Lecture 2: Linearization, high frequency asymptotics, and the convolutional model

- The least squares function, its gradient, the linearized map, and its adjoint

- Adjoint state method, role of time reversal

- High frequency asymptotic approximation of the Green's function

- Linearization + hf asymptotics + layered assumption $\Rightarrow$ *convolutional model* - basis of standard processing

Insights gleaned from 1D theory which may carry over in some form:

- variation of coefficients controlled $\Rightarrow$ material is "acoustically transparent" - pulses of arbitrary frequency pass through, no wave localization occurs
- to have any hope of extracting $c(\mathbf{x})$ and other coefficients without unrealistic constraints, must use (formally) overdetermined data

A setting for further analysis: set of source receiver positions $\{\mathbf{x}_r, \mathbf{x}_s\}$ either continuous or discrete - use summation notation, understand integration tacitly when appropriate.

Since ρ plays a relatively minor role and varies less dramatically than c in upper crust, until further notice set $\rho \equiv 1$ and forget about it

Acoustic potential $u(\mathbf{x}, t; \mathbf{x}_s)$ satisfies $u \equiv 0$, $t \leq 0$, Dirichlet conditions on a (distant) spatial boundary, and

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u(\mathbf{x}, t; \mathbf{x}_s) = w(t)\delta(\mathbf{x} - \mathbf{x}_s)$$

Predicted seismic data $S[c] = \{p(\mathbf{x}_r, t; \mathbf{x}_s)\}$

Reflection seismic inverse problem, least squares data fitting version: given $S^{\rm obs}$, find c to minimize

$$J[c; S^{\rm obs}] = \frac{1}{2}\|S[c] - S^{\rm obs}\|^2$$

$$= \sum_{\mathbf{x}_r, \mathbf{x}_s} \int_0^T dt\, |(p - p^{\rm obs}|^2(\mathbf{x}_r, t; \mathbf{x}_s)$$

If S is differentiable (in what sense?) then

$$\nabla_c J[c; S^{\rm obs}] = DS[c]^*(S[c] - S^{\rm obs})$$

Local optimization approach to data fitting uses gradient, perhaps Hessian to update c

Gauss-Newton algorithm (basic):

$$c \leftarrow c + \delta c$$

where δc solves *normal equation*

$$DS[c]^* DS[c] \delta c = DS[c]^* (S[c] - S^{\mathrm{obs}})$$

Analysis of data fitting *via* local optimization $\Rightarrow$ study of differentiability of S, structure of $DS[c], DS[c]^*, DS[c]^* DS[c]$. *Most of standard seismic processing methodology based on single Gauss-Newton step!*

Formal linearization: $c \mapsto c + \delta c$, $S \mapsto S + \delta S$

$$\delta S(\mathbf{x}_r, t; \mathbf{x}_s) = \delta p(\mathbf{x}_r, t; \mathbf{x}_s)$$

where $\delta p = \frac{\partial \delta u}{\partial t} = 0, t \leq 0$ and

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \delta u = \frac{2 \delta c}{c^3} \frac{\partial^2 u}{\partial t^2} = \frac{2 \delta c}{c} \nabla^2 u$$

Note: linear influence of δc

If there is any justice $DS[c][\delta c] \equiv \delta S$. Accuracy of linearization: error

$$S[c+\delta c] - (S[c] + DS[c]\delta c)$$

- *small* when c smooth, δc rough or oscillatory on wavelength scale - well-separated scales
- *large* when c not smooth and/or δc not oscillatory - poorly separated scales

This conclusion backed up by a little mathematics, a lot of simulation - much more analysis needed

Implications:

- Some geologies have well-separated scales - cf. sonic logs - and methods based on linearization will work well there. Other geologies do not - expect trouble!
- c smooth, δc oscillatory $\Rightarrow DS[c]\delta c$ approximates *primary reflection* = result of wave interacting with material heterogeneity only once (single scattering)

Analysis of linearization error for 1D problem (R. M. Lewis & WWS, 1991), $f(x,t) = \delta(x)\delta(t)$

- $S[c] \in L^2_{\text{loc}}$ well defined for $c \in H^1_{\text{loc}}$, defines diff'ble map : $H^3[0,Z] \to L^2[0,T]$ for Z suff large rel T
-
$$\|S[c+\delta c] - S[c] - DS[c]\delta c\|_{L^2[0,T]}$$
$$\leq K(\|c\|_{H^3[0,Z]})\|\delta c\|_{L^2[0,Z]}\|\delta c\|_{H^2[0,Z]}$$
- So: $c, \delta c \in H^3$, δc oscillatory ($\|\delta c\|_{L^2} << \|\delta c\|_{H^2}$) $\Rightarrow$ linearization error is small - even if $\|\delta c\|_{H^3} \to \infty$!
- Scaling realization:
$$c(z) = c_0(z)\left(1 + \epsilon^2 \alpha\left(\frac{z}{\epsilon}\right)\right)$$
Then $\|\delta c\|_{H^3} = O(\epsilon^{-1})$ but linearization error $=O(\epsilon^2)$

nD, n >1: various "elliptic" results (eg. Santos et al. 1994) for dependence of solution of Helmholtz equation

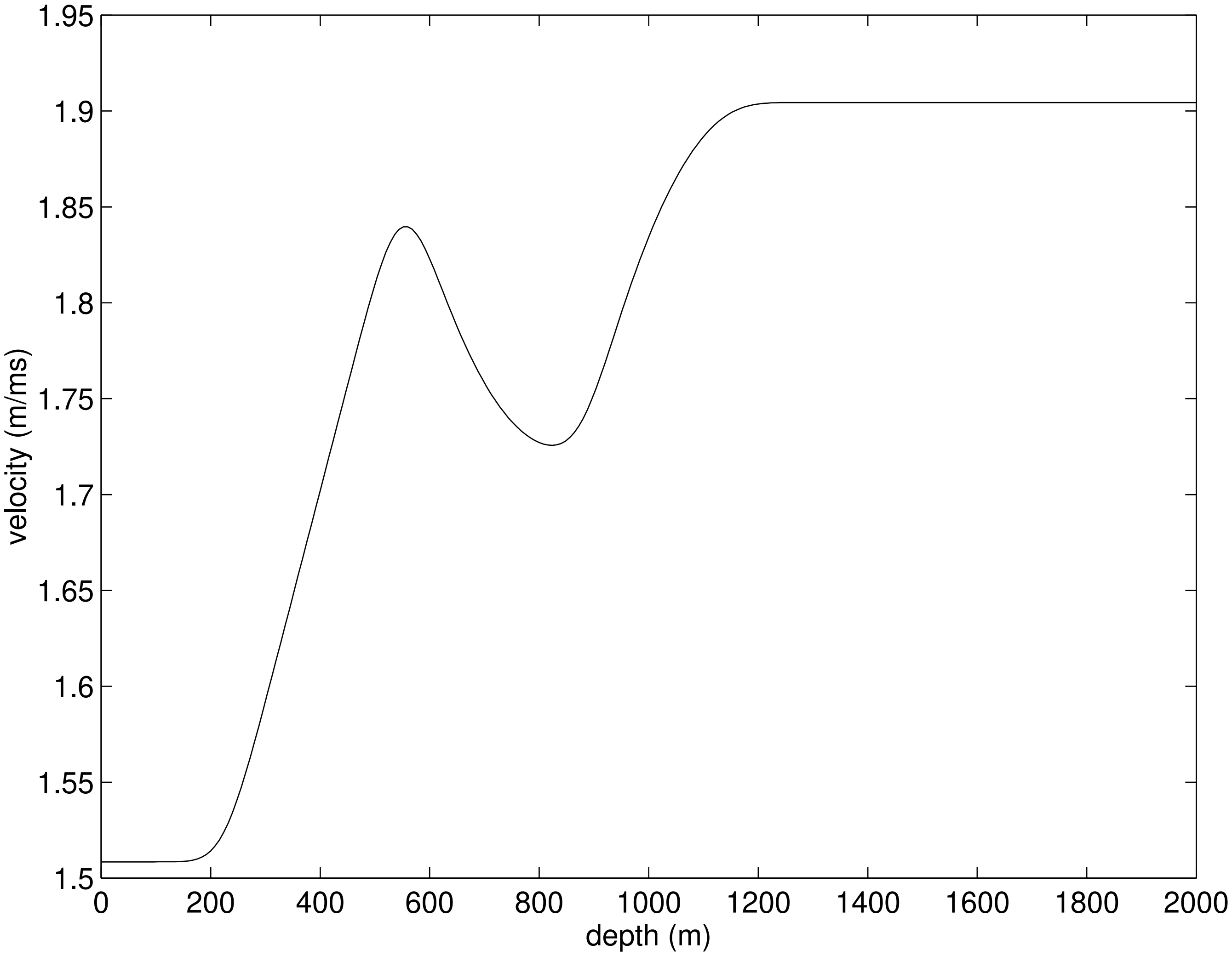
$$\nabla^2 \hat{u} + \frac{\omega^2}{c^2}\hat{u} = \hat{f}$$

on coefficient. Fixed frequency, do not reveal role of scale separation or relation between scales in wave velocity, solution

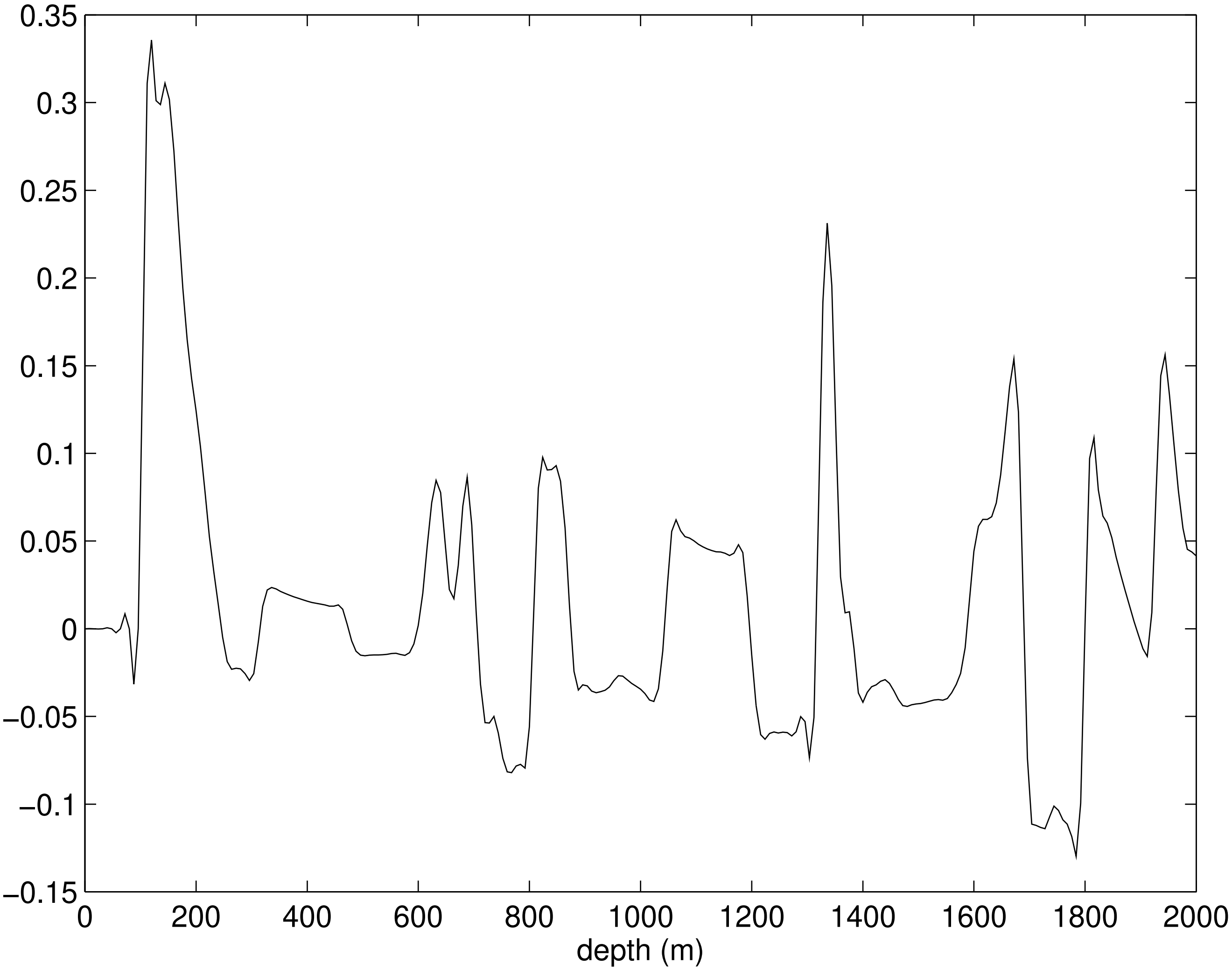
otherwise little known for nD, n >1

Numerical evidence based on finite difference simulation.

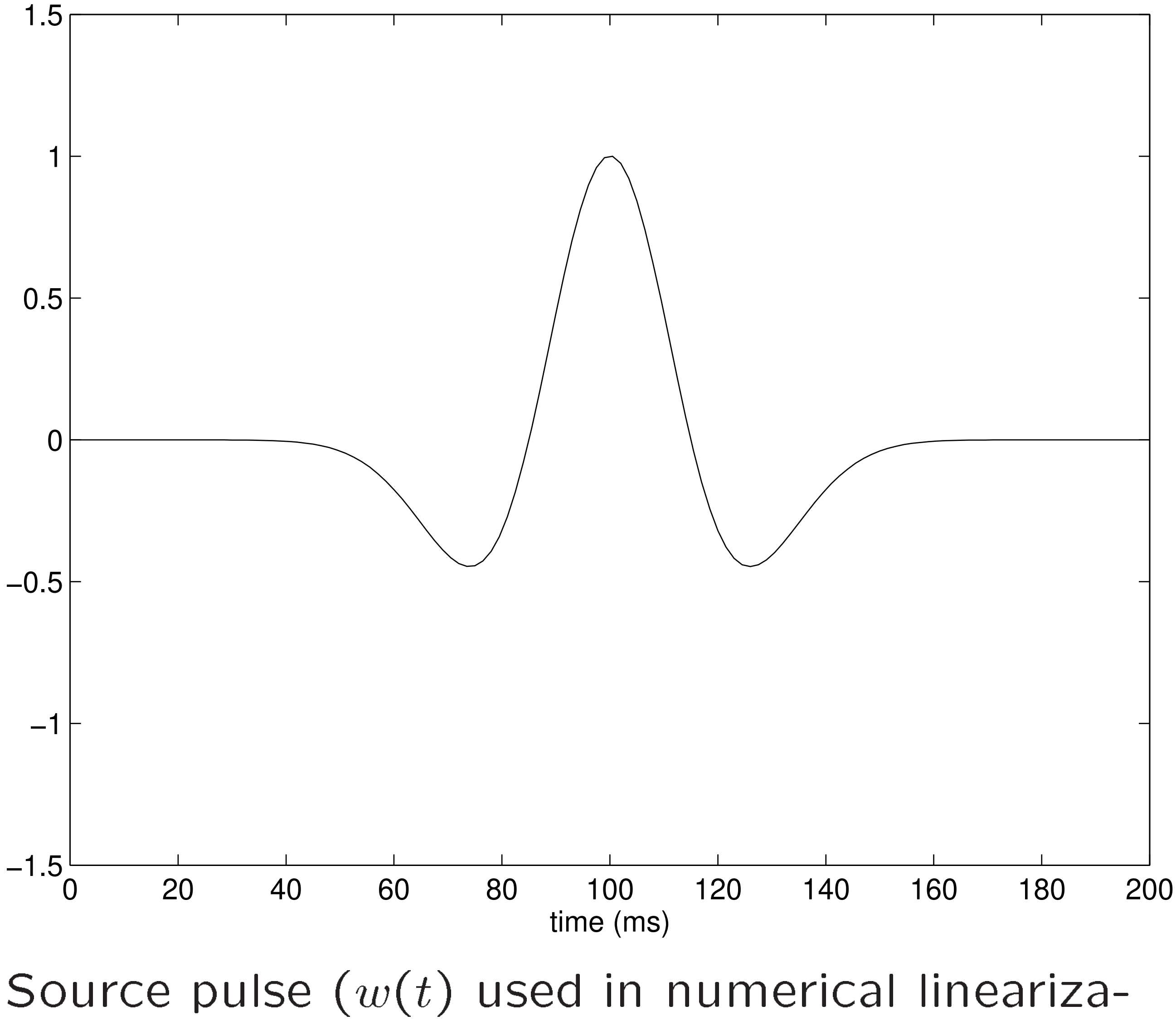
2D Examples: simulated shot gathers with typical "2D" marine seismic geometry. Smooth $c(z)$, oscillatory $\delta c(z)$ extracted from "Marmousi" synthetic data set (Grau & Versteeg, 1994)



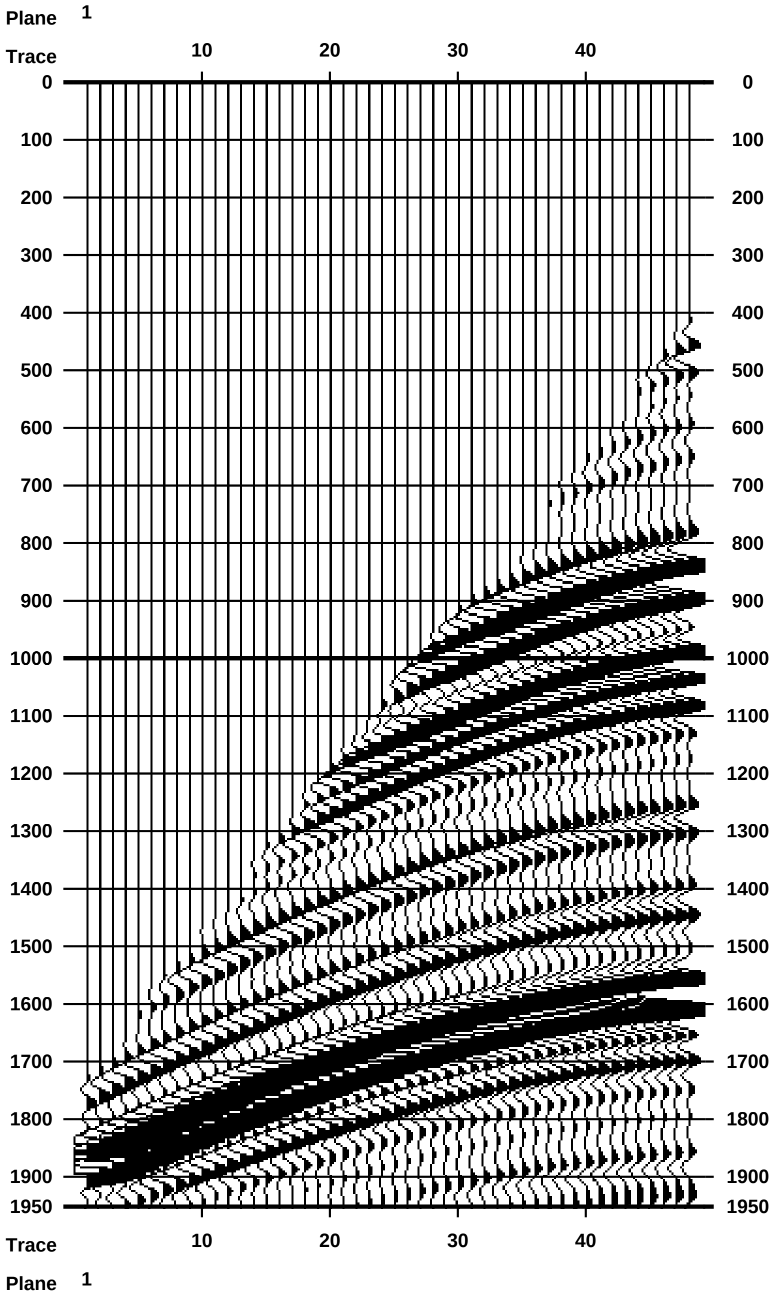
Velocity (function of depth only) used in numerical linearization study



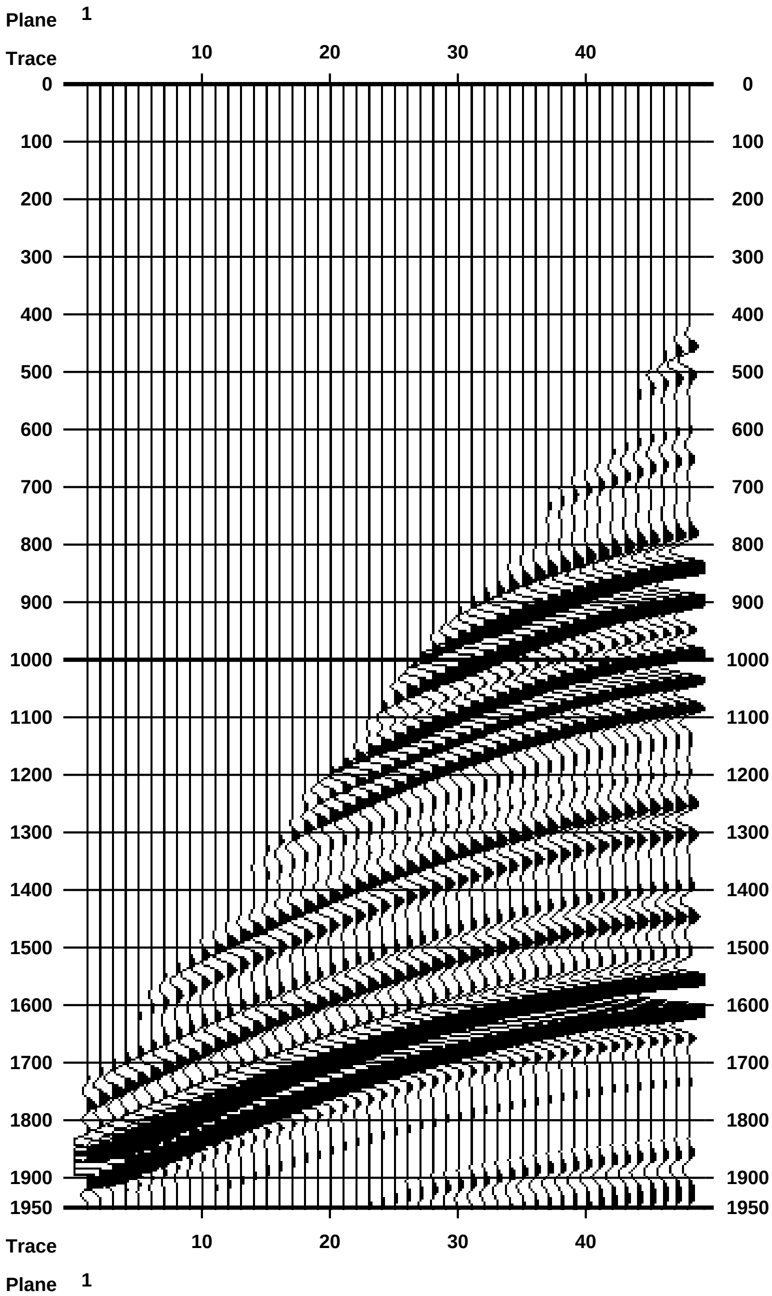
Relative velocity fluctuations ($\delta c/c$) used in numerical linearization study



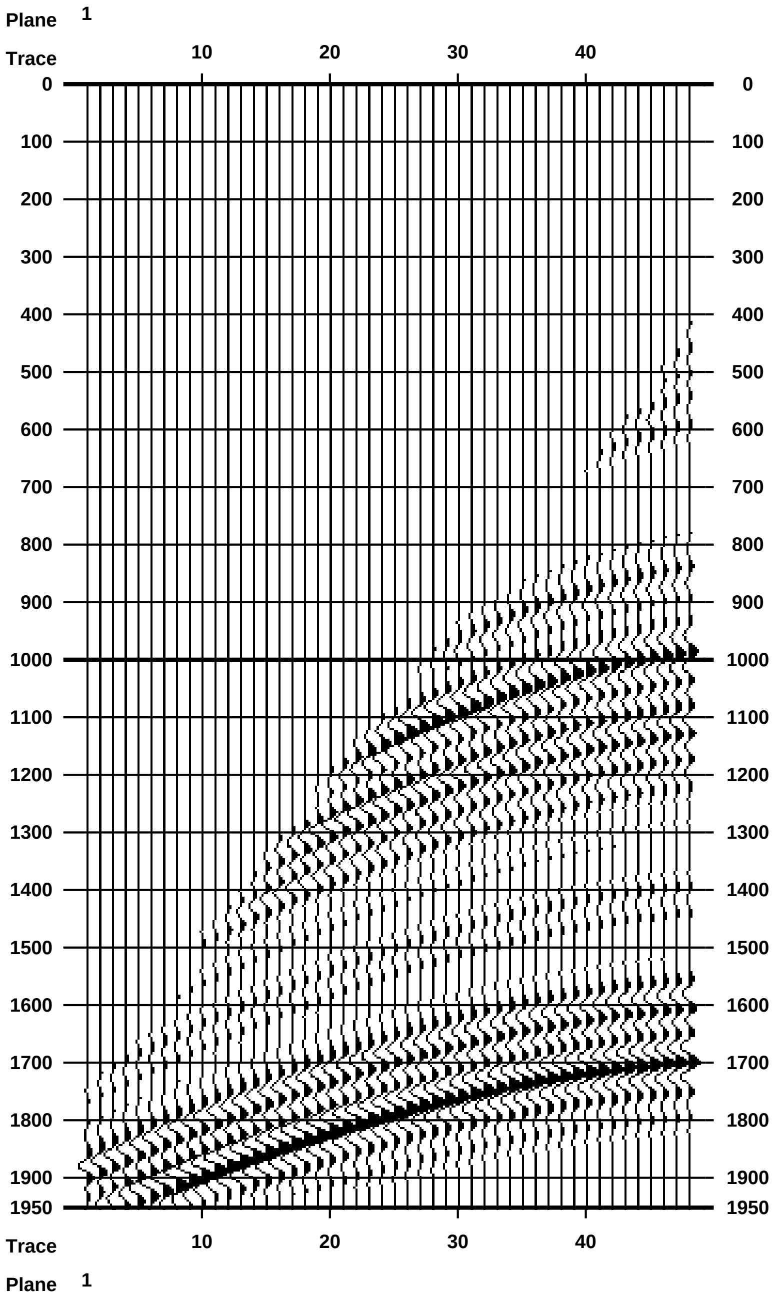
Source pulse ($w(t)$ used in numerical linearization study



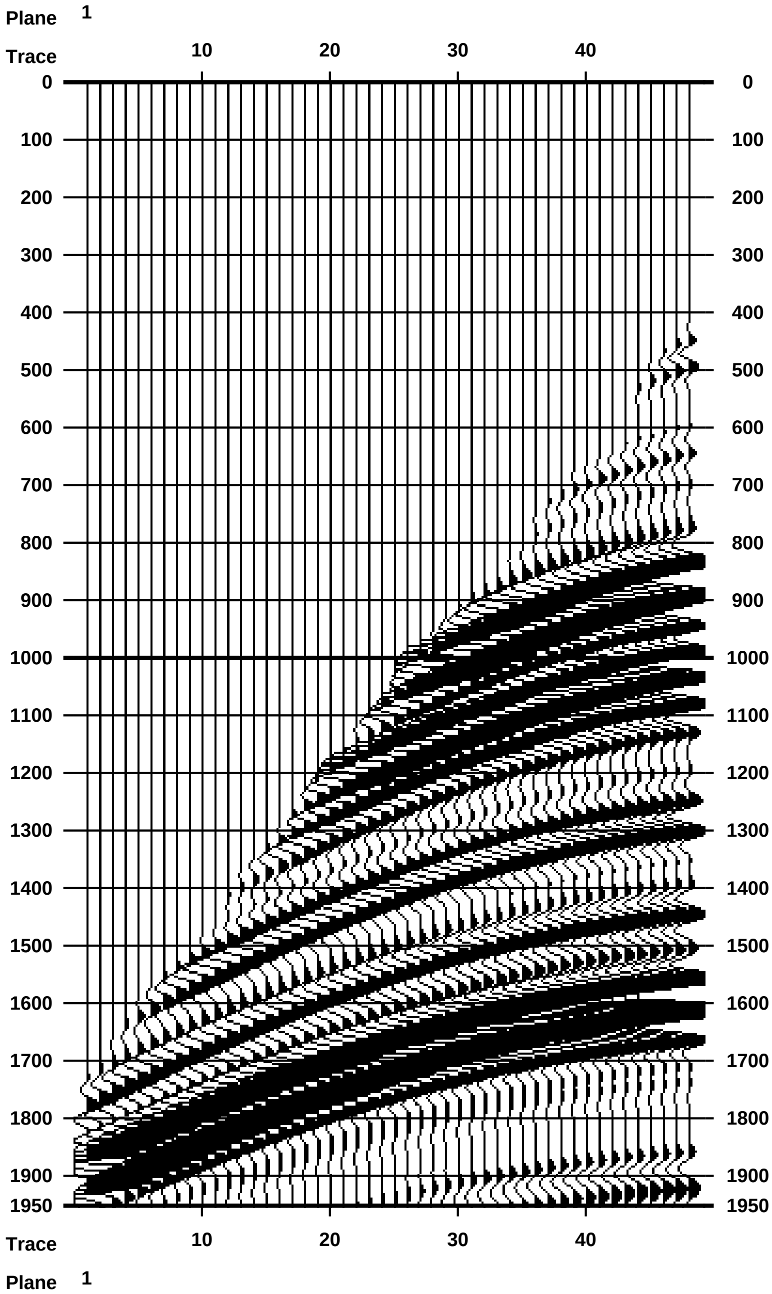
Reference common source seismogram $S[c + \delta c]$, after mute - offsets 150-1800 m 25 m.



Linearized seismogram $S[c] + DS[c]\delta c]$ muted in the same way.



Error seismogram $S[c+\delta c]-S[c]-S[c]\delta c]$ - note multiple refls.



Effect of smooth perturbation: $S[1.02c+\delta c]-S[c+\delta c]-0.02DS[c+\delta c]c$ plotted on same scale.

Computation of adjoint $DS[c]^*$: based on observation ("adjoint state method", Lions 1968, Chavent 1974):

$$DS[c]^*r(\mathbf{x}) = -\sum_{\mathbf{x}_s} \frac{2}{c(\mathbf{x})} \int_0^T dt \, \left(\frac{\partial v}{\partial t} \nabla^2 u\right) (\mathbf{x}, t; \mathbf{x}_s)$$

where $v \equiv 0$, $t \geq T$ and

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) v(\mathbf{x}, t; \mathbf{x}_s) = \sum_{\mathbf{x}_r} r(\mathbf{x}_r, t; \mathbf{x}_s)\delta(\mathbf{x}-\mathbf{x}_r)$$

Proof:

$$< DS[c]^*r, \delta c >=< r, DS[c]\delta c >$$

$$= \sum_{\mathbf{x}_s,\mathbf{x}_r} \int_0^T dt \, r(\mathbf{x}_r, t; \mathbf{x}_s) \frac{\partial \delta u}{\partial t}(\mathbf{x}_r, t; \mathbf{x}_s)$$

$$= \sum_{\mathbf{x}_s} \int dx \int_0^T dt \left\{ \sum_{\mathbf{x}_r} r(\mathbf{x}_r, t; \mathbf{x}_s)\delta(\mathbf{x} - \mathbf{x}_r) \right\}$$

$$\times \frac{\partial \delta u}{\partial t}(\mathbf{x}, t; \mathbf{x}_s)$$

$$= \sum_{\mathbf{x}_s} \int dx \int_0^T dt \left[\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) v\right] \frac{\partial \delta u}{\partial t}(\mathbf{x}, t; \mathbf{x}_s)$$

$$= -\sum_{\mathbf{x}_s} \int dx \int_0^T dt \left[\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta u\right] \frac{\partial v}{\partial t}(\mathbf{x}, t; \mathbf{x}_s)$$

(boundary terms in integration by parts vanish because (i) $\delta u \equiv 0$, $t << 0$; (ii) $v \equiv 0$, $t >> 0$; (iii) Dirchlet boundary is disjoint from support of either for $0 \leq t \leq T$)

$$= -\sum_{\mathbf{x}_s} \int dx \int_0^T dt \left(\frac{2\delta c}{c}\nabla^2 u \frac{\partial v}{\partial t}\right)(\mathbf{x}, t; \mathbf{x}_s)$$

$$= -\sum_{\mathbf{x}_s} \int dx \, \delta c(\mathbf{x}) \frac{2}{c(\mathbf{x})} \int_0^T dt \left(\nabla^2 u \frac{\partial v}{\partial t}\right)(\mathbf{x}, t; \mathbf{x}_s)$$

$$= < \delta c, DS[c]^* r >$$

q.e.d.

Exercise: Revive ρ, work out the formal linearization $DS[c,\rho](\delta c,\delta\rho)$ of the scattering operator, and construct an adjoint state computation of $DS[c,\rho]^*$.

Note that the problem for v is a *final value problem*.

Alternate description in terms of *time reversed* adjoint state $\tilde{v}(\mathbf{x},t;\mathbf{x}_s) = v(\mathbf{x},T-t;\mathbf{x}_s) \equiv 0$, $t \leq 0$,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\tilde{v}(\mathbf{x},t;\mathbf{x}_s) = \sum_{\mathbf{x}_r} r(\mathbf{x}_r, T-t;\mathbf{x}_s)\delta(\mathbf{x}-\mathbf{x}_r)$$

and

$$DS[c]^* r(\mathbf{x})$$

$$= -\sum_{\mathbf{x}_s}\frac{2}{c(\mathbf{x})}\int_0^T dt\,\frac{\partial\tilde{v}}{\partial t}(\mathbf{x},t;\mathbf{x}_s)\nabla^2 u(\mathbf{x},T-t;\mathbf{x}_s)$$

So adjoint process is: (i) time-reverse data, (ii) rebroadcast ("backpropagate"), and (iii) crosscorrelate time derv. with time-reversed Laplacian of incident field.

Most fruitful analysis of Gauss-Newton step ("linearized" inverse problem) based on *integral representation* of $DS[c]$.

Causal Green's function $G(\mathbf{x},t;\mathbf{y})$ = retarded fundamental solution:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\mathbf{x},t;\mathbf{y}) = \delta(t)\delta(\mathbf{x}-\mathbf{y})$$

and $G \equiv 0, t < 0$. Then $p(\mathbf{x},t;\mathbf{x}_s) = w *_t \frac{\partial G}{\partial t}(\mathbf{x},t;\mathbf{x}_s)$ and similarly for δp. Linearity $\Rightarrow$

$$\delta p(\mathbf{x}_r,t;\mathbf{x}_s)$$

$$= \int dx \int ds \, \frac{\partial G}{\partial t}(\mathbf{x}_r, t-s;\mathbf{x}) \frac{2\delta c(\mathbf{x})}{c(\mathbf{x})^3} \frac{\partial^2 u}{\partial t^2}(\mathbf{x},s;\mathbf{x}_s)$$

$$= w *_t \int dx \, \frac{2\delta c(\mathbf{x})}{c(\mathbf{x})^3} \int ds \, \frac{\partial G}{\partial t}(\mathbf{x}_r, t-s;\mathbf{x}) \frac{\partial^2 G}{\partial t^2}(\mathbf{x},s;\mathbf{x}_s)$$

so the distribution kernel of $DS[c]$ is

$$K(\mathbf{x}_r,t,\mathbf{x}_s;\mathbf{x}) = w *_t \int ds \, \frac{\partial G}{\partial t}(\mathbf{x}_r, t-s;\mathbf{x}) \frac{\partial^2 G}{\partial t^2}(\mathbf{x},s;\mathbf{x}_s)$$

To evaluate this kernel, recall:

- linearization accurate when c smooth, δc oscillatory on wavelength scale
- if c is smooth on wavelength scale, then G well approximated by *geometric optics*

Agenda:

- geometric optics approximation of G
- generalized Radon representation of $DS[c]$

- convolutional model of layered medium seismogram $\Rightarrow$ standard seismic processing (estimation of $c(z), \delta c(z)$)
- oscillatory integral representation of normal operator $\Rightarrow$ prestack migration (estimation of $c(\mathbf{x}), \delta c(\mathbf{x})$)

Geometric optics, local version: if $\mathbf{x}$ "not too far" from $\mathbf{x}_s$ ($|\mathbf{x}-\mathbf{x}_s| < r_{\mathsf{go}},\ t < t_{\mathsf{go}}$) then

$$G(\mathbf{x},t;\mathbf{x}_s) = a(\mathbf{x};\mathbf{x}_s)\delta(t-\tau(\mathbf{x};\mathbf{x}_s)) + R(\mathbf{x},t;\mathbf{x}_s)$$

where the traveltime $\tau(\mathbf{x};\mathbf{x}_s)$ solves the eikonal equation

$$\nabla\tau(\mathbf{x};\mathbf{x}_s)\cdot\nabla\tau(\mathbf{x};\mathbf{x}_s) = \frac{1}{c^2(\mathbf{x})}$$

$$\tau(\mathbf{x};\mathbf{x}_s) \sim \frac{|\mathbf{x}-\mathbf{x}_s|}{c(\mathbf{x}_s)},\ \ \mathbf{x}\to\mathbf{x}_s$$

and the amplitude $a(\mathbf{x};\mathbf{x}_s)$ solves the transport equation

$$\nabla\cdot(a^2\nabla\tau) = 0$$

All of this is meaningful only if the remainder R is small in a suitable sense: energy estimate (**Exercise!**) $\Rightarrow$

$$\int_{|\mathbf{x}-\mathbf{x}_s|<r_{\mathsf{go}}} dx \int_0^{t_{\mathsf{go}}} dt|R(\mathbf{x},t;\mathbf{x}_s)|^2 \le C\|c\|_{C^4}$$

Recap of Hamilton-Jacobi theory (see last year's notes for details):

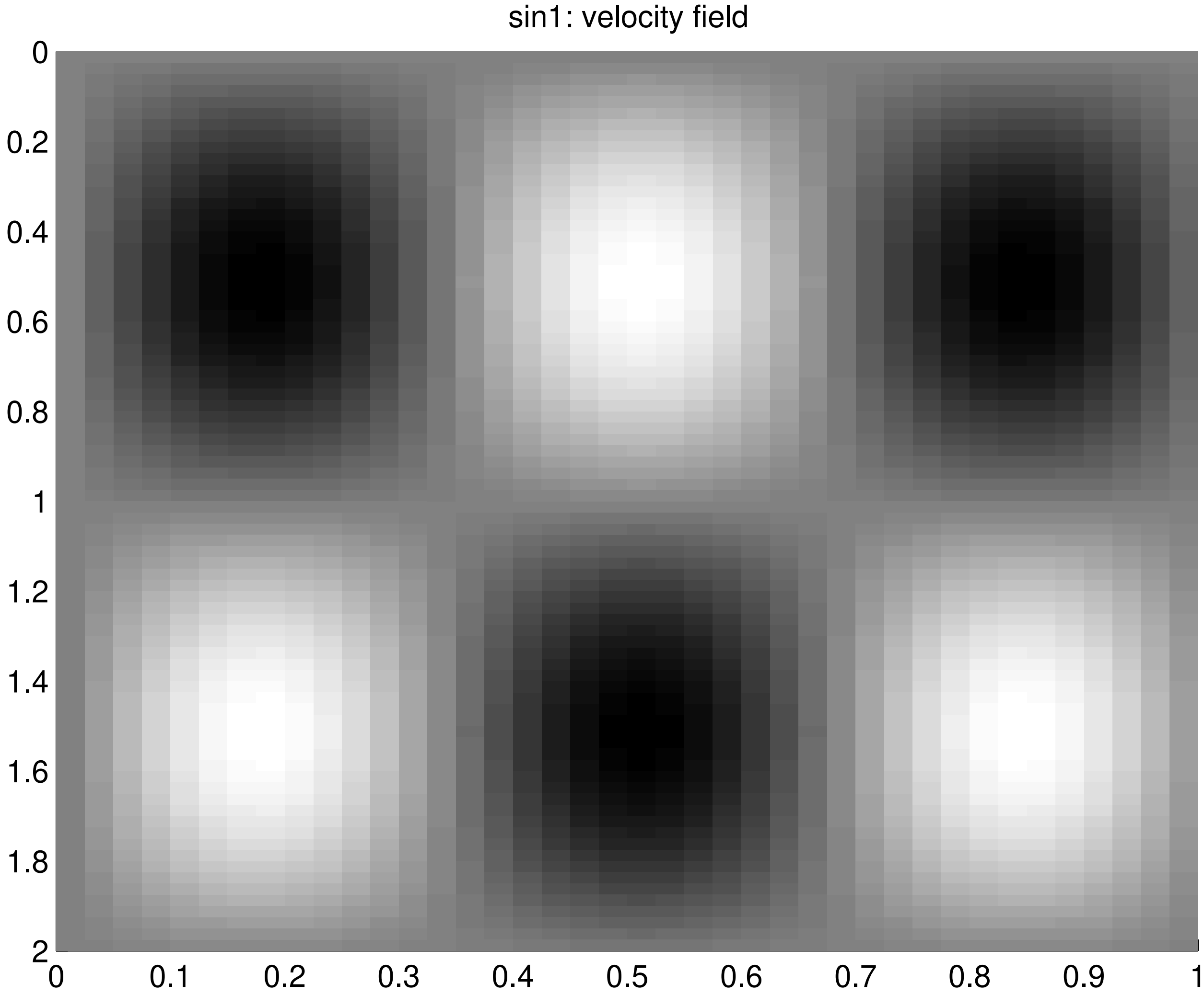
A unique smooth solution of the eikonal equation exists on

$$\{\mathbf{x} : 0 < |\mathbf{x} - \mathbf{x}_s| < r_{\mathrm{go}} = r_{\mathrm{go}}[\| \log c\|_{C^2}]\}$$

in which it equals the geodesic distance from $\mathbf{x}$ to $\mathbf{x}_s$ in the metric $c(\mathbf{x})^{-2}(dx^2 + dy^2 + dz^2)$; geodesics = *rays*. In this domain, the exponential map $(r, \Theta) \in (0, \infty) \times S^2 \mapsto \mathbf{x}(r, \Theta)$, defined by geodesic distance r from $\mathbf{x}_s$ along the ray leaving with (solid) angle Θ, is a diffeomorphism.

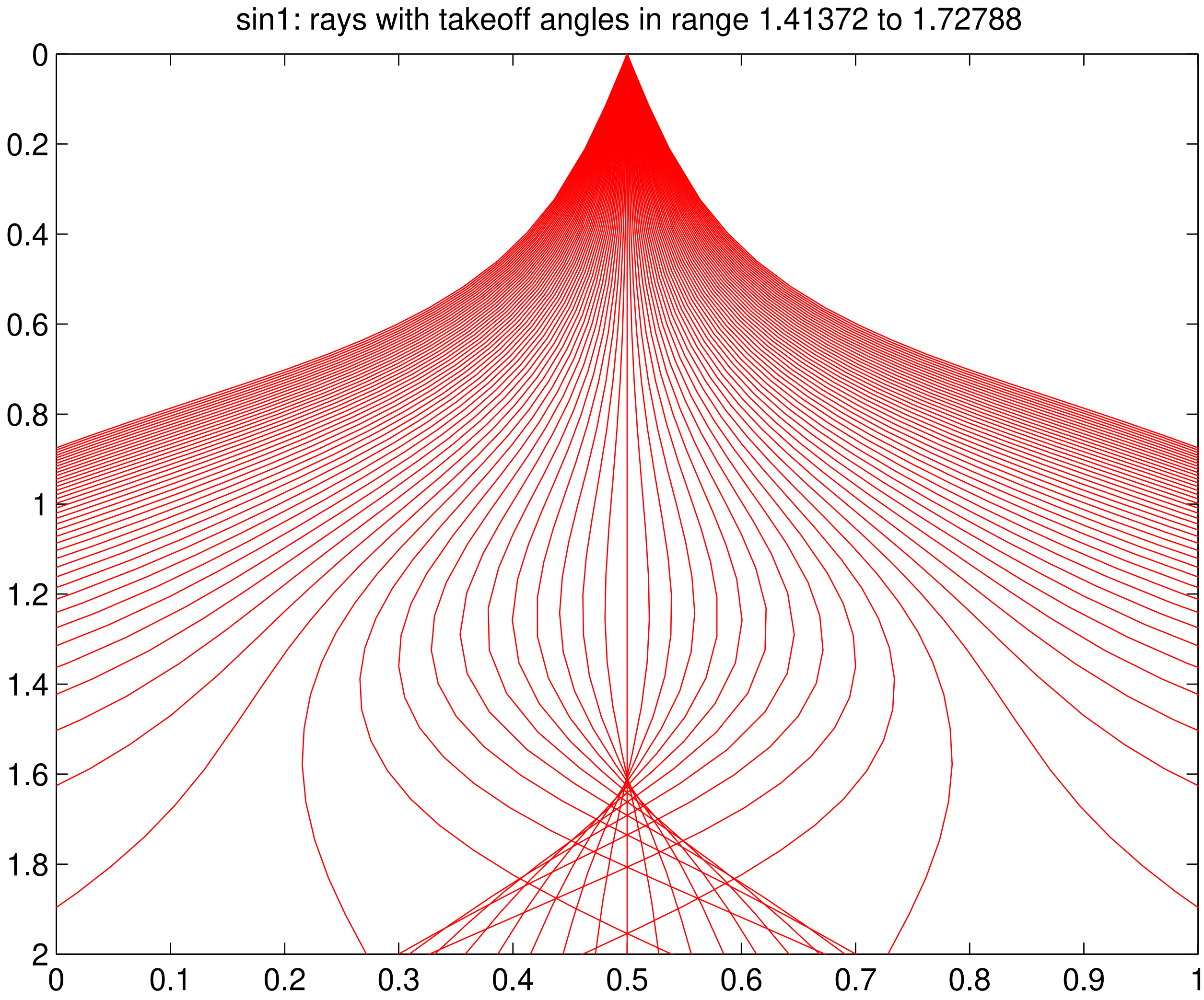
For "random but smooth" $c(\mathbf{x})$ with variance σ, the exponential map fails to be a diffeomorphism within $r_{\mathrm{go}} \sim \sigma^{-3/2}$. Mode of failure: formation of ray envelope or *caustic* (White, 1982) ($\Leftrightarrow$ *multipathing*)

Upon caustic formation, the simple geometric optics field description above is no longer correct (Ludwig, 1966).



Sinusoidal velocity field

$$c(x,z) = 1 + 0.2\sin\frac{\pi z}{2}\sin 3\pi x$$



Rays in sinusoidal velocity field, source point = origin. Note formation of caustic, multivalued time in lower center.

Assume: supp δc contained in simple geometric optics domain (each point reached by unique ray from any source point $\mathbf{x}_s$).

Then

$$K(\mathbf{x}_r,t,\mathbf{x}_s;\mathbf{x}) = w*_t\int ds\,\frac{\partial G}{\partial t}(\mathbf{x}_r,t-s;\mathbf{x})\frac{\partial^2 G}{\partial t^2}(\mathbf{x},s;\mathbf{x}_s)$$

$$\simeq w*_t$$

$$\int ds\,a(\mathbf{x}_r,\mathbf{x})a(\mathbf{x},\mathbf{x}_s)\delta'(t-s-\tau(\mathbf{x}_r,\mathbf{x}))\delta''(s-\tau(\mathbf{x},\mathbf{x}_s))$$

$$= w *_t a(\mathbf{x},\mathbf{x}_r)a(\mathbf{x},\mathbf{x}_s)\delta'''(t-\tau(\mathbf{x},\mathbf{x}_r)-\tau(\mathbf{x},\mathbf{x}_s))$$

provided that this expression is well defined as a distribution.

Gel'fand and Shilov, 1958:

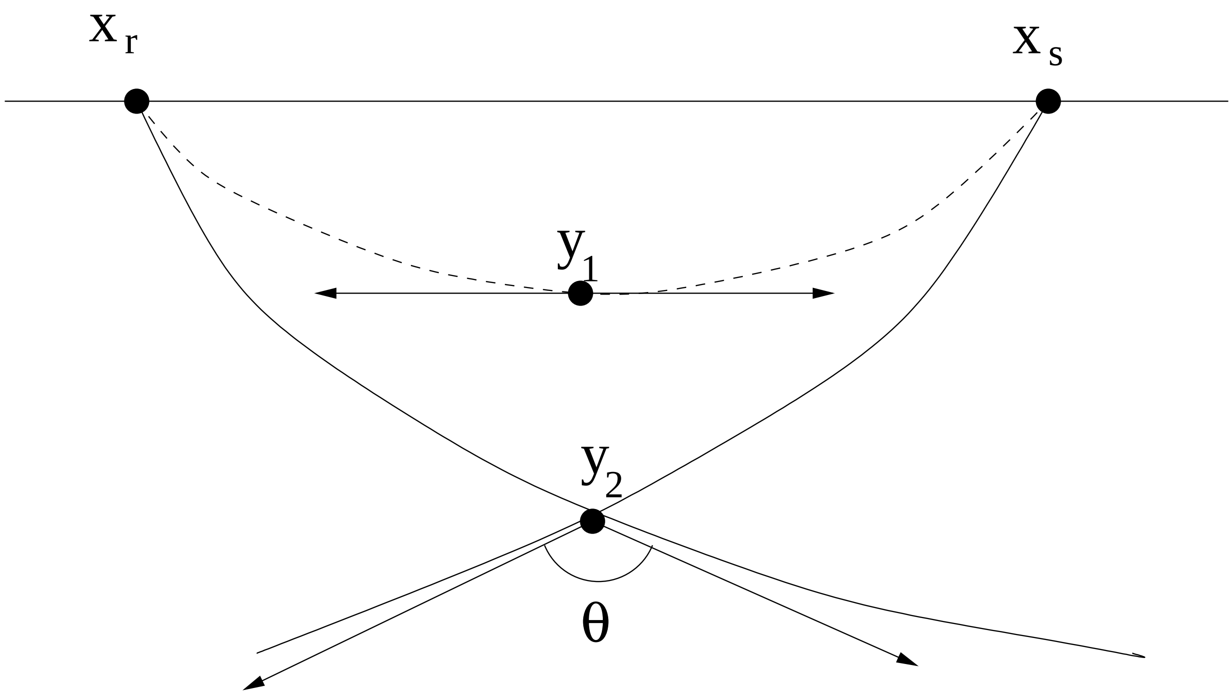
(i) *pullback* (= composition) of distribution $d \in \mathcal{D}'(\mathbf{R})$ by function $\phi : \Omega \subset \mathbf{R}^n \rightarrow \mathbf{R}$ defines member of $\mathcal{D}'(\Omega)$ if $\nabla\phi(\mathbf{x}) \neq 0$ at every $\mathbf{x} \in \Omega$ for which $\phi(\mathbf{x}) \in \operatorname{supp} d$

Applied to kernel of $DS[c]$:

$\nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{x}_r)+\nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{x}_s)\neq 0$

$\Leftrightarrow$ velocity at $\mathbf{x}$ of ray from $\mathbf{x}_s$ **not** negative of velocity of ray from $\mathbf{x}_r$

$\Leftrightarrow$ *no forward scattering*



Forward scattering at $\mathbf{x}_1$, scattering through *opening angle* $\theta<\pi$ at $\mathbf{x}_2$

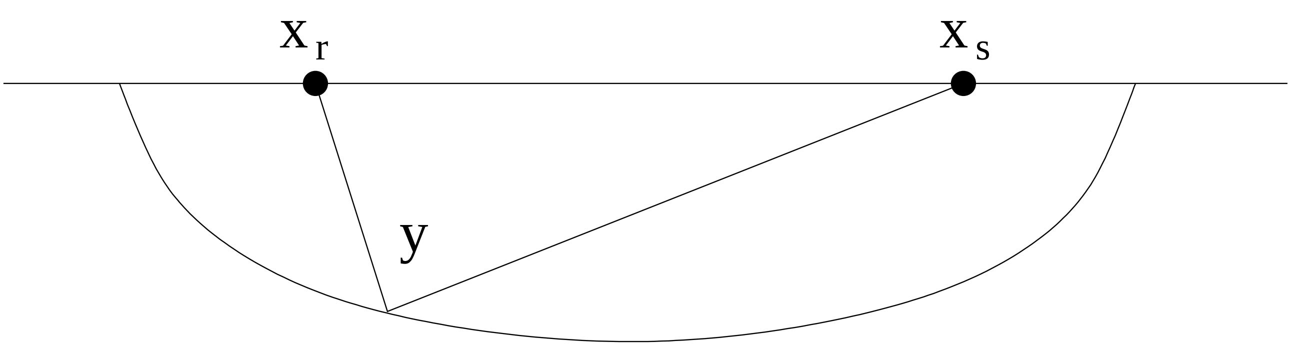
So: for δc supported in simple geometric optics domain, no forward scattering $\Rightarrow$

$$\delta p(\mathbf{x}_r, t; \mathbf{x}_s) = w''' *_t$$

$$\int dx \, \frac{2\delta c(\mathbf{x})}{c^3(\mathbf{x})} a(\mathbf{x}, \mathbf{x}_r) a(\mathbf{x}, \mathbf{x}_s) \delta(t - \tau(\mathbf{x}, \mathbf{x}_r) - \tau(\mathbf{x}, \mathbf{x}_s))$$

Interpretation: pressure perturbation is filtered, weighted integral of δc over *pseudoellipsoid* $\{\mathbf{x} : t = \tau(\mathbf{x}, \mathbf{x}_r) + \tau(\mathbf{x}, \mathbf{x}_s)\}$

Note: if $c =$const. then it really is an ellipsoid, as $\tau(\mathbf{y}, \mathbf{x}) = c^-1|\mathbf{y} - \mathbf{x}|$!



t= τ(y,x r)+ τ(y,x s)

Layered theory:

- c,δc functions of $z = x_3$ only
- $\mathbf{x}_r = (x_r, 0, 0), \mathbf{x}_s = (x_s, 0, 0), x_{\text{min}} \le x_r, x_s \le x_{\text{max}}, 0 \le t \le T$ (line recording geometry)
- global bound on $c \Rightarrow$ for $|\mathbf{x}| > X$, $\tau(\mathbf{x}, \mathbf{x}_r) + \tau(\mathbf{x}, \mathbf{x}_s) > T$

$\Rightarrow$ integrand vanishes outside compact set $\Rightarrow$ for suitable $\phi \in C_0^\infty(\mathbf{R}^3)$,

$$DS[c]\delta c(x_r, t; x_s) \simeq w''' *_t \int dx$$

$$\phi(\mathbf{x})\frac{2\delta c(z)}{c^3(z)}a(\mathbf{x}, x_r)a(\mathbf{x}, x_s)\delta(t-\tau(\mathbf{x}, x_r)-\tau(\mathbf{x}, x_s))$$

$$= w''' *_t \int dz \, \frac{2\delta c(z)}{c^3(z)} \int d\omega$$

$$\int dx \int dy \phi(\mathbf{x})a(\mathbf{x}, x_r)a(\mathbf{x}, x_s)e^{i\omega(t-\tau(\mathbf{x},x_r)-\tau(\mathbf{x},x_s))}$$

Since we have already thrown away smoother (lower frequency) terms, do it again using *stationary phase:*

$$\int\int dx\,dy\,\phi(\mathbf{x})a(\mathbf{x},x_r)a(\mathbf{x},x_s)e^{i\omega(t-\tau(\mathbf{x},x_r)-\tau(\mathbf{x},x_s))}$$

$$=\sum_i A(\mathbf{x}_i,x_r,x_s)\omega^{-2}e^{i\omega(t-\tau(\mathbf{x}_i,x_r)-\tau(\mathbf{x}_i,x_s))}+O(\omega^{-3})$$

where the sum is over the stationary points $\mathbf{x}_i=(x_i,y_i,z_i)$ of the phase $\psi(x,y)=\tau(\mathbf{x},x_r)+\tau(\mathbf{x},x_s)$:

$$\nabla_{x,y}(t-\psi(x,y))=0,\ i=1,...$$

and the amplitude A is a combination of the a's and other stuff.

Key Assumption: all stationary points are isolated, and even more that the phase *Hessian*

$$\begin{pmatrix} \frac{\partial^2\psi}{\partial x^2} & \frac{\partial^2\psi}{\partial x\partial y} \\ \frac{\partial^2\psi}{\partial x\partial y} & \frac{\partial^2\psi}{\partial y^2} \end{pmatrix}$$

is nonsingular at each stationary point.

Must find stationary points and verify nonsingularity of Hessian!

Because $c = c(z)$ solutions of the eikonal equation are translation invariant: $\tau(\mathbf{x}, x_s) \equiv \tau_s(x, y, z) = \tau(r_s, 0, z, 0) \equiv \bar{\tau}(r_s, z)$,
$r_s = \sqrt{(x - x_s)^2 + y^2}$, and similarly for x_r. Stationarity conditions are

$$0 = \frac{\partial \psi}{\partial x} = \frac{x - x_s}{r_s}\frac{\partial \bar{\tau}}{\partial r}(r_s, z) + \frac{x - x_r}{r_r}\frac{\partial \bar{\tau}}{\partial r}(r_r, z)$$

$$0 = \frac{\partial \psi}{\partial y} = \frac{y}{r_s}\frac{\partial \bar{\tau}}{\partial r}(r_s, z) + \frac{y}{r_r}\frac{\partial \bar{\tau}}{\partial r}(r_r, z)$$

Sufficent conditions: $y = 0$ and $x = \frac{x_s + x_r}{2} \equiv x_m$, the *midpoint*.

Interesting exercise in H-J theory: these conditions are also *necessary*.

Upshot: exactly one stationary point, beneath midpoint.

Hessian components:

$$\frac{\partial^2\psi}{\partial x^2} = \frac{y^2}{r_s^3}\frac{\partial\bar\tau}{\partial r}(r_s,z) + \frac{(x-x_s)^2}{r_s^2}\frac{\partial^2\bar\tau}{\partial r^2}(r_s,z)$$

$$+\frac{y^2}{r_r^3}\frac{\partial\bar\tau}{\partial r}(r_r,z) + \frac{(x-x_r)^2}{r_r^2}\frac{\partial^2\bar\tau}{\partial r^2}(r_r,z)$$

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{y(x-x_s)}{r_s^3}\frac{\partial\bar\tau}{\partial r}(r_s,z) + \frac{y(x-x_s)}{r_s^2}\frac{\partial^2\bar\tau}{\partial r^2}(r_s,z)$$

$$+\frac{y(x-x_r)}{r_r^3}\frac{\partial\bar\tau}{\partial r}(r_r,z) + \frac{y(x-x_r)}{r_r^2}\frac{\partial^2\bar\tau}{\partial r^2}(r_r,z)$$

$$\frac{\partial^2\psi}{\partial y^2} = \frac{(x-x_s)^2}{r_s^3}\frac{\partial\bar\tau}{\partial r}(r_s,z) + \frac{y^2}{r_s^2}\frac{\partial^2\bar\tau}{\partial r^2}(r_s,z)$$

$$+\frac{(x-x_r)^2}{r_r^3}\frac{\partial\bar\tau}{\partial r}(r_r,z) + \frac{y^2}{r_r^2}\frac{\partial^2\bar\tau}{\partial r^2}(r_r,z)$$

At $x = x_m, y = 0$ also $|x-x_s| = |x-x_r| = r_s = r_r = \frac{|x_s-x_r|}{2} \equiv h$, the half-offset, and most of this simplifies:

At $x = x_m, y = 0$,

$$\begin{pmatrix} \frac{\partial^2\psi}{\partial x^2} & \frac{\partial^2\psi}{\partial x\partial y} \\ \frac{\partial^2\psi}{\partial x\partial y} & \frac{\partial^2\psi}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2\bar\tau}{\partial r^2}(h,z) & 0 \\ 0 & \frac{1}{h}\frac{\partial\bar\tau}{\partial r}(h,z) \end{pmatrix}$$

In the quarter-plane $r, z \geq 0$, $\bar\tau$ satisfies the 2D eikonal

$$\left(\frac{\partial\bar\tau}{\partial z}\right)^2 + \left(\frac{\partial\bar\tau}{\partial r}\right)^2 = \frac{1}{c(z)^2}$$

Differentiating,

$$\nabla\bar\tau \cdot \nabla\left(\frac{\partial\bar\tau}{\partial r}\right) = 0$$

which is *Snell's law:* the horizontal ray slowness $\frac{\partial\bar\tau}{\partial r} = \xi = \frac{\sin\gamma}{c(z)}$ is constant along rays.

Assume for simplicity $c(z) = const.$ for small z; then τ is proportional to distance near (0,0), so $\xi = \frac{\partial\bar\tau}{\partial r} > 0$ for $r > 0, z, r$ small. However since ξ is constant along rays, $\frac{\partial\bar\tau}{\partial r} > 0$ for $r > 0$, period.

The simple geometric optics assumption (exponential map is a diffeomorphism onto a domain containing all scattering points) means: $(\bar{\tau}, \xi) \mapsto (r, z)$ is an invertible differentiable map in the region of interest. In particular,

$$\det \begin{pmatrix} \frac{\partial \bar{\tau}}{\partial r} & \frac{\partial \xi}{\partial r} \\ \frac{\partial \bar{\tau}}{\partial z} & \frac{\partial \xi}{\partial z} \end{pmatrix}$$

has one sign throughout. Near the origin, $\bar{\tau}$ is proportional to distance, and ξ to $r/\sqrt{r^2 + z^2}$, which makes it easy to check that the sign is negative:

$$0 > \frac{\partial \bar{\tau}}{\partial r}\frac{\partial \xi}{\partial z} - \frac{\partial \bar{\tau}}{\partial z}\frac{\partial \xi}{\partial r}$$

However

$$\frac{\partial \bar{\tau}}{\partial r}\frac{\partial \xi}{\partial r} + \frac{\partial \bar{\tau}}{\partial z}\frac{\partial \xi}{\partial z} = 0$$

(ξ constant along rays) whence

$$\frac{\partial \xi}{\partial z} = -\frac{\partial \bar{\tau}}{\partial r}\frac{\partial \xi}{\partial r}\left(\frac{\partial \bar{\tau}}{\partial z}\right)^{-1}$$

so

$$0 > -\left(\left(\frac{\partial\bar{\tau}}{\partial r}\right)^2 \left(\frac{\partial\bar{\tau}}{\partial z}\right)^{-1} + \frac{\partial\bar{\tau}}{\partial z}\right)\frac{\partial\xi}{\partial r}$$

$$= -\left(c(z)^2\frac{\partial\bar{\tau}}{\partial z}\right)^{-1}\frac{\partial\xi}{\partial r}$$

using the eikonal equation.

Excercise: the no-forward-scattering assumption implies that $\frac{\partial\bar{\tau}}{\partial z} > 0$ throughout the scattering domain.

Therefore

$$\frac{\partial\xi}{\partial r} = \frac{\partial^2\bar{\tau}}{\partial r^2} > 0$$

Finally, since $\frac{\partial\bar{\tau}}{\partial r}(0,z) = 0$,

$$\lim_{r\to 0}\frac{1}{r}\frac{\partial\bar{\tau}}{\partial r}(r,z) = \frac{\partial^2\bar{\tau}}{\partial r^2}(0,z) > 0$$

Upshot: the determinant

$$\det \begin{pmatrix} \frac{\partial^2 \psi}{\partial x^2} & \frac{\partial^2 \psi}{\partial x \partial y} \\ \frac{\partial^2 \psi}{\partial x \partial y} & \frac{\partial^2 \psi}{\partial y^2} \end{pmatrix} = \frac{1}{h}\frac{\partial \bar{\tau}}{\partial r}(h,z)\frac{\partial^2 \bar{\tau}}{\partial r^2}(h,z)$$

is positive throughout the scattering domain - the stationary phase Hessian is nonsingular, justifying (mostly) the use of the stationary phase formula, and giving

$$DS[c]\delta c(x_r, t; x_s) \simeq w''' *_t \int dz\, \frac{2\delta c(z)}{c^3(z)} \int d\omega$$

$$A(z, x_r, x_s)\omega^{-2} e^{i\omega(t-\tau(x_m,0,z,x_r)-\tau(x_m,0,z,x_s))}$$

Easy: $\tau(x_m, 0, z, x_r) - \tau(x_m, 0, z, x_s) = 2\bar{\tau}(h, z)$. Transfer derivatives from w to integrand, absorb sign change in A, recognize Fourier representation of $\delta(t)$:

$$= w *_t \int dz\, \frac{2\delta c(z)}{c^3(z)} A(z, x_r, x_s)\delta'(t - 2\bar{\tau}(h, z))$$

Deeper in calc: A depends only on $h = (x_r - x_s)/2$

On to the convolutional model:

- Stuff c^{-2} into A
- change variables $z \mapsto 2\bar{\tau}(h,z)$ (when is that justified? discuss tomorrow...), inverse $t \mapsto \bar{\zeta}(h,t)$
- absorb Jacobian in A

$$DS[c]\delta c(x_r,t;x_s) \simeq w *_t (A(\bar{\zeta}(h,t),h))R(\bar{\zeta}(h,t)))$$

This is the **convolutional model**, which is the basis of standard seismic processing. R is the "reflectivity"

$$R(z) = \frac{1}{2}\frac{d}{dz}\frac{\delta c(z)}{c(z)}$$

NB: have dropped terms which do not involve the derivative of δc - these are minor perturbations if δc is oscillatory

Conventional: since

$$(x_s, x_r) \mapsto \left(x_m = \frac{x_r + x_s}{2},\, h = \frac{x_r - x_s}{2}\right)$$

is a change of variables, express data in "mid-point, offset" coordinates:

$$DS[c]\delta c(h, t; x_m) \simeq w *_t (A(\bar{\zeta}(h,t), h) R(\bar{\zeta}(h,t)))$$

Data with fixed midpoint x_m, varying offset = "common midpoint gather".

A curiousity: why does x_m appear on the left side of this formula, but not the right?

Answer tomorrow...