Extensions and Nonlinear Inverse Scattering

William W. Symes

IPRPI, April 2004
Outline

1. The acoustic model of reflection seismology
2. Linearization
3. Why Least Squares doesn’t work
4. Extensions
5. Annihilators
7. References
1. The Acoustic Model of Reflection Seismology
Marine Acquisition
90%+ of all data collected worldwide
Data parameters: time $t$, source location $x_s$, and receiver location $x_r$, (vector) \textit{half offset} $h = \frac{x_r - x_s}{2}$, scalar half offset $h = |h|$. Experiment = \textit{shot}, single experiment data = \textit{shot record}.
Typical Marine Record

Shot record, Gulf of Mexico (thanks: Exxon)
Mechanical Characteristics of Sedimentary Rock

Well logs from North Sea borehole. Top curve: compressional wave velocity (m/s); middle curve: density (kg/m$^3$); bottom curve: shear wave velocity (m/s). (thanks: Mobil R&D, Viking Graben)
Constant Density Acoustic Model

**Acoustic potential** $u(x, t)$ related to pressure $p$ and particle velocity $v$ by

$$p = \frac{\partial u}{\partial t}, \quad v = \frac{1}{\rho} \nabla u$$

Second order wave equation for potential

$$\left( \frac{1}{c(x)^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) u(x, t) = w(t) \delta(x - x_s)$$

plus initial, boundary conditions. RHS models localized energy source, “no low frequencies” - many wavelengths between source and target. **Useful idealization:** $w(t) = \delta(t)$.

**Forward map:** $F[c] \equiv p|_Y$, where $Y = \{(t, x_r, x_s) : 0 \leq t \leq T, ... \}$ is acquisition manifold.
2. Linearization
Nonlinear inverse scattering

Inverse problem: given \( d \in L^2(Y) \) find \( c \in C \) s. t. \( \mathcal{F}[c] \simeq d \).

Many difficulties:

- What is \( C \)?
- What is \( \simeq \)?
- If \( \simeq \) means “close in \( L^2 \)”, could pose as *least squares* problem: find \( c \in C \) to minimize \( \| \mathcal{F}[c] - d \|^2 \).
- Results of numerical experimentation mixed.
- Theoretical foundation inadequate - few results re relevant properties of \( \mathcal{F} \).
(Partly) linearized inverse scattering

Formally, $\mathcal{F}[v(1 + r)] \simeq \mathcal{F}[v] + F[v]r$ where $F[\cdot]$ is linearized forward map defined by

$$\left(\frac{1}{v(x)^2} \frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta u(x, t) = 2 \frac{r(x)}{v^2(x)} \frac{\partial^2 u}{\partial t^2}(x, t)$$

$$F[v]r = \delta p|_Y$$

- basis of most practical data processing procedures.
- Beylkin (1985) and many others: good understanding of the linear map $r \mapsto F[v]r$ and the associated linear inverse problem for $r$ given $v$;
- $v$ is no more known than $r$, inverse problem for $[v, r]$ still nonlinear!
Linearization error

Critical question: If there is any justice $F[v]r = \text{directional derivative} \ DF[v][vr]$ of $\mathcal{F}$ - but in what sense? Physical intuition, numerical simulation, and not nearly enough mathematics: linearization error

$$\mathcal{F}[v(1 + r)] - (\mathcal{F}[v] + F[v]r)$$

- **small** when $v$ smooth, $r$ rough or oscillatory on wavelength scale - well-separated scales
- **large** when $v$ not smooth and/or $r$ not oscillatory - poorly separated scales

2D finite difference simulation: shot gathers with typical marine seismic geometry. Smooth (linear) $v(x, z)$, oscillatory (random) $r(x, z)$ depending only on $z$ (“layered medium”). Source wavelet $w(t) = \text{bandpass filter.}$
Left: Total velocity $c = v(1 + r)$ with smooth (linear) background $v(x, z)$, oscillatory (random) $r(x, z)$. Std dev of $r = 5\%$.

Right: Simulated seismic response ($\mathcal{F}[v(1 + r)]$), wavelet = bandpass filter 4-10-30-45 Hz. Simulator is (2,4) finite difference scheme.
Model in previous slide as smooth background (left, $v(x, z)$) plus rough perturbation (right, $r(x, z)$).
Left: Simulated seismic response of smooth model ($\mathcal{F}[v]$),
Right: Simulated linearized response, rough perturbation of smooth model ($F[v]r$)
Model in previous slide as rough background (left, $v(x, z)$) plus smooth 5% perturbation ($r(x, z)$).
Left: Simulated seismic response of rough model ($\mathcal{F}[v]$),
Right: Simulated linearized response, smooth perturbation of rough model ($F[v]r$)
Left: linearization error \( \mathcal{F}[v(1 + r)] - \mathcal{F}[v] - F[v]r \), rough perturbation of smooth background

Right: linearization error, smooth perturbation of rough background (plotted with same grey scale).
Implications

• Some geologies have well-separated scales - cf. sonic logs - linearization-based methods work well there. Other geologies do not - expect trouble!

• \( v \) smooth, \( r \) oscillatory \( \Rightarrow F[v]r \) approximates \textbf{primary reflection} = result of wave interacting with material heterogeneity only once (single scattering); error consists of \textbf{multiple reflections}, which are “not too large” if \( r \) is “not too big”, and sometimes can be suppressed.

• \( v \) nonsmooth, \( r \) smooth \( \Rightarrow \) error consists of \textit{time shifts} in waves which are very large perturbations as waves are oscillatory.

\textit{No mathematical results are known which justify/explain these observations in any rigorous way, except in 1D.}
3. Why Least Squares doesn’t work
\[
\min_c \| \mathcal{F}[c] - d \|^2
\]

- Problems are so large that iterative methods (variants of Newton) are only option (3D: millions of unknowns, billions of equations) \(\Rightarrow\) can only find stationary points;
- For any choice of norm in domain, \( \mathcal{D} \mathcal{F} \) has very poor condition - very large, very small singular values (cf. examples);
- Poor approximation of \( \mathcal{F} \) by linearization \(\Rightarrow\) poor approximation of least squares function by quadratic;
- Observed behaviour is \textit{nonconvex} \(\Rightarrow\) many stationary points exist with large residuals.
- Same remarks apply (and are a bit easier to justify) for \textit{partially linearized least squares} \( \min_{v,r} \| F[v]r - (d - F[v]) \|^2 \).
The good news...

We actually know something about $F[v]$, besides its representation when $w(t) = \delta(t)$:

$$F[v]r(t, x_r, x_s) = \frac{\partial^2}{\partial t^2} \int dx \int d\tau G(x, x_r, t - \tau)G'(x, x_s, \tau) \frac{2r(x)}{v^2(x)}$$

- $F[v] : \mathcal{E}'(X) \rightarrow \mathcal{D}'(Y)$ ($X = \text{Earth}$) is a Fourier Integral Operator associated to a canonical relation (Lagrangian submanifold of $T^*(X \times Y)$) (Rakesh, 1988);
- when canonical relation is graph, representation as Generalized Radon Transform (Beylkin, 1985) $\Rightarrow$ many practical computations;
- when canonical relation is a graph (Beylkin 1985, Rakesh 1988) and sometimes even when it isn’t (Smit, Verdel, tenKroode 1998, Nolan 1997, Stolk 2000), $F[v]^*F[v]$ is pseudodifferential operator $\Rightarrow$ construction of left parametrix or approximate microlocal inverse.
\[ \min r \| F[v]r - (d - \mathcal{F}[v]) \|^2, \text{ given } v \]

Approximate linear least squares solution après Beylkin (“GRT inversion”), Mississippi Canyon, Gulf of Mexico, 2D survey (750 MB, 500 shots). Thanks: Exxon.
4. Extensions
Extended models

Extension of $F[v]$ (aka extended model): manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \to \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Y)$ so that

\[
\begin{array}{ccc}
\mathcal{E}'(\bar{X}) & \rightarrow & \mathcal{D}'(Y) \\
\chi & \uparrow & \uparrow \text{id} \\
\mathcal{E}'(X) & \rightarrow & \mathcal{D}'(Y) \\
F[v] & & \\
\end{array}
\]

commutes, i.e.

$$\bar{F}[v] \chi r = F[v] r$$

Extension is “invertible” iff $\bar{F}[v]$ has a right parametrix $\bar{G}[v]$, i.e. $I - \bar{F}[v] \bar{G}[v]$ is smoothing, or more generally if $\bar{F}[v] \bar{G}[v]$ is pseudodifferential (“inverse except for wrong amplitudes”). Also require existence of a left inverse $\eta$ for $\chi$: $\eta \chi = \text{id}$.

**NB:** The trivial extension - $\bar{X} = X$, $\bar{F} = F$ - is virtually never invertible.
Grand Example

The Standard Extended Model:

- \( \bar{X} = X \times H, \ H = \text{offset range.} \)
- \( \chi_r(x, h) = r(x) \) (so \( \bar{r} \in \text{range of } \chi \) \( \iff \) plots of \( \bar{r}(\cdot, \cdot, z, h) \) (“image gathers”) appear flat)
- \( \bar{F}[v] \bar{r}(x_r, x_s, t) = \frac{\partial^2}{\partial t^2} \int dx \int d\tau \ G(x, x_r, t - \tau)G(x, x_s, \tau) \frac{2\bar{r}(x, h)}{v^2(x)} \)

(recall \( h = (x_r - x_s)/2 \))

**NB:** \( \bar{F} \) is “block diagonal” - family of operators (FIOs) parametrized by \( h \).
Reformulation of inverse problem

Given $d$, find $v$ so that $\bar{G}[v]d \in$ the range of $\chi$.

Claim: if $v$ is so chosen, then $[v, r]$ solves partially linearized inverse problem with $r = \eta \bar{G}[v]d$.

Proof: Hypothesis means

$$\bar{G}[v]d = \chi r$$

for some $r$ (whence necessarily $r = \eta \bar{G}[v]d$), so

$$d \simeq \bar{F}[v] \bar{G}[v]d = \bar{F}[v] \chi r = F[v]r$$

Q. E. D.
Application: Migration Velocity Analysis

Membership in range of $\chi$ is *visually evident*

$\Rightarrow$ industrial practice: adjust parameters of $v$ *by hand* (!) until visual characteristics of $\mathcal{R}(\chi)$ satisfied - “flatten the image gathers”.

For the Standard Extended Model, this means: until $\tilde{G}[v]d$ is independent of $h$.

Practically: insist only that $\tilde{F}[v]\tilde{G}[v]$ be pseudodifferential, so adjust $v$ until $\tilde{G}[v]d$ is “smooth” in $h$. 
Left: shot record \( (d) \) from North Sea survey (thanks: Shell Research), lightly pre-processed.

Right: restriction of \( \bar{G}[v]d^{obs} \) to \( x, y = \text{const} \) (function of depth, offset): shows rel. sm’ness in \( h \) (offset) for properly chosen \( v \).
5. Annihilators
Automating the reformulation

Suppose $W : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Z)$ annihilates range of $\chi$:

$$
\begin{align*}
\chi \quad & W \\
\mathcal{E}'(X) \to \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Z) \to 0
\end{align*}
$$

and moreover $W$ is bounded on $L^2(\bar{X})$. Then

$$
J[v; d] = \frac{1}{2} \| W \bar{G}[v] d \|^2
$$

minimized when $[v, \eta \bar{G}[v] d]$ solves partially linearized inverse problem.

Construction of annihilator of $\mathcal{R}(F[v])$ (Guillemin, 1985 - cf deHoop’s talks):

$$
d \in \mathcal{R}(F[v]) \Leftrightarrow \bar{G}[v] d \in \mathcal{R}(\chi) \Leftrightarrow W \bar{G}[v] d = 0
$$
Annihilators, annihilators everywhere...

For Standard Extended Model, several popular choices:

•

\[ W = (I - \Delta)^{-\frac{1}{2}} \nabla_h \]

(“differential semblance” - WWS, 1986)

•

\[ W = I - \frac{1}{|H|} \int dh \]

(“stack power” - Toldi, 1985)

•

\[ W = I - \chi F[v]^\dagger \bar{F}[v] \]

⇒ minimizing \( J[v, d] \) equivalent to least squares.
But not many are good for much...

Since problem is huge, only $W$ giving rise to differentiable $v \mapsto J[v, d]$ are useful - must be able to use Newton!!! Once again, idealize $w(t) = \delta(t)$.

**Theorem** (Stolk & WWS, 2003): $v \mapsto J[v, d]$ smooth $\iff W$ pseudodifferential.

i.e. only differential semblance gives rise to smooth optimization problem, regardless of source bandwidth.

NB: Least squares embedded in larger family of optimization formulations, some (others) of which are tractable.

Invertible Extensions

- Beylkin (1985), Rakesh (1988): if $\|\nabla^2 v\|_{C^0}$ “not too big” (no caustics appear), then the Standard Extended Model is invertible.

- Nolan & WWS 1997, Stolk & WWS 2004: if $\|\nabla^2 v\|_{C^0}$ is too big (caustics, multipathing), Standard Extended Model is not invertible! Not in any version - common offset, common source, common scattering angle,...

- Stolk & deHoop 2001: *Claerbout extension* is invertible under much weaker condition (absence of turning rays).

- WWS, Stolk, Biondi 2003: generalized Claerbout extension to accommodate turning rays.
Beyond Born

Nonlinear effects not included in linearized model: *multiple reflections*. Conventional approach: treat as *coherent noise*, attempt to eliminate - active area of research going back 40+ years, with recent important developments.

Why not model this “noise”?

Proposal: *nonlinear extensions* with $F[v]r$ replaced by $\mathcal{F}[c]$. Create annihilators in same way (now also nonlinear), optimize differential semblance.

Nonlinear analog of Standard Extended Model appears to be *invertible* - in fact extended nonlinear inverse problem is *underdetermined*.

Open problems: no theory. Also must determine $w(t)$ (Delprat & Lailly 2003).
And so on...


- Anisotropy - see deHoop’s talk, this meeting.

- Anelasticity - in the sedimentary section, $Q = 100 - 1000$, lower in gassy sediments and near surface. No results.


- ...
Conclusion

• Least error formulation of (waveform) reflection seismic inverse problem *intractable* - very irregular with large residual stationary points ⇒ *no influence on practice*.

• Linearized *extended models* provide framework for both (industry standard) interpretive velocity analysis and automated techniques based on construction of *range annihilators*.

• Only *(pseudo)differential annihilators* yield smooth objective functions.

• Not all extensions suitable for use in “complex structure” (strong refraction).

• May be able to account for more nonlinearity (multiple reflections) via nonlinear extensions.
Thanks to...

National Science Foundation
Department of Energy
Sponsors of The Rice Inversion Project
Veritas, Schlumberger, and Seismic Consultants for graphics
Exxon, Mobil, and Shell for data
The Organizers, for inviting me.

http://www.trip.caam.rice.edu
7. Selected References


Dobrin, M. and Savit, C.: *Introduction to Geophysical Prospecting*

