# Estimating Wave Velocity
# in
# Heterogeneous Media

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Agenda: what can be understood about the reflection seismic inverse problem, and contemporary industrial solutions, using *deterministic* models

Lecture 1: Layered medium models, theoretical perfection, and practical disaster

- what reflection seismograms are like
- elastic and acoustic models, energy, and power
- reduction to layered and 1D cases
- $L^2$ theory of the 1D inverse problem
- necessity, difficulties of the data fitting approach

Reflection seismology:

- probe Earth with artificially generated sound ("seismic") waves

- Important tool in oil & gas exploration and production, civil engineering, environmental remediation, tectonic studies

- Similar technologies: radar, sonar, biomedical ultrasonics, ultrasonic materials testing

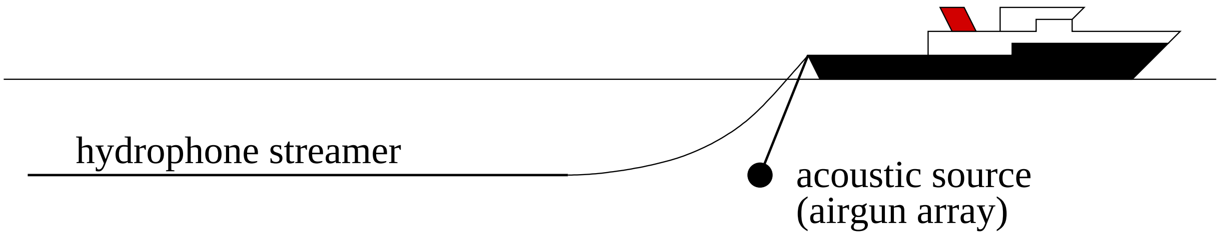
Reflection seismic inverse problem:

From measurements of ground motion near surface of Earth, determine structure of Earth's interior.

Marine reflection seismology apparatus:

- source (airgun array, explosives,...)
- receivers (hydrophone streamer, ocean bottom cable,...)

- recording and onboard processing




Land acquisition similar in principle, but coupling of source, receiver to material is more variable, near-surface more heterogeneous, so data tends to pose more complex processing and interpretation problems.

Also vast bulk (90%+) of data acquired each year is marine.

Typical airgun array geometry, streamer structure

Bolt airgun

Key structural feature: velocity of waves

"It all depends on $v(x,y,z)$"

- Jon Claerbout, SEP Report No. 100, 1999

"...It should be noted that the most critical and time consuming aspect of the 3-D PreStack Depth Migration imaging project was building the velocity model."

- Davis W. Ratcliff and David J. Weber, "Geophysical imaging of subsalt geology", *The Leading Edge*, February 1997

Conundrum: main source of information about wave velocity, needed to interpret data for Earth structure, is data itself!

Displaying reflection seismograms: “Wiggle trace variable area plot” or grey scale

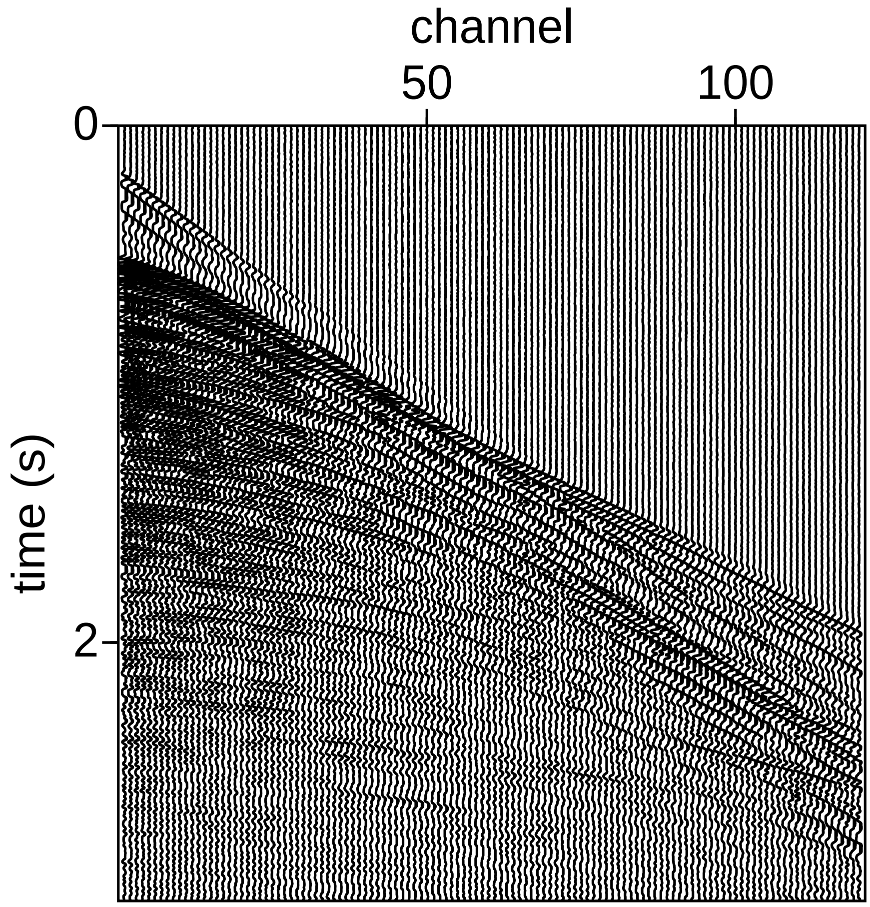
- **trace** = single channel recording, typically sampled at 1, 2, or 4 ms; plot time axis *down* (time $\sim$ depth)

- plot many traces together, each with its own copy of the time axis = **gather** or **ensemble** (several possibilities)

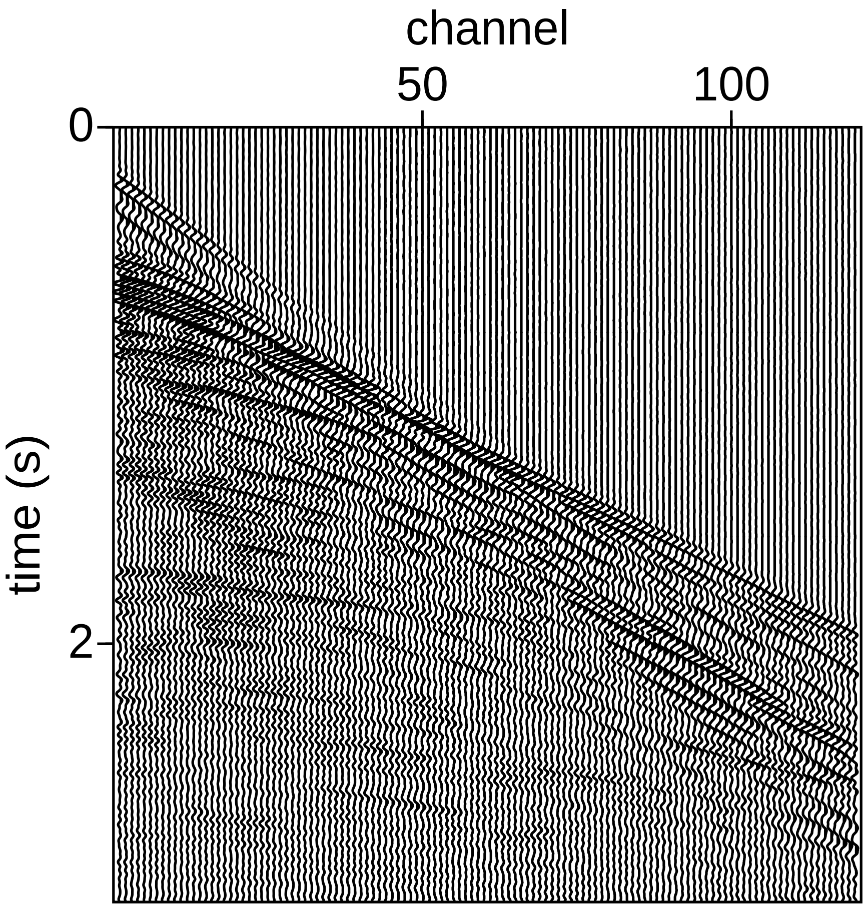
- horizontal coordinate *within* single trace = **amplitude**

- horizontal coordinate *between* traces = (or is related to) **offset** = source - receiver distance or some other trace parameter

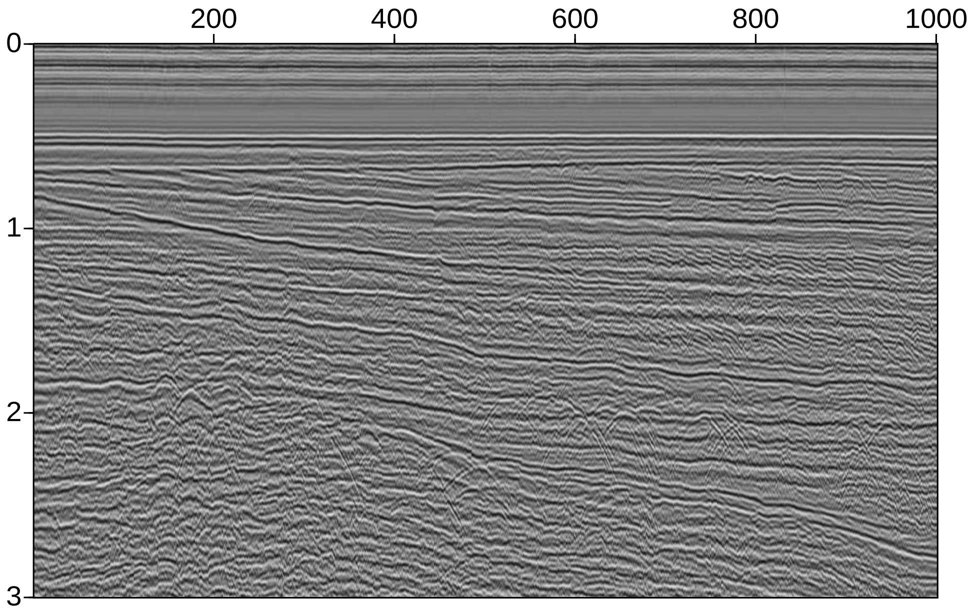
- WTVA: shade in excursion to one side of each time axis to emphasize space-time coherence = **events** or **reflections**



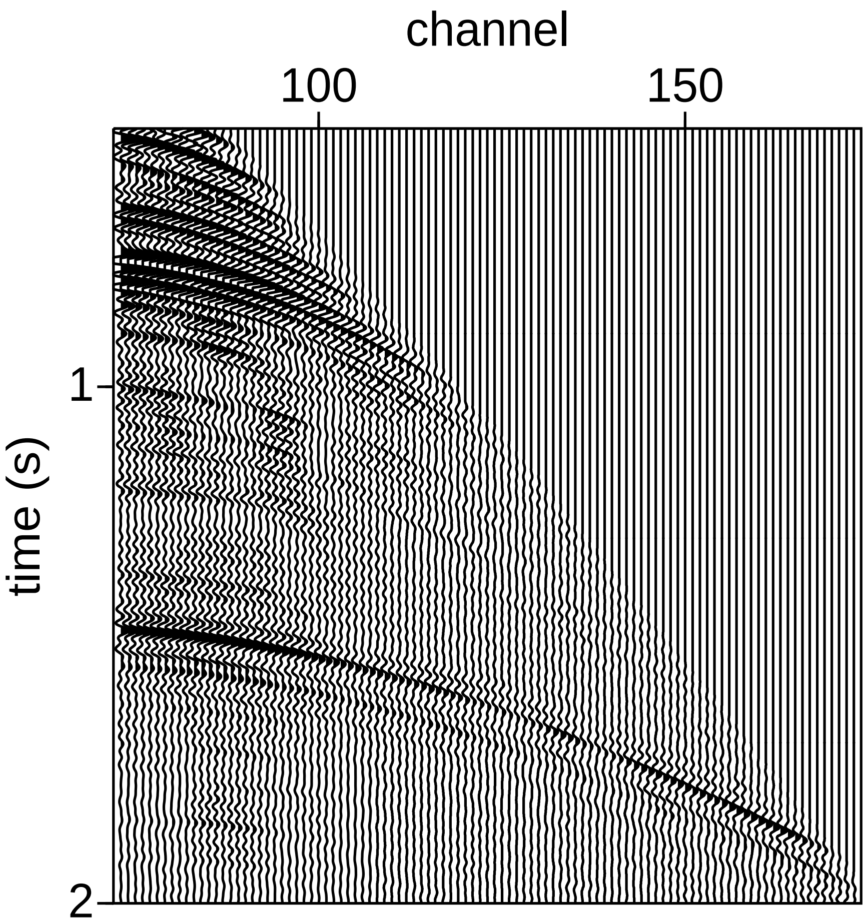

A field shot record from the North Sea, single streamer survey: “Mobil AVO”, 25 m shot and recvr intervals, near offset 160 m, far offset 3000 m, source depth 4 m, recvr depth 6 m, 120 channels (thanks: Mobil R & D)




Same shot record, bandpass filtered 10-20-40-50




Near offset gather from "Mobil AVO"




Another shot record from the North Sea: 12.5 m receiver interval = channel spacing, 240 receiver channels in cable, near offset = 146 m, far offset = 3111 m, shot interval = 18.75 m, source depth = 5 m, receiver depth = 6 m (Thanks: Shell Research)

Contemporary "3D" surveys

- Simultaneous recording by multiple streamers (up to 12!)

- Many (roughly) parallel lines, areal coverage

- *Much* more data - terabytes not uncommon

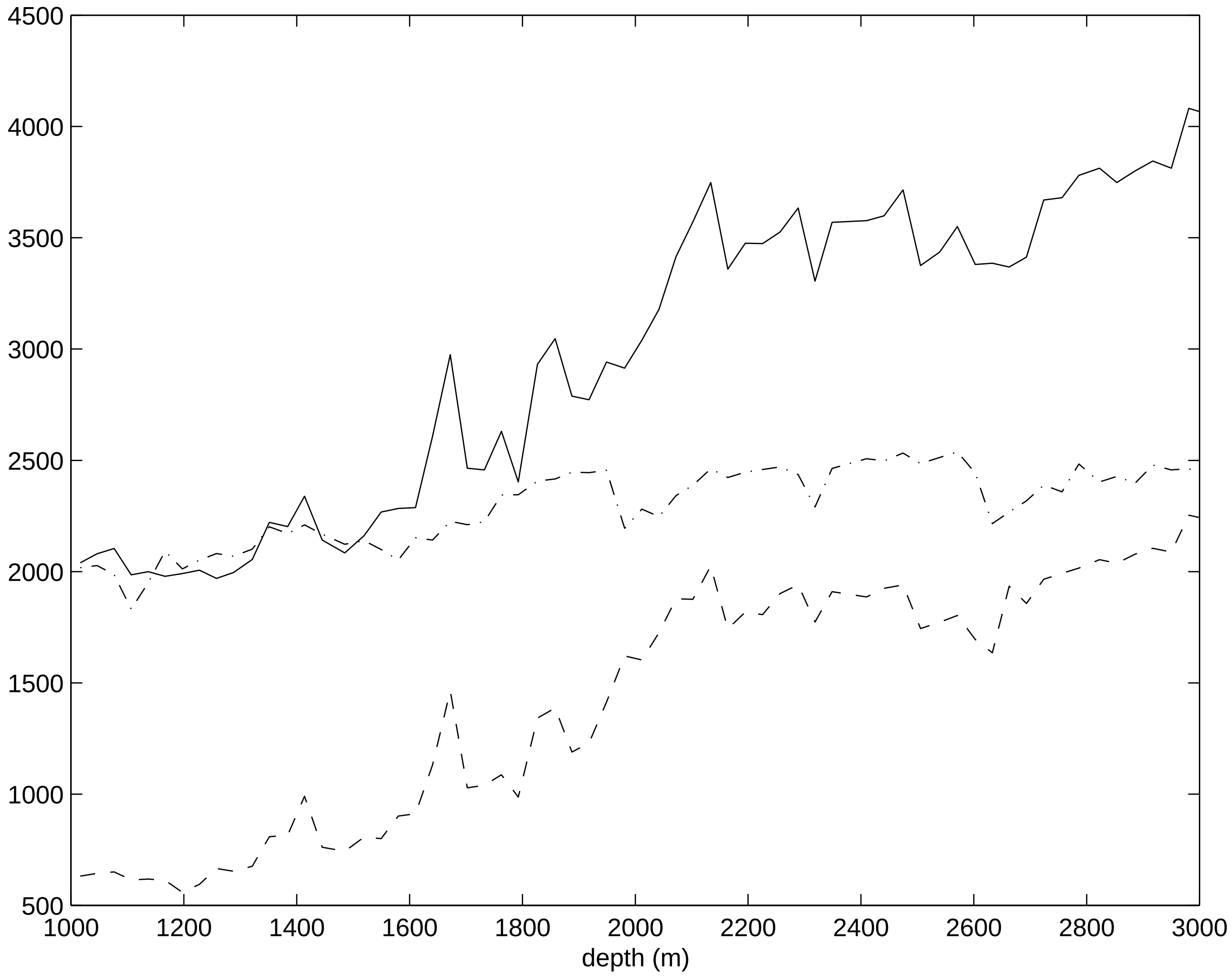
- used to generate 3D Earth model

- Much can be learned from 1D, 2D idealizations, though 3D problem presents its own difficulties

- will concentrate on "2D" (and "1D"!) surveys in this talk - sources, receivers in line.

Characteristics of crustal rocks

- depths of interest in science and technology: 100 m = environmental engineering, 10 km = petroleum exploration, 100 km = crustal geophysics
- approximately linearly elastic - small forces, displacements proportional, but some frequency dependence
- direct, indirect measurement of some elastic parameters in wells - **logging**
- heterogeneity at all scales - km to mm to $\mu$m; wave velocities vary by factor of 2 - 5 over region of interest, density somewhat less
- sedimentary rocks contain many localized zones of rapid mechanical property changes along surfaces = **reflectors**




Blocked logs from well near “Mobil AVO” line. Solid: p-wave velocity (m/s), dashed: s-wave velocity (m/s), dash-dot: density (kg/m$^3$). “Blocked” means “averaged”, in this case over 30 m windows. Original sample rate of log tool $< 1$ m.

The Modeling Task Any model of the reflection seismogram must predict

- wave motion
- reflections from reflectors
- significant variation of wave velocity, material density,...

A *really good* model will also accomodate

- multiple wave modes, speeds
- material anisotropy
- attenuation, frequency dispersion of waves
- complex source, receiver characteristics

A pretty good model: Linear Elasticity

Relates particle velocity $\mathbf{v}(\mathbf{x}, t)$ and displacement from equilibrium $\mathbf{u}(\mathbf{x}, t)$ at position $x$, time $t$, $\rho(x)$ = mass density, $\sigma$ = stress tensor (force/area), $\mathbf{f}$ = force density

$$\rho\frac{\partial \mathbf{v}}{\partial t} = \nabla \cdot \sigma + \mathbf{f},\ \mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}\ (+\,\text{b.c.}'\text{s})$$

Elastic constitutive law:

$$\sigma_{ij} = \sum_{kl} C_{ijkl}\epsilon_{kl},\ \epsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$$

$C$ = Hooke tensor, $\epsilon$ = strain tensor

$\epsilon$ symmetric by defn, $\sigma$ symmetric due to conservation of angular momentum, $\Rightarrow C_{ijkl} = C_{jikl} = C_{ijlk}$. Conservation of Energy $\Rightarrow C_{ijkl} = C_{klij}$. Upshot: 21 independent $C$'s.

Strain energy in $\Omega \subset \mathbf{R}^3$

$$E(t) = \frac{1}{2}\int_\Omega dx\ \{\rho \mathbf{v}\cdot\mathbf{v} + \mathrm{tr}\,\sigma\epsilon\}$$

Conservation of energy:

$$\frac{dE}{dt} = P$$

where $P$ is the power (rate of energy) transmitted through $\partial\Omega$:

$$P(t) = \int_{\partial\Omega} dS(\mathbf{x})\mathbf{n}^T\sigma(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)$$

**Exercise:** if you've never done this computation, do it now.

Thought experiment: suppose that sources, receivers are outside of target region $\Omega$. Then total power through $\partial\Omega$ dominates measured signal strength $\Rightarrow$ *deterministic localization must not occur, else no measurements!*

Linear elasticity as model of seismic waves:

- predicts several types of waves, each with characteristic speed
- speeds can be anisotropic (dependent on direction of travel)
- does not predict attenuation, dispersion of waves

Common idealization: material isotropy

$$C_{ijkl}(\mathbf{x}) = \lambda(\mathbf{x})\delta_{ij}\delta_{kl} + 2\mu(\mathbf{x})\delta_{ik}\delta_{jl}$$

yields compressional mode with speed $c_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$, shear mode with speed $c_s = \sqrt{\frac{\mu}{\rho}}$

Reflection seismic inverse problem, elastic version: From measurements of stress and/or particle velocity components near surface of Earth, determine distribution of elastic parameters, i.e. $\rho(\mathbf{x}), c_p(\mathbf{x}), c_s(\mathbf{x})$

If near-surface is fluid ($\mu = 0$) as in marine seismic, natural measurement is **pressure**

$$p = -\frac{1}{3}\mathrm{tr}\,\sigma = -\lambda\nabla\cdot\mathbf{u}$$

Simple model of seismic survey:

Experiments parametrized by source position $\mathbf{x}_s$, eg. point dilatational model

$$\mathbf{f}(\mathbf{x},t;\mathbf{x}_s) = w(t)\nabla\delta(\mathbf{x}-\mathbf{x}_s)$$

Correspondingly, velocity and stress fields depend on $\mathbf{x}_s$

Set of source, receiver positions = $\{\mathbf{x}_s, \mathbf{x}_r\}$

Data = $\{p(\mathbf{x}_r,t;\mathbf{x}_s) : 0 \le t \le T\}$

Reflection seismic inverse problem, marine version: from pressure measurements near surface for variety of source, receiver positions, determine $\rho(\mathbf{x}), c_p(\mathbf{x}), c_s(\mathbf{x})$.

Acoustic Model: $\mu = 0$ throughout subsurface
$\Rightarrow$ only compressional waves

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{f},$$

$$\frac{\partial p}{\partial t} = -\lambda \nabla \cdot \mathbf{v} \ (+\,\text{b.c.}'\text{s})$$

(compressional) wave speed $c = \sqrt{\frac{\lambda}{\rho}}$

Clearly wrong, but very useful - basis of most seismic processing.

*acoustic field potential* $u(\mathbf{x},t) = \int_{-\infty}^{t} ds\, p(\mathbf{x},s)$:

$$p = \frac{\partial u}{\partial t},\ \mathbf{v} = \frac{1}{\rho}\nabla u$$

Equivalent form: second order wave equation for potential

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla u = \int_{-\infty}^{t} dt\, \nabla \cdot \left(\frac{\mathbf{f}}{\rho}\right) \equiv \frac{f}{\rho}$$

**Exercise:** in terms of the acoustic potential, the strain energy and power densities are

$$\frac{1}{2\rho}\left\{\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 + \nabla u \cdot \nabla u\right\} \text{ and } \frac{1}{\rho}\frac{\partial u}{\partial t}\theta \cdot \nabla u$$

Application of energy conservation: existence of *weak solutions* of Dirichlet problem =

$$u \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H^1_0(\Omega))$$

satisfying: for any $\phi \in C_0^\infty((0,T) \times \Omega)$,

$$\int_0^T \int_\Omega dt\, dx \left\{\frac{1}{\rho c^2}\frac{\partial u}{\partial t}\frac{\partial \phi}{\partial t} - \frac{1}{\rho}\nabla u \cdot \nabla \phi - \frac{1}{\rho} f\phi\right\} = 0$$

**Theorem** (Lions, 1972): Suppose that $\log \rho, \log c \in L^\infty(\Omega), f \in L^2(\Omega)$. Then weak solutions of Dirichlet problem exist; initial data

$$u(\cdot,0) \in H^1_0(\Omega),\ \frac{\partial u}{\partial t}(\cdot,0) \in L^2(\Omega)$$

uniquely determine them.

Layered fluid: $z = x_3, \rho = \rho(z), c = c(z)$. Plane wave potential: $u = u(t - p_1 x_1 + p_2 x_2, z)$, slowness vector $(p_1, p_2)$. Normally incident: $p_1 = p_2 = 0 \Rightarrow u(t,z)$ satisfies

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial z}\frac{1}{\rho}\frac{\partial u}{\partial z} = \frac{f}{\rho}$$

**Assumptions for layered medium, plane wave problems:**

- boundaries are far away: spatial domain $\Omega = \mathbf{R} = \{z : -\infty < z < \infty\}$

- coefficients are constant for $z < 0$: $\rho(z) = \rho_0, c = c_0$

- support of energy source (i.e. $f$) contained in $z, t << 0$ ("seismology at the bottom of a very deep ocean"), so effectively...

- $u$ satisfies homogeneous wave equation for $z \sim 0$ with *downgoing wave initial data*

$$u(z,t) = w(t - c_0 z),\ t < 0$$

for a pulse $w(t)$ with $w(t) = 0, t > 0$ and $w' \in L^2(\mathbf{R})$ (so initial data has finite energy);

- so $u(z,t)$ is an *upcoming* wave in $z < 0, t > 0$ and satisfies the radiation boundary condition

$$\frac{\partial u}{\partial z}(0,t) = c_0 \frac{\partial u}{\partial t}(0,t),\ t > 0$$

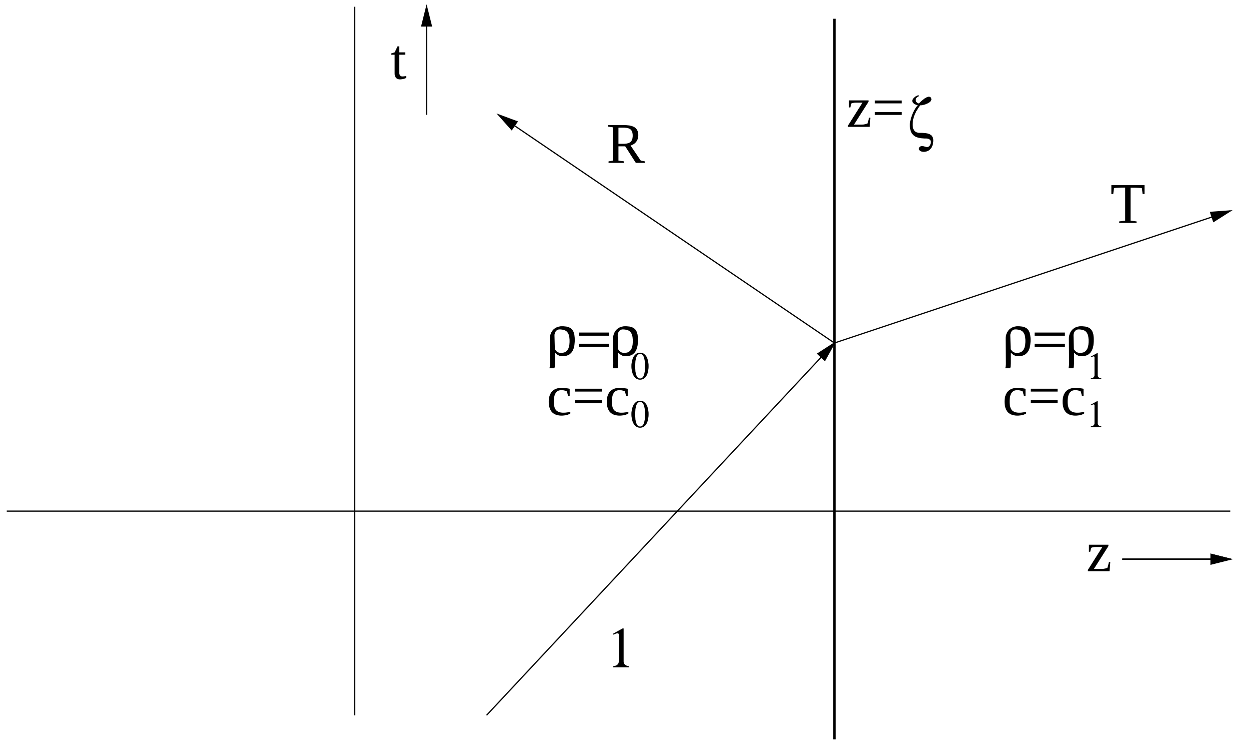
- $\log \rho, \log c \in L^\infty(\mathbf{R})$

From Lions: weak solutions exist.

Reflection seismic inverse problem, normally incident plane wave ("1D") version: given pressure record at $z = 0$: $\frac{\partial u}{\partial t}(0,t), 0 \le t \le T$, determine $\rho(z), c(z)$ for $z > 0$.

Can learn a lot from a very special case: piecewise homogeneous material, interface at $\zeta$:

$$\rho(z) = \begin{cases} \rho_0, & z < \zeta \\ \rho_1, & z > \zeta \end{cases} \quad c(z) = \begin{cases} c_0, & z < \zeta \\ c_1, & z > \zeta \end{cases}$$



Physics of wave interaction with interface:

- on either side of interface, potential is sum of up- and down-going waves
- across interface, pressure ($\frac{\partial u}{\partial t}$) and particle velocity ($\frac{1}{\rho}\frac{\partial u}{\partial z}$) are continuous

**Exercise:** Show that potential satisfies these "physical" conditions iff it is a weak solution of wave equation.

Leads to reflection and transmission coefficients:

$$u(z,t) = \begin{cases} w(t - c_0 z) + Rw(t + c_0(z - 2\zeta)), & z < \zeta \\ Tw(t - c_1 z + (c_0 - c_1)\zeta), & z > \zeta \end{cases}$$

where

$$R = \frac{\rho_1 c_1 - \rho_0 c_0}{\rho_1 c_1 + \rho_0 c_0},\ T = \frac{2\rho_1 c_1}{\rho_1 c_1 - \rho_0 c_0}$$

Only three quantities appear in predicted "seismogram"

$$\frac{\partial u}{\partial t}(0,t) = w'(t) + Rw'(t - 2c_0\zeta)$$

namely

- the pulse $w(t)$
- the reflection coefficient $R$
- the *two way traveltime* to the interface $\tau \equiv c_0\zeta$

Immediate conclusions:

- from 1D seismogram, *can determine neither wave velocity* $c_0$ *nor interface depth* $\zeta$ separately, unless you know one of them *a priori*
- the data tells you nothing about $\rho_0$
- suppose you know $\rho_0, c_0$, then data determines $\zeta, R$ - but knowledge of $R$ gives only the *impedance* $\rho_1 c_1$, *not* $\rho_1$ and $c_1$ separately.

Natural idea: adjust $w(t), R, \tau$ to fit predicted to observed data.

[Contrary to what you might think, the pulse $w(t)$ is *not* generally observed directly - the “data” is best modeled as the second term above!]

Predicted, observed data:

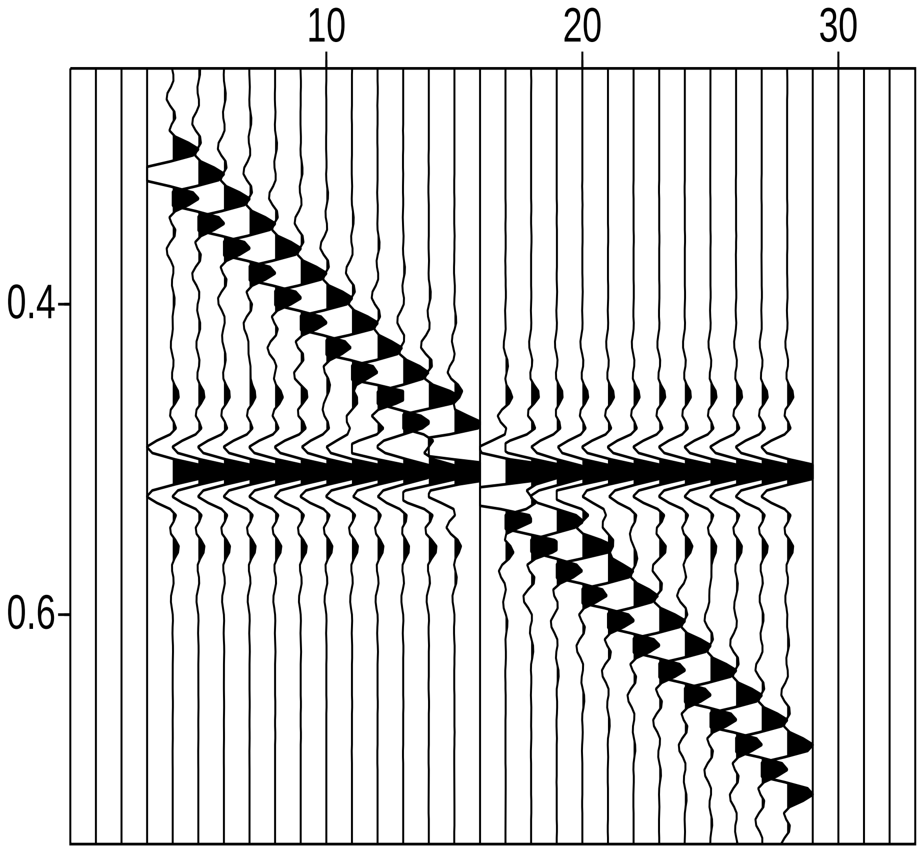
$$S[w,R,\tau](t) = Rw'(t-\tau),\ S_{\text{obs}}(t) = R^*(w^*)'(t-\tau^*)$$

Mean-square error function:

$$J[w,R,\tau] = \frac{1}{2}\int dt\,|S[w,R,\tau](t) - S_{\text{obs}}(t)|^2$$

Note that $J$ is *linear* in $w, R$, quite *nonlinear* in $\tau$: if $\tau$ differs by more than the pulse width from $\tau^*$, than $J$ becomes independent of $\tau$!.

Upshot: local (Newton, descent) optimization doesn't work unless you know the answer already! Global optimization no big deal for 1 parameter, but for larger scale problems becomes prohibitive.




Example of saturation phenomenon: two identical signals, except for sign and phase. As second is shifted to match first, no change in size of difference until the phases are within a wavelength of matching! Moral: the unitary group of shifts on $L^2$ is strongly, but not even locally uniformly, continuous.

Back to general 1D: suppose you change the depth scale, replacing $z$ by $\tau(z), \tau' > 0, \tau(0) = 0$ (notational foresight!).

**Exercise:** Set $\bar{u}(\tau,t) = u(z(\tau),t)$; then

$$\frac{1}{\bar{\rho}\bar{c}^2}\frac{\partial^2 \bar{u}}{\partial t^2} - \frac{\partial}{\partial \tau}\frac{1}{\bar{\rho}}\frac{\partial \bar{u}}{\partial \tau} = \frac{\bar{f}}{\bar{\rho}}$$

where

$$\bar{\rho}(\tau) = \rho(z(\tau))j(\tau),\ \bar{c}(\tau) = \frac{c(z(\tau))}{j(\tau)},$$

$$\bar{f}(\tau,t) = f(z(\tau),t) \quad j(\tau) = z'(\tau)$$

Also note that $\bar{u}(0,t) = u(0,t)$ - i.e. the data is independent of the length scale.

Conclusion: it is impossible to extract $c$ and/or $\rho$ as functions of depth from the data of the 1D reflection seismic inverse problem.

What can be extracted?

First make sure that data prediction is well-defined!

Note that weak existence result for $\rho, c \in L^\infty$ does not assure that the pressure trace $\frac{\partial u}{\partial t}(0,t)$ exists, even as a distribution.

The $L^2$ Theory: begin with class of coefficients for which pressure trace is well-defined, develop continuity results permitting a maximal extension.

Where to start: $\log \rho, \log c \in C^\infty$. Make unique choice of new depth scale $\tau(z)$ for which $\bar{c} \equiv 1$:

$$\tau(z) = \int_0^z \frac{dz}{c(z)}$$

i.e. traveltime. Set $a(\tau) = \bar{\rho}(\tau) =$ *impedance as function of traveltime*. Then (dropping overbars)

$$\frac{1}{a}\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial \tau}\frac{1}{a}\frac{\partial u}{\partial \tau} = 0$$

Alternate form in terms of *reflectivity* $r = (\log a)' \in C^\infty$:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial \tau^2} + r\frac{\partial u}{\partial \tau} = 0$$

Recall that boundaries, source support are "far away", coeffs constant for $z$ or $\tau < 0$, so

- $r = 0$, $a = 1$ for $\tau < 0$
- for $t < 0$, $u(\tau, t) = w(t - \tau)$ (downgoing incident wave), $w' \in L^2, w(t) = 0$ for $t > 0$.

Choose $T > 0$ and define $S_w : C_0^\infty(0, T) \rightarrow L^2[0, 2T]$ by

$$S_w[r](t) = \frac{\partial u}{\partial t}(0, t),\ 0 < t < 2T$$

Note that

$$S_w[r] = w * S_\delta[r] = w' * S_H[r]$$

Plan: study $S_H$ ("Green's function, impulse response") first; then properties of $S_w$ follow from well-known properties of convolution

Main result:

$S_H$ extends to a $C^1$ diffeomorphism: $L^2[0,T] \to L^2[0,2T]$ (WWS, 1986)

- inverse problem for impedance as function of travel time, with impulsive data, is as well-posed as possible
- bandlimited pulse will cause effective nonuniqueness
- nothing else is determined

Main ideas of proof:

- energy estimates show equivalence of boundary, initial/other data
- Characteristic initial data of $u_H$ directly from impedance (geometric optics)

**Energy estimates for initial value problem:** similar to, but simpler than, estimates used to establish that $S_H$ is a diffeo. For the moment, $u$ denotes a solution of the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial \tau^2} + r\frac{\partial u}{\partial \tau} = 0$$

in $t, \tau > 0$, initial data

$$u(\tau, t) = u_0(\tau),\ \frac{\partial u}{\partial t}(\tau, 0) = v_0(\tau)$$

and for simplicity assume that $u_0, v_0$ and $r \in C_0^\infty(\mathbf{R})$, whence $u \in C^\infty$ also - extensions to larger classes will follow.

Pick $T > 0$, and for $0 < t, \tau < T$ set

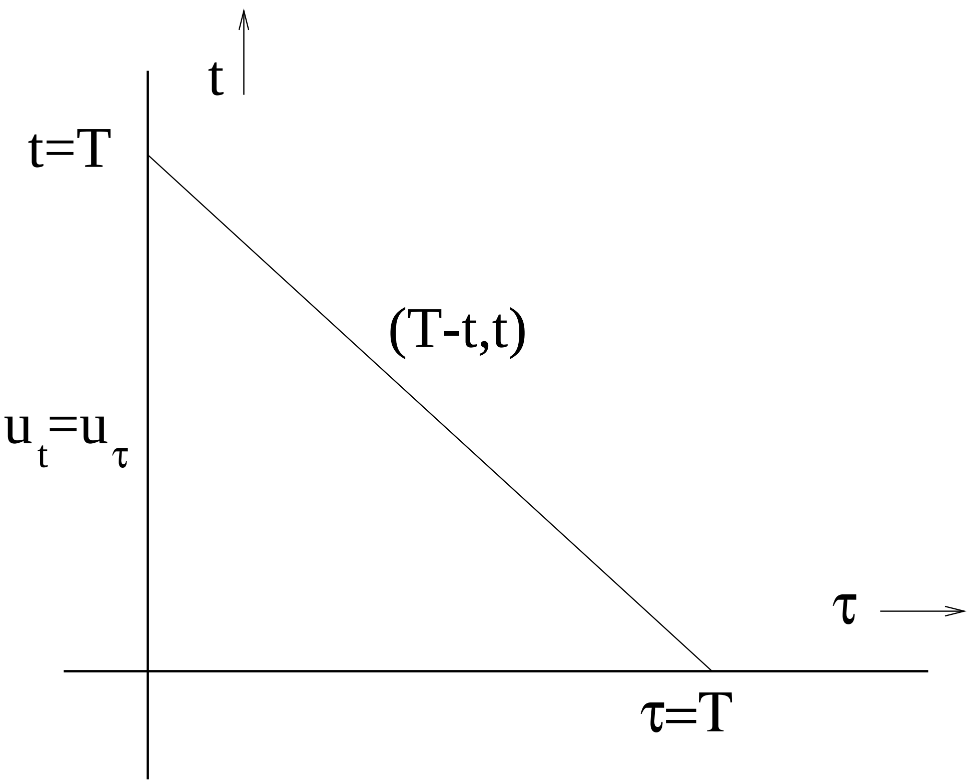
$$E(t) = \frac{1}{2}\int_0^{T-t} d\tau\, \frac{1}{a(\tau)} \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial \tau}\right)^2 \right\} (\tau, t)$$

$$Q(\tau) = \frac{1}{2}\int_0^{T-t} dt\, \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial \tau}\right)^2 \right\} (\tau, t)$$

Note that $E(0)$ is the energy of the initial data:

$$E(0) = \frac{1}{2}\int_0^T d\tau \, \frac{1}{a(\tau)} \left\{ v_0^2 + \left(\frac{\partial u_0}{\partial \tau}\right)^2 \right\}(\tau)$$

Because $u(\tau, 0) \equiv 0$ for $\tau < 0$, $u$ is upcoming in $\tau < 0$, so $u_t = u_\tau$ on $\tau = 0, 0 < t < T$.



For soln of homogeneous eqn, energy in initial data bounds power through $\tau = 0$:

$$\frac{dE}{dt}(t) = -\frac{1}{2a(T-t)}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial \tau}\right)^2\right\}(T-t,t)$$

$$+\int_0^{T-t} d\tau\, \frac{1}{a}\left\{\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial \tau}\frac{\partial^2 u}{\partial \tau \partial t}\right\}$$

$$= -\frac{1}{2a(T-t)}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial \tau}\right)^2\right\}(T-t,t)$$

$$+\left[\frac{1}{a}\frac{\partial u}{\partial t}\frac{\partial u}{\partial \tau}\right]_{\tau=0}^{\tau=T-t}$$

$$\leq -\frac{1}{2a(0)}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial \tau}\right)^2\right\}(0,t)$$

where we have used the upcoming boundary condition at $\tau = 0$.

Integrate from 0 to $T$ and use $E(T) = 0$ to get

$$-E(0) \leq -\frac{1}{a(0)}Q(0), \text{ i.e. } Q(0) \leq a(0)E(0)$$

Power through $\tau = 0$ dominates initial energy - i.e. deterministic localization does not occur!

Must add assumption to ensure that we don't create a purely downgoing wave:

$$\int_0^T d\tau \, \frac{\partial u}{\partial t}\frac{\partial u}{\partial \tau}(\tau, 0) \geq CE(0)$$

for some $C > 0$.

**Exercise:** ("sideways energy inequality") By differentiating $Q$, integrating by parts, using the wave equation and Gronwall's inequality, show that

$$Q(\tau) \leq Q(0) - \int_0^\tau d\sigma \, e^{-2\int_\sigma^\tau |r|}\frac{\partial u}{\partial t}\frac{\partial u}{\partial \tau}(\sigma, 0)$$

Since $Q(T) = 0$, setting $\tau = T$ get

$$Q(0) \geq Ce^{-2\int_0^T |r|}E(0)$$

Note that this argument works because the 1D wave equation is hyperbolic sideways as well as right side up. Fast numerical methods for IP based on this fact (Bube & Burridge, 1983).

Perspective on non-localization: critical quantity is

$$\int_0^T |r| = \| \log a \|_{BV}$$

As this goes $\to \infty$, power output relative to initial energy may go to zero.

In a family $a_\epsilon(\tau) = a_0(\tau)\left(1 + \epsilon^\alpha \eta\left(\frac{\tau}{\epsilon}\right)\right)$, $\| \log a_\epsilon \|_{BV} \to \infty$ if $\alpha < 1$ so the lower bound for $Q(0)/E(0) \to 0$ as one would expect from stochastic localization theory.

Estimates for the Green's function: for smooth $a$, has a jump along $t = \tau$ which is algebraic in $a$ (transport equation from geometric optics).

Similar energy estimates for characteristic initial value problem, and for related problems involving differences and 1st order Taylor remainders for Green's functions, complete proof of diffeo result. See reference for details.

Back to the multidimensional problem, reconsider "layer over half space" example:

$$\rho(z) = \begin{cases} \rho_0, & z < \zeta \\ \rho_1, & z > \zeta \end{cases} \quad c(z) = \begin{cases} c_0, & z < \zeta \\ c_1, & z > \zeta \end{cases}$$

To have any hope of determining $c_0$, $\zeta$ individually, must have data for more than one plane wave.

Wave equation for plane wave $u(z,t;p)$ with slowness $(p,0)$:

$$\frac{1}{\rho}\left(\frac{1}{c^2} - p^2\right)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial z}\frac{1}{\rho}\frac{\partial u}{\partial z} = 0$$

so the vertical wavespeed for this plane wave is

$$\frac{c}{\sqrt{1 - c^2 p^2}} = \left(\frac{1}{c^2} - p^2\right)^{-\frac{1}{2}}$$

(faster!) and the pressure trace at $x = y = z = 0$ is

$$\frac{\partial u}{\partial t}(0,t;p) = w'(t) + R(p)w'\left(t - \zeta\sqrt{\frac{1}{c_0^2} - p^2}\right)$$

**Exercise:** Show that knowledge of the arrival time

$$\tau(p) = \zeta\sqrt{\frac{1}{c_0^2} - p^2}$$

for two different values of $p$ (say $p = 0$ and $p_1 > 0$ determines $c_0$ and $\zeta$ uniquely. Estimate the condition number of this problem (i.e. of its linearization) and show that it $\to \infty$ as $p_1 \to 0$.

How do you get your hands on wave data: naively,

$$u(z, t; p_1, p_2) =$$

$$\int\int dx_{s,1}\, dx_{s,2} u(0, 0, z, t - p_1 x_{s,1} - p_2 x_{s,2}; \mathbf{x}_s)$$

Data sampled and not available for all $x_{s,1}, x_{s,2}$ $\Rightarrow$ production of plane wave data from point source data is nontrivial.

Perhaps easier to stick with point source data.

Enough data to determine $c(z) \Rightarrow$ overdetermined, so data fitting formulation is natural.

For example: output least squares

$$J[\rho, c; p^{\text{obs}}] =$$

$$\int\int\int dx_r\, dx_s\, dt\, |p(\mathbf{x}_r, t; \mathbf{x}_s) - p^{\text{obs}}(\mathbf{x}_r, t; \mathbf{x}_s)|^2$$

Predicted pressure field $p$ depends on $\rho, c$ through wave equation; solve this numerically, approximate integrals with quadratures, sample $\rho, c$ appropriately $\Rightarrow$ large scale numerical optimization problem. Perhaps add constraints or regularization.

**Such approaches have not been successful.** Beautiful example: Kolb et al, 1986. "Layer over half space example" already hints at reason. For more effective understanding, need more analysis...