{x}, \vec{p}_r; \vec{x}) + \psi(\vec{x}, \vec{p}_s; \vec{x}) - t)$ is clearly homogeneous with respect to $\vec{\theta} = (\omega \vec{p}_r, \omega \vec{p}_s, \omega)$.
- Immersivity of the map $i_\Phi$:

\[
i_\Phi(\vec{x}, \vec{y}, \vec{p}_r, \vec{p}_s, \omega) = (\vec{x}, \omega \nabla_{\vec{x}} \left[ \psi(\vec{x}, \vec{x}_r, \vec{p}_r) + \psi(\vec{x}, \vec{x}_s, \vec{p}_s) \right], \vec{x}_r, \vec{x}_s, t, \omega \nabla_{(x_r, y_r)} \psi(\vec{x}, \vec{x}_r, \vec{p}_r), \omega \nabla_{(x_s, y_s)} \psi(\vec{x}, \vec{x}_s, \vec{p}_s), -\omega),\]

holds because of the non-degeneracy of the phase functions $\psi$ and the “no-grazing rays” assumption.
- The third condition is satisfied if $\nabla_{\vec{x}} \psi(\vec{x}, \vec{x}_r, \vec{p}_r) + \nabla_{\vec{x}} \psi(\vec{x}, \vec{x}_s, \vec{p}_s) \neq 0$. This is the “no scattering over $\pi$” assumption.
Parameterization wavefront relation $\Lambda_\Phi$

\[
\Lambda_\Phi = \left\{(\vec{x}, \omega \nabla_{\vec{x}} \psi(\vec{x}; \vec{x}_r, \vec{p}_r) + \psi(\vec{x}; \vec{x}_s, \vec{p}_s), \vec{x}_r, \vec{x}_s, t, \omega \nabla_{(x_r,y_r)} \psi(\vec{x}; \vec{x}_r, \vec{p}_r), \omega \nabla_{(x_s,y_s)} \psi(\vec{x}; \vec{x}_s, \vec{p}_s), -\omega) \mid (\vec{x}, \vec{x}_r, \vec{x}_s, t, \vec{p}_r, \vec{p}_s, \omega) \in S_\Phi \right\}.
\]

- Stationarity of $\Phi$ with respect to $\vec{p}_r$ means that there is a ray from $\vec{x}$ to $\vec{x}_r$. Its initial slowness is $\nabla_{\vec{x}} \psi(\vec{x}; \vec{x}_r, \vec{p}_r)$ and its arrival slowness is $\vec{p}_r$.
- Similar for stationarity with respect to $\vec{p}_s$.
- Stationarity with respect to $\omega$ means that
  \[
t = \psi(\vec{x}; \vec{x}_r, \vec{p}_r) + \psi(\vec{x}; \vec{x}_s, \vec{p}_s).
  \]
- $\vec{e}(\alpha)$: unit vector defined by angles $\alpha := (\alpha_1, \alpha_2)$.

- $\vec{r}(\vec{x}, \alpha)$: intersection of the ray leaving $\vec{x}$ in the direction $\vec{e}(\alpha)$ with $z = 0$.

- $\phi_r(\vec{x}, \alpha)$: traveltime from $\vec{x}$ to $\vec{r}(\vec{x}, \alpha)$ along this ray.

- $\vec{\rho}(\vec{x}, \alpha)$: projection of slowness vector at $\vec{r}(\vec{x}, \alpha)$ on $z = 0$.

- $\vec{e}(\beta)$, $\vec{s}(\vec{x}, \beta)$, $\phi_s(\vec{x}, \beta)$, $\vec{\sigma}(\vec{x}, \beta)$: defined similarly.
\[ \Lambda_\Phi = \left\{ \left( \vec{x}, \omega \nabla_{\vec{x}} \left[ \psi(\vec{x}; \vec{x}_r, \vec{p}_r) + \psi(\vec{x}; \vec{x}_s, \vec{p}_s) \right], \vec{x}_r, \vec{x}_s, t, \right. \right. \\
\omega \nabla_{(x_r, y_r)} \psi(\vec{x}; \vec{x}_r, \vec{p}_r), \omega \nabla_{(x_s, y_s)} \psi(\vec{x}; \vec{x}_s, \vec{p}_s), -\omega \right), \\
\left| (\vec{x}, \vec{x}_r, \vec{x}_s, t, \vec{p}_r, \vec{p}_s, \omega) \in S_\Phi \right\} \]

\[ = \left\{ \left( \vec{x}, \frac{\omega}{c(\vec{x})} \left( \vec{e}(\alpha) + \vec{e}(\beta) \right), \vec{r}(\vec{x}, \alpha), \vec{s}(\vec{x}, \beta), \right. \left. \right. \left. \phi_r(\vec{x}, \alpha) + \phi_s(\vec{x}, \beta), \omega \vec{\rho}(\vec{x}, \alpha), \omega \vec{\sigma}(\vec{x}, \beta), -\omega \right) \right. \right. \left. \left. \left| \vec{x} \in X, \alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \omega > 0 \right\} \right. \right. \]
The normal operator

- The product $F^* F : C_0^\infty(X) \to D'(X)$ of the asymptotic linearized forward map with its adjoint is called the *normal operator*. If this operator is invertible, the operator $(F^* F)^{-1} F^*$ is a left inverse of $F$.

The normal operator is an integral operator of the form

$$F^* F = \int d\theta d\theta' d\bar{y} A^*(\bar{x}', \bar{y}, \bar{\theta}') A(\bar{x}, \bar{y}, \bar{\theta}) e^{i\Phi(\bar{x}, \bar{y}, \bar{\theta}) - i\Phi(\bar{x}', \bar{y}, \bar{\theta}')}.$$

The stationarity conditions with respect to $\vec{\Theta} := (\vec{\theta}, \vec{\theta}', \vec{y})$ are

1. $\nabla_{\vec{\theta}} \Phi(\vec{x}, \vec{y}, \vec{\theta}) = 0,$
2. $\nabla_{\vec{\theta}} \Phi(\vec{x}', \vec{y}, \vec{\theta}') = 0,$
3. $\nabla_{\vec{y}} \Phi(\vec{x}, \vec{y}, \vec{\theta}) = \nabla_{\vec{y}} \Phi(\vec{x}', \vec{y}, \vec{\theta}').$
From 1 and 2 it follows that

\[ 1' \quad i_{\Phi}(\vec{x}, \vec{y}, \vec{\theta}) = \left( \vec{x}, \frac{\omega}{c(\vec{x})} \left( \vec{e}'(\alpha) + \vec{e}'(\beta) \right), \vec{r}(\vec{x}, \alpha), \vec{s}(\vec{x}, \beta), \right. \]
\[ \left. \phi_r(\vec{x}, \alpha) + \phi_s(\vec{x}, \beta), \omega \vec{\rho}(\vec{x}, \alpha), \omega \vec{\sigma}(\vec{x}, \beta), -\omega \right), \]

\[ 2' \quad i_{\Phi}(\vec{x}', \vec{y}', \vec{\theta}') = \left( \vec{x}', \frac{\omega}{c(\vec{x}')} \left( \vec{e}'(\alpha') + \vec{e}'(\beta') \right), \vec{r}(\vec{x}', \alpha'), \vec{s}(\vec{x}', \beta'), \right. \]
\[ \left. \phi_r(\vec{x}', \alpha') + \phi_s(\vec{x}', \beta'), \omega' \vec{\rho}(\vec{x}', \alpha'), \omega' \vec{\sigma}(\vec{x}', \beta'), -\omega' \right). \]

Combining this with 3, we find

\[ \vec{r}(\vec{x}, \alpha) = \vec{r}(\vec{x}', \alpha'), \]
\[ \vec{s}(\vec{x}, \beta) = \vec{s}(\vec{x}', \beta'), \]
\[ \vec{\rho}(\vec{x}, \alpha) = \vec{\rho}(\vec{x}', \alpha'), \]
\[ \vec{\sigma}(\vec{x}, \beta) = \vec{\sigma}(\vec{x}', \beta'), \]
\[ \phi_r(\vec{x}, \alpha) + \phi_s(\vec{x}, \beta) = \phi_r(\vec{x}', \alpha') + \phi_s(\vec{x}', \beta'). \]
The traveltime injectivity condition (TIC)

- We conclude that
  - There exists a ray (say ray 1) connecting \( \vec{x}, \vec{x}' \) and \( \vec{r} = r(\vec{x}, \vec{x}) = r(\vec{x}', \vec{x}') \).
  - There exists a ray (say ray 2) connecting \( \vec{x}, \vec{x}' \) and \( \vec{s} = s(\vec{x}, \vec{\beta}) = s(\vec{x}', \vec{\beta}') \).
  - The total traveltime for the trajectory \( \vec{s} \xrightarrow{\text{ray 2}} \vec{x} \xrightarrow{\text{ray 1}} \vec{r} \) is the same as the total traveltime for the trajectory \( \vec{s} \xrightarrow{\text{ray 2}} \vec{x}' \xrightarrow{\text{ray 1}} \vec{r} \).

Assumption 1 (Traveltime injectivity condition) Under these circumstances the tuples \( (\vec{x}, \vec{\alpha}, \vec{\beta}) \) and \( (\vec{x}', \vec{\alpha}', \vec{\beta}') \) are identical.
Examples of non-simple ray systems

This situation can occur but does not satisfy all stationarity equations.

This situation is excluded by the TIC.

This situation is excluded by the TIC.
The normal operator is a $\Psi$DO

- **Theorem 2** Under the traveltime injectivity condition the normal operator $F^* F$ is an elliptic pseudo-differential operator. Its principal symbol is given by

$$\int \omega^4 d\omega d\alpha d\beta \psi' C(\vec{x}, \alpha, \beta) e^{i\omega \frac{\vec{e}(\alpha) + \vec{e}(\beta)}{|\vec{e}(\alpha) + \vec{e}(\beta)|} \cdot (\vec{x} - \vec{x}')}. $$

Sketch of the proof:

- We have already argued that under the TIC the wavefront relation of $F^* F$ is (part of the) diagonal.
- The only thing left to show is that the composition of the FIOs $F^*$ and $F$ is again an FIO. This is done using the so-called **clean intersection calculus** developed by Hörmander and Duistermaat.
Ingredients for clean intersection calculus

- Consider the so-called *double fibration* (Gel’fand, Guillemin)

\[
\begin{array}{c}
\Lambda_{\Phi} \\
\cap \\
T^* X \times T^* Y
\end{array}
\]

\[
\begin{array}{ccc}
\pi & & \rho \\
\downarrow & & \downarrow \\
T^* X & \rightarrow & T^* Y
\end{array}
\]

where \(\pi\) and \(\rho\) are the natural projections on \(T^* X\) and \(T^* Y\).

1. Under the “no scattering over \(\pi\)” assumption \(\rho\) is an immersion \(\Rightarrow\) decomposition \(F^* F = \Psi + G\) into a local and a nonlocal part.

2. Under the TIC \(\rho : \Lambda_{\Phi} \rightarrow L_{\Phi}\) is a diffeomorphism onto a submanifold \(L_{\Phi} \subset T^* Y \Rightarrow F^* F\) is an FIO.
The inverse operator in angular coordinates

- No simple expression for the inverse operator \((F^* F)^{-1} F^*\) exists because of the caustics. A fair approximation is

\[
V(\vec{x}) \approx \int d\alpha d\beta \text{Re } B(\vec{x}, \alpha, \beta) \delta G(\vec{r}(\vec{x}, \alpha), \vec{s}(\vec{x}, \beta), t = \phi(\vec{x}, \alpha, \beta))
\]

\[
+ \int d\alpha d\beta \text{Im } B(\vec{x}, \alpha, \beta) (H\delta G)(\vec{r}(\vec{x}, \alpha), \vec{s}(\vec{x}, \beta), t = \phi(\vec{x}, \alpha, \beta)),
\]

- \(\phi(\vec{x}, \alpha, \beta) = \phi_r(\vec{x}, \alpha) + \phi_s(\vec{x}, \beta),\)
- \(H\delta G\): Hilbert transform of the reflection data,
- \(B(\vec{x}, \alpha, \beta) = \frac{|\hat{e}(\alpha) + \hat{e}(\beta)|^3}{A_r(\vec{x}, \alpha) A_s(\vec{x}, \beta)}.\)

This approximation breaks down if \(\vec{x}_r\) or \(\vec{x}_s\) is caustic with respect to \(\vec{x}\), since the amplitudes \(B\) go to zero in a non-smooth manner there.
The inverse operator in acquisition coordinates

Alternatively, we can write

\[
V(\vec{x}) \sim \sum_{\text{even branches}} \int d\vec{x}_r d\vec{x}_s \tilde{B}^i(\vec{x}, \vec{x}_r, \vec{x}_s) \delta G(\vec{x}_r, \vec{x}_s, t = \phi^i(\vec{x}_r, \vec{x}, \vec{x}_s))
\]

\[
+ \sum_{\text{odd branches}} \int d\vec{x}_r d\vec{x}_s \tilde{B}^i(\vec{x}, \vec{x}_r, \vec{x}_s) (\mathcal{H} \delta G)(\vec{x}_r, \vec{x}_s, t = \phi^i(\vec{x}_r, \vec{x}, \vec{x}_s)),
\]

where \(\phi^i(\vec{x}_r, \vec{x}, \vec{x}_s)\) is the total traveltime on a branch.
Example: minimum time migration for Marmousi
Example: multivalued migration for Marmousi
Example: minimum time migration for salt dome
Example: multivalued migration for salt dome